Optimal control of quadratic functionals for affine nonlinear systems

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Abstract In this paper we analyze the optimal control problem for a class of affine nonlinear systems under the assumption that the associated Lie algebra is nilpotent. The Lie brackets generated by the vector fields which define the nonlinear system represent a remarkable mathematical instrument for the control of affine systems. We determine the optimal control which corresponds to the nilpotent operator of the first order. In particular, we obtain the control that minimizes the energy of the given nonlinear system. Applications of this control to bilinear systems with first order nilpotent operator are considered. © 2012 The Chinese Society of Theoretical and Applied Mechanics. [doi:10.1063/2.1204310]

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The optimal control problem for nonlinear affine systems is to determine the optimal control functions which minimize the quadratic functionals submitted to the differential constraints represented by the considered dynamic system.

The theory of optimal control offers modern methods for the control of nonlinear systems and plays an important role in the analysis of the linear control characterizing the quadratic linear regulators and the Gaussian quadratic linear control. Using optimal control for the linear systems class will substantially diminish the effort in computing the laws of optimal control. The Lie brackets generated by the vector fields which define the nonlinear system represent a remarkable mathematical instrument for the control of affine systems. For instance, Siguerdidjane has applied the Lie algebra methods for the optimal control of satellites. Banks and Yew have studied the optimal control problem with the scope of minimizing the energy for a class of bilinear systems and Liu et al. have further generalized their results to the class of nonlinear affine systems.

We consider the class of affine nonlinear dynamic systems

\[ \dot{x} = Y^0(x) + \sum_{i=1}^{m} Y^i(x)u_i = Y^0(x) + Y(x)u, \]

where \( x \in \mathbb{R}^n \), \( u_i \in \mathbb{R}^1 \), \( i = 1, 2, \cdots, m \), \( Y^i(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) are assumed continuous functions of \( x \). The control problem associated to the differential constraints represented by the dynamic system (1) can be formulated as follows. Determine the optimal control functions \( u^*_i (i = 1, 2, \cdots, m) \) which minimize the quadratic functionals

\[ J = \frac{1}{2} \int_{t_0}^{t_f} (x^TQx + u^TRu) \, dt, \]

where \( Q = (g_{ij}) \) and \( R = (r_{ij}) \) are constant symmetric matrices of order \( (n \times m) \) and \( (m \times m) \), respectively, and \( t_f \) is an arbitrary chosen finite time.

Let us associate to the nonlinear system (1) the Lie algebra \( L \) generated by the system of vector fields \( \{Y^0, Y^1, \cdots, Y^m\} \).

It is assumed that \( L \) is the set \( Y^0, Y^1, \cdots, Y^m \) and all the Lie brackets generated by \( Y^0, Y^1, \cdots, Y^m \) with their linear combinations form a Lie algebra.

In the sequel, we use the following notations

\[ \text{ad}_L^0, \text{ad}_L \]

where \( [M, N] = \{[M, N] : M \in L, N \in L\} \),

\[ \text{ad}^{k+1}_L = \text{ad}_L \text{ad}^k_L, \]

(3)

We say that the Lie algebra \( L \) is nilpotent if there exists an integer number \( k > 0 \) such that \( \text{ad}^k_L = 0 \).

The complex structure of the system (1) produces difficulties in solving the problems of optimal control, and therefore, their approximation by systems with simpler structures often becomes a necessity. For example, Popescu shows that, under certain assumptions, the affine system (1), with or without the passivity of the term \( Y^0(x) \) can be locally approximated by a nilpotent system of the same type. The considered nonlinear system is nilpotent, if the associated system of the Lie algebra \( L \) is nilpotent.

The Hamiltonian associated to the optimal problem is

\[ H = p^T \left[ Y^0(x) + Y(x)u \right] - \frac{1}{2} (x^TQx + u^TRu), \]

where \( p \) is the adjoint vector \( (n \times 1) \).

The Hamiltonian system given by

\[ \dot{x} = Y^0(x) + Y(x)u, \quad x(0) = 0, \]
\[ \dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial}{\partial x} \left[ p^T (Y^0 + Y^i u) \right] \]
\[ \frac{1}{2} \frac{\partial}{\partial x} \left( x^T Q x \right) = Q x - \left[ p^T \frac{\partial}{\partial x} (Y^0 + Y^i u) \right]^T = \frac{\partial}{\partial x} \left( Y^0 + Y^i u \right)^T p + Q x, \quad p(t) = 0. \] (5)

This corresponds to the Hamiltonian \( H \) and the vector \((m \times 1)\) which is added to (3), yielding
\[ y = \frac{\partial H}{\partial u} = [p^T Y^i - (Ru)_i]^T, \quad i = 1, 2, \ldots, m. \] (6)

where \((Ru)_i, i = 1, 2, \ldots, m\) is the \(i\)th element of the \(m \times 1\) vector \((Ru)\).

The necessary conditions for the optimal control \( u^* \) are given by
\[ y = \frac{\partial H}{\partial u} \bigg|_{u^*} = 0, \] (7)
and it can be componentwise written as
\[ y_i = p^T Y^i - (Ru)_i, \quad i = 1, 2, \ldots, m. \] (8)

By derivation of Eq. (8), we obtain
\[ \dot{y}_i = p^T Y^i + p^T \dot{Y}^i - (u^T R_i)', \] (9)

where \(R_i\) is the \(i\)th line of the \(m \times m\) matrix \(R\).

With the notation \([F, Y^i] = \text{ad}_F Y^i\), hence
\[ \frac{d}{dt} (Ru)_i = p^T [F, Y^i] + x^T Q Y^i - \dot{y}_i. \] (10)

Furthermore, \([F, Y^i] = \text{ad}_F Y^i\), hence
\[ \frac{d}{dt} (Ru)_i = p^T \text{ad}_F Y^i + x^T Q Y^i - \dot{y}_i. \] (11)

In order to perform the derivation, we have used the following Lemma.

**Lemma 1**: Let \(V\) a vector and \(p\) the adjoint optimal vector. Then
\[ \frac{d}{dt} (p^T V) = p^T \text{ad}_F V + (x^T Q)V. \] (12)

Substituting the optimal control \(u^*\) in relation (4) of \(H\), we obtain the optimal Hamiltonian
\[ H^*(x, p) = H(x, p, u^*), \]
and the Hamiltonian system (5) is modified replacing \(H\) with \(H^*\).

Using the necessary condition of optimality \(y^{(k)} = 0, k = 0, 1, 2, \ldots, m\), yields.

**Proposition 1**: The necessary conditions for optimality of \(u^*_i\) are expressed by satisfying, along \(H^*\), the equations
\[ \left( Ru^*_i \right)^{(k)} = p^T \text{ad}_F^k Y^i + (x^T Q) \text{ad}_F^{k-1} Y^i + \]
\[ \frac{d}{dt} \frac{d}{dt} \left( x^T Q \right) \text{ad}_F^k Y^i + \]
\[ \cdots + \frac{d^{k-2}}{dt^{k-2}} \frac{d}{dt} \left( x^T Q \right) \text{ad}_F Y^i, \]
\[ k, i = 1, 2, \ldots, m. \] (13)

Thus, the properties of the optimal control can be written as
\[ (Ru^*_i) = p^T Y^i, \quad i = 1, 2, \ldots, m, \] (14)
\[ \frac{d}{dt} \left( Ru^*_i \right) = p^T [Y^i + Y u, Y^i] + \]
\[ (x^T Q)Y^i, \quad i = 1, 2, \ldots, m. \] (15)

In the sequel, we consider that the nonlinear systems are affine and have a nilpotent structure.

**Corollary 1**: If the Lie algebra \(L\) satisfies the condition of nilpotency
\[ \text{ad}_L = 0, \] (16)
for a natural number \(k\), then
\[ \frac{d}{dt} \left( x^T Q \right) = (x^T Q)V, \] (17)
for all vectors \(V \in \text{ad}_L^{k-1} L\).

Next, we analyze the following cases.

**Case 1**: (Commutative, \(k = 1\))

This case corresponds to
\[ \text{ad}_L L = \left\{ [M, N], M, N \in L \right\} = 0. \] (18)

Since the vector field is \(\{Y^0, Y^1, \ldots, Y^m\}\) by explicitly computing (18), we obtain
\[ [Y^0, Y^j] = 0, \quad [Y^i, Y^j] = 0, \]
\[ i, j = 1, 2, \ldots, m. \] (19)

The relations (19) express the commutativity of the operations that define the Lie algebras.

As \(\text{ad}_F Y^i = 0\), Eq. (11) becomes
\[ u^*_i = \frac{\Delta_i}{\det(R)} + C_i^1, \quad i = 1, 2, \ldots, m, \] (20)
where \(C_i^1\) are constants of integration and \(\Delta_i\) are the determinants, obtained from \(\det(R)\) by replacing the column \((i)\) with column \(\int x^T Q Y^i dt\).

Let \(\alpha_i\) be the minor of the terms \(\int x^T Q Y^i dt\) from \(\Delta_i\). Using Corollary 1, the expression of the optimal control \(u^*_i\) becomes
\[ u^*_i = \frac{1}{\det(R)} \sum_{k=1}^{m} \alpha_k (p^T Y^k) + C_i^1, \] (21)
\[ i = 1, 2, \ldots, m. \]
Integrating the system
\[ \dot{x} = Y^0(x) + Y(x)u^*, \quad x(t_0) = x_0, \quad (22) \]
we obtain the solution \( x^* \) corresponding to the optimal control \( u^* \). Thus, the minimal functional is given by
\[ J^*(x_0, u^*) = \frac{1}{2} \int_{t_0}^{t_f} (x^T Q x^* + u^T R u^*) \, dt. \quad (23) \]

The computation of the constants for which Eq. (23) is optimal results from the conditions
\[ \frac{d J^*}{d C_i} = 0, \quad i = 1, 2, \cdots, m. \quad (24) \]

The formulation of optimal control problem is the minimization performance index
\[ J(x_0, u) = \frac{1}{2} \int_{0}^{t_f} u^T u \, dt, \quad (25) \]
associated to the constraints
\[ \dot{x} = Y^0(x) + \sum_{i=1}^{m} Y^i(x) u_i. \]

We analyze the cases which correspond to the first, second and third degree nilpotent operators.

**Case 2:** (Commutative, \( k = 1 \))

For the functional Eq. (25), Corollary 1 becomes
\[ p^T V(x) = \text{constant}, \quad \text{where } V \in \text{ad}_L^{k-1} L. \]

From Eq. (14), we have \( \dot{u}^*_i = p^T \text{ad}_F Y^i = 0 \).

Thus, the control of the minimum of the cost function for the nilpotent system with \( \text{ad}_F = 0 \) is the constant vector \( \dot{u}^* = C_1 \).

This result can be directly obtained from Eq. (13) in the particular case \( Q = 0 \).

The constant \( C_1 \) is determined using Corollary 1, yielding
\[ u^* = p^T(t) Y(x) = p^T(t) Y(x) \bigg|_{t=t_0} = C_1 = 0. \quad (26) \]

The solution of the problem of optimum \( x^* \), is obtained by solving the system
\[ \dot{x} = Y^0(x). \]

**Case 3:** \( (k = 2, \text{ad}_L^2 = 0) \)

As \( \text{ad}_L^2 Y^i = 0, \quad i = 1, 2, \cdots, m, \) from Eq. (13) we have \( \dot{u}^*_i = p^T \text{ad}_L^2 Y^i = 0 \), which has the solution
\[ u^*_i(t) = C^2_1 + C^2_2 t, \quad i = 1, 2, \cdots, m, \]
with \( C^2_k, \quad k = 1, 2 \) constants.

**Case 4:** \( (k = 3, \text{ad}_L^3 = 0) \)

This case assumes that \( \text{ad}_L^3 Y^i = 0, \quad i = 1, 2, \cdots, m, \) and from Eq. (13) we shall obtain
\[ u^{*(3)}_i(t) = 0, \quad i = 1, 2, \cdots, m. \]

Utilizing Corollary 1, one obtains
\[ \dot{u}^*_i = a^i + \sum_{j=1}^{m} b^i_j u^*_j + \sum_{k=1}^{m} c^i_k u^*_k + \]
\[ \sum_{j=1}^{m} \sum_{k=1}^{m} d^i_{jk} u^*_j u^*_k, \quad i = 1, 2, \cdots, m, \quad (27) \]
where \( a^i, b^i_j, c^i_k, d^i_{jk} \) are constants defined by
\[ a^i = p^T[Y^0, [Y^0, Y^i]], \quad b^i_j = p^T[Y^0, [Y^j, Y^i]], \quad c^i_k = p^T[Y^k, [Y^0, Y^i]], \quad d^i_{jk} = p^T[Y^j, [Y^k, Y^i]]. \]

For the scalar control with one single component \( (m = 1) \), we have \( b^i_j = d^i_{jk} = 0 \), and therefore the optimal control is given by
\[ u^* = c_1 \exp(c_3 t) + c_2 \exp(-c_3 t) + c_4, \]
with \( c_i \) \( (i = 1, 2, 3, 4) \), constants.

Systems of the form
\[ \dot{x} = \sum_{i=1}^{m} u_i Y^i(x) \quad (28) \]
belong to the class of affine nonlinear systems (1), with \( Y^0 = 0 \). Hence, the optimal control for systems (28) which minimize the quadratic functionals is obtained as a particular case of the previous results.

As an application, the optimal control problem for bilinear systems with first order nilpotent operators is studied. Consider the following bilinear system \(^{1,6-8} \)
\[ \dot{x} = Ax + Bu, \quad (29) \]
where \( x \in \mathbb{R}^n \), the control \( u \) is a scalar signal and \( A \) and \( B \) are \( n \times n \) constant matrices. The optimization problem is formulated as follows: Given a bilinear system (29) and assuming that
\[ [B, A] = BA - AB = 0, \]
(i.e., \( A \) and \( B \) commute with respect to the Lie bracket), find the optimal pair \( (u^*, x^*) \) for the bilinear system (29) that minimizes the functional
\[ J = \frac{1}{2} \int_{t_0}^{t_f} (x^T x + u^2) \, dt. \quad (30) \]

Assuming that \( x \in \mathbb{R}^2 \), let
\[ M = \left\{ \begin{array}{c} \alpha_i, 2\beta_i \\ \beta_i, 2\alpha_i \end{array} \right| \alpha_i, \beta_i \in \mathbb{Q}, \alpha_i \neq 0, \beta_i \neq 0, \quad i = 1, 2, \cdots \right\} \quad (31) \]
be a set of matrices. Multiplication makes the set \( M \) a commutative group such that, if \( A, B \in M \) then
\[ [B, A] = BA - AB = 0. \quad (32) \]

According to the result for first order nilpotent operators we have
\[ u^* = p^T B x + c_1, \quad (33) \]
such that, the integration of the bilinear system (29) yields

\[ x^*(t) = \exp \left\{ (A + c_1 B)(t - t_0) \right\} x_0 + \int_{t_0}^{t} \exp \left\{ (A + c_1 B)(t - \tau) \right\} p^T(\tau) [B x^*(\tau)]^2 d\tau, \]

(34)

where \( x^*(t) \) is the optimal solution.

The problem of finding the controls that minimize quadratic functionals for the class of affine nonlinear systems is analyzed for the case when the associated Lie algebra is nilpotent. We have obtained the equations representing the necessary conditions of optimality of the control. The three cases we have mentioned and corresponding to Eq. (13) represent the orders \( k = 1, 2, 3 \) respectively of the nilpotent operator. Finally we have studied the application of the method to the case of bilinear systems.

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