On the oscillatory properties of certain fourth order nonlinear difference equations

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Abstract

Some new criteria for the oscillation of fourth order nonlinear difference equations of the form

\[ \Delta \left( \frac{1}{a_3(n)} \left( \Delta \frac{1}{a_2(n)} \left( \Delta \frac{1}{a_1(n)}(\Delta x(n))^{\alpha_1} \right)^{\alpha_2} \right)^{\alpha_3} \right) + \delta q(n)f(x[g(n)]) = 0, \]

where \( \delta = \pm 1 \) are investigated.

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1. Introduction

Consider the fourth order nonlinear difference equation

\[ L_4 x(n) + \delta q(n)f(x[g(n)]) = 0, \quad (E_\delta) \]

where

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where a recent years. For contributions to this study, we refer to [1–7,10] and the references cited therein.

The second order half-linear difference equation

\[ L_0 x(n) = x(n), \quad L_k x(n) = \frac{1}{a_k(n)}(\Delta L_{k-1} x(n))^{\alpha_k}, \quad k = 1, 2, 3, \]

\[ L_4 x(n) = \Delta L_3 x(n), \]

\[ \delta = \pm 1, \text{ and } \Delta \text{ is the forward difference operator defined by } \Delta x(n) = x(n+1) - x(n). \]

In what follows we will assume that

(i) \( a_i(n), q(n) : \mathbb{N} = \{1, 2, \ldots\} \to \mathbb{R}^+ = (0, \infty), i = 1, 2, 3, \text{ and} \)

\[ \sum_{j=n_0 \in \mathbb{N}}^{\infty} a_i^{1/\alpha_i}(j) = \infty, \quad i = 1, 2, 3; \]

(ii) \( g(n) : \mathbb{N} \to \mathbb{R} = (-\infty, \infty), \Delta g(n) \geq 0 \text{ for } n \geq n_0 \text{ and } \lim_{n \to \infty} g(n) = \infty; \)

(iii) \( f \in C(\mathbb{R}, \mathbb{R}), xf(x) > 0 \text{ and } f'(x) \geq 0 \text{ for } x \neq 0; \)

(iv) \( \alpha_i, i = 1, 2, 3, \text{ are the ratios of positive odd integers, while we denote by } \alpha = \alpha_1\alpha_2\alpha_3. \)

The domain \( \mathcal{D}(L_4) \) of \( L_4 \) is defined to be the set of all functions \( x(n) : [n_x, \infty) \to \mathbb{R}, n_x \geq n_0 \in \mathbb{N} \) such that \( L_j x(n), 0 \leq j \leq 4, \text{ exist on } [n_x, \infty). \) Our attention is restricted to those solutions \( \{x(n)\} \in \mathcal{D}(L_4) \) which satisfy \( \sup_{n \geq N} |x(n)| > 0 \text{ for } N \geq n_x. \)

A solution \( \{x(n)\} \) of Eq. (E_δ) is called oscillatory if for every \( m \in \mathbb{N} \) there exist \( m_1, m_2 \in \mathbb{N}, m \leq m_1 < m_2 \text{ such that } x(m_1) x(m_2) < 0, \text{ otherwise, it is nonoscillatory. Equation } (E_\delta) \text{ is called oscillatory if all its solutions are oscillatory.} \)

The second order difference equation

\[ \Delta \left( \frac{1}{a_1(n)}(\Delta x(n)) \right) + \delta q(n)f(x[g(n)]) = 0 \]

as well as the second order half-linear difference equation

\[ \Delta \left( \frac{1}{a_1(n)}(\Delta x(n))^{\alpha_1} \right) + \delta q(n)f(x[g(n)]) = 0, \]

where \( a_1, \alpha, \delta, q, g \text{ and } f \) are as in Eq. (E_δ) have been the subject of the intensive studies in the recent years. For contributions to this study, we refer to [1–7,10] and the references cited therein.

The behavioral properties and in particular the oscillatory behavior of Eq. (E_δ) when \( \alpha_i = 1 \text{ or } \alpha_i \neq 1, i = 1, 2, 3, \text{ has received considerably less attention. Therefore, the purpose of this paper is to establish a systematic study dealing with the behavioral properties of solutions of Eq. (E_δ).} \)

In Section 2, we present some of the behavioral properties of Eq. (E_1) and we end with some sufficient conditions for the oscillation of Eq. (E_1) via comparison with first and second order difference equations whose oscillatory behavior are known. In Section 3, we present some of the behavioral properties as well as some sufficient conditions for the oscillation of Eq. (E_−1). We note that the obtained results are presented in a form which is essentially new even when \( \alpha_i = 1, i = 1, 2, 3. \)

2. Properties of solutions of equation (E_1)

We first state the result on the signs of \( L_ix(n), i = 0, 1, 2, 3, 4, \text{ for a nonoscillatory solution } \{x(n)\} \text{ of Eq. (E_1)}. \) Namely, one of the following two cases holds:

\[ (-1)^{i+1} L_i x(n) x(n) > 0, \quad i = 1, 2, 3, \text{ and } x(n)L_4 x(n) \leq 0 \text{ eventually,} \]

\[ x(n)L_i x(n) > 0, \quad i = 1, 2, 3, \text{ and } x(n)L_4 x(n) \leq 0 \text{ eventually.} \]
We shall say that the solution \( \{x(n)\} \) is of type \( B_1 \) if (2.1) holds and that it is of type \( B_3 \) if (2.2) holds.

In order to prove our main result we will compare the following inequalities:

\[
\Delta \left( \frac{1}{a_1(n)} \left( \Delta x(n) \right)^{\alpha_1} \right) + q(n) f \left( x \left[ g(n) \right] \right) \leq 0, \\
\Delta \left( \frac{1}{a_1(n)} \left( \Delta x(n) \right)^{\alpha_1} \right) + q(n) f \left( x \left[ g(n) \right] \right) \geq 0,
\]

with the equation

\[
\Delta \left( \frac{1}{a_1(n)} \left( \Delta x(n) \right)^{\alpha_1} \right) + q(n) f \left( x \left[ g(n) \right] \right) = 0,
\]

where \( a_1(n), q(n), g(n), f(x) \) and \( \alpha_1 \) satisfy (i)–(iv).

We shall employ the following lemma which is a special case of Lemma 2.3 in [1,10].

**Lemma 2.1.** If inequality (2.3) (inequality (2.4)) has an eventually positive (negative) solution, then Eq. (2.5) also has eventually positive (negative) solution.

We shall also need the following two lemmas which are given in [1,3,8] and [8,9], respectively.

**Lemma 2.2.** If the inequality

\[
\Delta \left( \frac{1}{a_1(n)} \left( \Delta x(n) \right)^{\alpha_1} \right) - q(n) f \left( x \left[ g(n) \right] \right) \geq 0,
\]

where \( a_1(n), q(n), g(n), f(x) \) and \( \alpha_1 \) satisfy (i)–(iv), has an eventually positive bounded (unbounded) solution, then the equation

\[
\Delta \left( \frac{1}{a_1(n)} \left( \Delta x(n) \right)^{\alpha_1} \right) - q(n) f \left( x \left[ g(n) \right] \right) = 0,
\]

also has eventually positive bounded (unbounded) solution.

**Lemma 2.3.** If the inequality

\[
\Delta y(n) + q(n) f \left( y \left[ g(n) \right] \right) \leq 0 \quad \text{or} \quad \Delta y(n) - q(n) f \left( y \left[ g(n) \right] \right) \geq 0,
\]

where \( g(n), q(n) \) and \( f(x) \) satisfy conditions (i)–(iii), has eventually positive solution, then the equation

\[
\Delta y(n) + q(n) f \left( y \left[ g(n) \right] \right) = 0 \quad \text{or} \quad \Delta y(n) - q(n) f \left( y \left[ g(n) \right] \right) = 0,
\]

also has eventually positive solution.

2.1. Nonexistence of solutions of type \( B_1 \)

We shall present some criteria for the nonexistence of solutions of type \( B_1 \) for Eq. \((E_1)\). In order to simplify notation we define sequence \( \{Q(n)\} \) and the function \( F(x) \) in the following way:
\[ Q(n) = \left( a_2(n) \sum_{s=n}^{\infty} \left( a_3(s) \sum_{j=s}^{\infty} q(j) \right)^{1/\alpha_3} \right)^{1/\alpha_2}, \quad n \geq n_0 \in \mathbb{N}, \]

\[ F(x) = \frac{1}{a_2(n)} f \left( x \left[ g(n) \right] \right), \quad x \in \mathbb{R}. \]

**Theorem 2.1.** Let conditions (i)–(iv) hold. If the half-linear equation

\[ \Delta \left( \frac{1}{a_1(n)} (\Delta x(n))^\alpha_1 \right) + Q(n) F(x[g(n)]) = 0 \quad (2.6) \]

is oscillatory, then equation (E1) has no solution of the type \( B_1 \).

**Proof.** Let \( \{x(n)\} \) be an eventually positive solution of equation (E1) of type \( B_1 \). There exists an \( n_0 \in \mathbb{N} \) such that (2.1) holds for \( n \geq n_0 \). Summing equation (E1) from \( n \) to \( m \geq n \geq n_0 \) and letting \( m \to \infty \), we obtain

\[ L_3 x(n) \geq \sum_{j=n}^{\infty} q(j) f \left( x \left[ g(j) \right] \right). \]

From the definition of the operator \( L_3 \), we get

\[ \Delta L_2 x(n) \geq \left( a_3(n) \sum_{j=n}^{\infty} q(j) \right)^{1/\alpha_3} f^{1/\alpha_3} \left( x \left[ g(n) \right] \right), \quad n \geq n_0. \quad (2.7) \]

Summing (2.7) from \( n \) to \( v \geq n \geq n_0 \) and letting \( v \to \infty \), we obtain

\[ -\Delta L_1 x(n) \geq Q(n) F(x[g(n)]) \quad \text{for} \quad n \geq n_0. \quad (2.8) \]

By applying Lemma 2.1, we see that Eq. (2.6) has an eventually positive solution, which contradicts the hypothesis and completes the proof. \( \square \)

**Theorem 2.2.** Let conditions (i)–(iv) hold,

\[ g(n) = n - \tau + 1, \quad \tau \geq 1 \text{ is a real number,} \quad (2.9) \]

\[ -f(-xy) \geq f(xy) \geq f(x)f(y) \quad \text{for} \quad xy > 0 \quad (2.10) \]

and

\[ \sum_{j=n \geq n_0}^{\infty} q(j) < \infty. \quad (2.11) \]

If the first order delay equation

\[ \Delta y(n) + Q(n) F \left( \sum_{j=n_0 \in \mathbb{N}}^{n-\tau} a_1^{1/\alpha_1} (j) \right) F \left( y^{1/\alpha_1} [n - \tau] \right) = 0 \quad (2.12) \]

is oscillatory, then Eq. (E1) has no solution of type \( B_1 \).

**Proof.** Let \( \{x(n)\} \) be an eventually positive solution of Eq. (E1) of the type \( B_1 \). There exists an \( n_0 \in \mathbb{N} \) such that (2.1) holds for \( n \geq n_0 \). Since we have that

\[ x(n) - x(n_0) = \sum_{i=n_0}^{n-1} \Delta x(i) = \sum_{i=n_0}^{n-1} a_1^{1/\alpha_1} (i) L_1^{1/\alpha_1} x(i), \]
using the fact that \( \{L_1x(n)\} \) is nonincreasing for \( n \geq n_0 \), we have
\[
x(n) \geq \left( \sum_{i=n_0}^{n-1} a_1^{1/\alpha_1}(i) \right) L_1^{1/\alpha_1} x(n-1) \quad \text{for } n \geq n_0.
\]
There exists an \( n_1 \in \mathbb{N} \), \( n_1 \geq n_0 + \tau + 1 \) such that
\[
x[n - \tau + 1] \geq \left( \sum_{i=n_0}^{n-\tau} a_1^{1/\alpha_1}(i) \right) z^{1/\alpha_1}[n - \tau] \quad \text{for } n \geq n_1, \tag{2.13}
\]
where \( L_1x(n) = z(n) \), \( n \geq n_1 \). As in the proof of Theorem 2.1, we obtain the inequality (2.8) for \( n \geq n_0 \) and using (2.13) in (2.8), we get
\[
\Delta z(n) + Q(n) F \left( \sum_{i=n_0}^{n-\tau} a_1^{1/\alpha_1}(i) \right) F(z^{1/\alpha_1}[n - \tau]) \leq 0 \quad \text{for } n \geq n_1.
\]
By applying Lemma 2.1, we see that Eq. (2.12) has eventually positive solution, which contradicts the hypothesis and completes the proof. \( \square \)

The following corollary is immediate.

**Corollary 2.1.** Let conditions (i)–(iv), (2.9)–(2.11) hold. Then, Eq. \((E_1)\) has no solution of type \( B_1 \) if one of the following conditions holds, either
\[
(I_1) \quad \liminf_{n \to \infty} \sum_{j=n-\tau}^{n-1} Q(j) F \left( \sum_{i=n_0}^{j-\tau} a_1^{1/\alpha_1}(i) \right) > \left( \frac{\tau}{\tau + 1} \right)^{\tau+1}, \tag{2.15}
\]
or
\[
(I_2) \quad \int_{\pm 0} du \frac{d u}{F(u^{1/\alpha_1})} < \infty, \tag{2.16}
\]

\[
\sum_{j=s}^{\infty} Q(j) F \left( \sum_{i=n_0}^{j-\tau} a_1^{1/\alpha_1}(i) \right) = \infty. \tag{2.17}
\]

**Theorem 2.3.** Let conditions (i)–(iv), (2.9) and (2.11) hold. If
\[
\int_{\pm 0} du \frac{d u}{F^{1/\alpha_1}(u)} < \infty \tag{2.18}
\]
and
\[
\sum_{j=s}^{\infty} Q(j) \left( a_1[s - \tau] \sum_{j=s}^{\infty} Q(j) \right) ^{1/\alpha_1} = \infty, \tag{2.19}
\]
then Eq. \((E_1)\) has no solution of type \( B_1 \).
Proof. Let \( \{x(n)\} \) be an eventually positive solution of Eq. (E1) of type \( B_1 \). As in the proof of Theorem 2.1, we obtain inequality (2.8) for \( n \geq n_0 \). Now, one can easily see that

\[
L_1x[n - \tau] \geq L_1x(n) \geq \left( \sum_{j=n}^{\infty} Q(j) \right) F(x[n - \tau + 1]),
\]

(2.20)
or

\[
\frac{\Delta x[n - \tau]}{F^{1/\alpha_1}(x[n - \tau + 1])} \geq \left( a_1[n - \tau] \sum_{j=n}^{\infty} Q(j) \right)^{1/\alpha_1} \text{ for } n \geq n_1 \geq n_0.
\]

Summing this inequality from \( n_1 \) to \( n \geq n_1 \) we have

\[
\sum_{s=n_1}^{n} \left( a_1[s - \tau] \sum_{j=s}^{\infty} Q(j) \right)^{1/\alpha_1} \leq \sum_{s=n_1}^{n} \frac{\Delta x[s - \tau]}{F^{1/\alpha_1}(x[s - \tau + 1])} \leq \int_{x[n_1 - \tau]}^{x[n_1 + 1]} \frac{dy}{F^{1/\alpha_1}(y)} < \infty.
\]

Thus,

\[
\sum_{s=n_1}^{\infty} \left( a_1[s - \tau] \sum_{j=s}^{\infty} Q(j) \right)^{1/\alpha_1} < \infty,
\]

contradicting condition (2.19). This completes the proof. \( \square \)

Theorem 2.4. Let conditions (i)–(iv) and (2.11) hold. If

\[
\sum_{n=0}^{\infty} Q(n) = \infty,
\]

(2.21)
then Eq. (E1) has no solution of type \( B_1 \).

Proof. Let \( \{x(n)\} \) be an eventually positive solution of Eq. (E1) of type \( B_1 \). Proceeding as in the proof of Theorem 2.1, we obtain inequality (2.8) for \( n \geq n_0 \). Now, there exist a constant \( b > 0 \) and an \( n_1 \in \mathbb{N} \) such that for \( n \geq n_1 \geq n_0 \),

\[
x[g(n)] \geq b.
\]

(2.22)
Using (2.22) in (2.8) and summing from \( n_1 \) to \( n \geq n_1 \) we have

\[
\infty > L_1x(n_1) \geq \left( \sum_{j=n_1}^{n} Q(j) \right) F(b) \to \infty \text{ as } n \to \infty,
\]
a contradiction. This completes the proof. \( \square \)

Using Theorems 2.3 and 2.4 we can establish sufficient conditions for all bounded solutions of Eq. (E1) to be oscillatory.
Theorem 2.5. Let conditions (i)–(iv) and (2.11) hold. If
\[ \sum_{j=n_0}^{\infty} Q(j) < \infty \]  
and
\[ \sum_{s=n}^{\infty} \left( a_1(s) \sum_{j=s}^{\infty} Q(j) \right)^{1/\alpha_1} = \infty, \]  
then all bounded solutions of Eq. (E₁) are oscillatory.

Proof. Let \( \{x(n)\} \) be a bounded and eventually positive solution of Eq. (E₁). As in the proof of Theorems 2.3 and 2.4 we obtain (2.20) and (2.22) for \( n \geq n_1 \). Now, by using (2.22) in inequality (2.20), we can easily find
\[ \Delta x(n) \geq \left( a_1(n) \sum_{j=n}^{\infty} Q(j) \right)^{1/\alpha_1} F^{1/\alpha_1}(b) \text{ for } n \geq n_1. \]
Summing the above inequality from \( n_1 \) to \( m - 1 \geq n_1 \) and applying condition (2.24), we arrive at the desired contradiction with the fact that the solution \( \{x(n)\} \) is bounded. \( \square \)

Remark 2.1. We note that Theorems 2.1–2.4 can be applied to give oscillation results for all bounded solutions of Eq. (E₁).

2.2. Nonexistence of solutions of type B₃

Next, we shall establish some criteria for the nonexistence of solutions of type B₃ for Eq. (E₁).

Theorem 2.6. Assume that conditions (i)–(iv) and (2.10) hold. If
\[ \sum_{s=n_0}^{\infty} q(s) \left( \sum_{m=n_0}^{g(s)-1} \left( a_1(m) \sum_{j=n_0}^{m-1} a_2^{1/\alpha_2}(j) \right)^{1/\alpha_1} \right) = \infty, \]  
then Eq. (E₁) has no solution of type B₃.

Proof. Let \( \{x(n)\} \) be an eventually positive solution of Eq. (E₁) of type B₃. Then there exists an \( n_0 \in \mathbb{N} \) such that (2.2) holds for \( n \geq n_0 \). Next, there exist a constant \( b > 0 \) and an \( n_1 \geq n_0 \) such that
\[ L_2 x(n) = \frac{1}{a_2(n)} \left( \Delta \left( \frac{1}{a_1(n)} (\Delta x(n))^{\alpha_1} \right)^{\alpha_2} \right) \geq b \text{ for } n \geq n_1 \]
or
\[ \Delta \left( \frac{1}{a_1(n)} (\Delta x(n))^{\alpha_1} \right) \geq (ba_2(n))^{1/\alpha_2} \text{ for } n \geq n_1. \]  
Summing (2.26) from \( n_1 \) to \( m - 1 \geq n_1 \) we have
\[ \Delta x(m) \geq b^{1/\alpha_2} \left( a_1(m) \sum_{n=n_1}^{m-1} a_2^{1/\alpha_2}(n) \right)^{1/\alpha_1}. \]
Once again, summing (2.27) from \( n_2 \geq n_1 \) to \( g(s) - 1 > n_2 \), we get
\[
x\left[g(s)\right] \geq b_1^{-\frac{1}{\alpha_1 \alpha_2}} \sum_{m=n_2}^{g(s)-1} a_1(m) \sum_{n=n_1}^{m-1} a_2^{1/\alpha_2}(n) \frac{1}{\alpha_1}.
\] (2.28)

Using condition (2.10) and inequality (2.28) in Eq. (E_1), we obtain
\[
-L_4 x(s) = q(s) f\left(x\left[g(s)\right]\right) \geq f\left(b_1^{-\frac{1}{\alpha_1 \alpha_2}} q(s) f\left(\sum_{m=n_2}^{g(s)-1} a_1(m) \sum_{n=n_1}^{m-1} a_2^{1/\alpha_2}(n) \right)^{1/\alpha_1}\right).
\]

Summing the above inequality from \( n_2 \) to \( j - 1 \), we have
\[
\sum_{n=n_2}^{\infty} L_3 x(n) \geq L_3 x(n_2) - L_3 x(j)
\]
\[
\geq f\left(b_1^{-\frac{1}{\alpha_1 \alpha_2}} \sum_{s=n_2}^{j-1} q(s) f\left(\sum_{m=n_2}^{g(s)-1} a_1(m) \sum_{n=n_1}^{m-1} a_2^{1/\alpha_2}(n) \right)^{1/\alpha_1}\right) \rightarrow \infty \quad \text{as } j \rightarrow \infty,
\]
a contradiction. This completes the proof. \( \square \)

**Theorem 2.7.** Let conditions (i)–(iv), (2.9) and (2.10) hold. If the first order delay equation
\[
\Delta y(n) + Q_1(n) f\left(y^{1/\alpha}[n - \tau]\right) = 0 \quad \text{(2.29)}
\]
is oscillatory, where
\[
Q_1(n) = q(n) f\left(\sum_{j=n_0}^{n-\tau} a_1(j) \sum_{m=n_0}^{j-1} a_2^{1/\alpha_2}(m) \sum_{s=n_0}^{m-1} a_3^{1/\alpha_3}(s) \right)^{1/\alpha_1},
\]
then Eq. (E_1) has no solution of type \( B_3 \).

**Proof.** Let \( \{x(n)\} \) be an eventually positive solution of Eq. (E_1) of type \( B_3 \). There exists \( n_0 \in \mathbb{N} \) such that (2.2) holds for \( n \geq n_0 \). Now,
\[
L_2 x(n) = L_2 x(n_0) + \sum_{j=n_0}^{n-1} a_3^{1/\alpha_3}(j) a_3^{-1/\alpha_3}(j) \Delta L_2 x(j)
\]
\[
= L_2 x(n_0) + \sum_{j=n_0}^{n-1} a_3^{1/\alpha_3}(j) L_3^{1/\alpha_3} x(j).
\]
Since \( \{L_3 x(n)\} \) is nonincreasing for \( n \geq n_0 \), we have
\[
L_2 x(n) \geq \left(\sum_{j=n_0}^{n-1} a_3^{1/\alpha_3}(j)\right) L_3^{1/\alpha_3} x(n) \quad \text{for } n \geq n_0,
\]
or
\[
\Delta L_1 x(n) \geq \left(a_2(n) \sum_{j=n_0}^{n-1} a_3^{1/\alpha_3}(j)\right)^{1/\alpha_2} L_3^{1/\alpha_3} x(n), \quad n \geq n_0.
\]
Summing the above inequality from \( n_0 \) to \( n-1 \), one can easily obtain
\[
\Delta x(n) \geq \left( a_1(n) \sum_{j=n_0}^{n-1} \left( a_2(j) \sum_{s=n_0}^{j-1} a_3^{1/\alpha_3}(s) \right)^{1/\alpha_2} \right)^{1/\alpha_1} L_3^{1/\alpha} x(n), \quad n \geq n_0.
\]

Once again, by summing this inequality from \( n_0 \) to \( n-\tau > n_0 \), we get for all \( n \geq n_1 \geq n_0 \) that
\[
x[n-\tau + 1] \geq \sum_{m=n_0}^{n-\tau} \left( a_1(m) \sum_{j=n_0}^{m-1} \left( a_2(j) \sum_{s=n_0}^{j-1} a_3^{1/\alpha_3}(s) \right)^{1/\alpha_2} \right)^{1/\alpha_1} L_3^{1/\alpha} x(n-\tau). \tag{2.30}
\]

Using (2.10) and (2.30) in Eq. (E1) and letting \( z(n) = L_3 x(n) \), \( n \geq n_1 \), we obtain
\[
\Delta z(n) + Q_1(n) f(z^{1/\alpha}[n-\tau]) \leq 0 \quad \text{for} \quad n \geq n_1.
\]
The rest of the proof is similar to that of Theorem 2.2 and hence omitted. \( \Box \)

**Corollary 2.2.** Let conditions (i)–(iv), (2.9) and (2.10) hold. Equation (E1) has no solution of type \( B_3 \) if one of the following conditions holds:

\[
(J_1) \quad f\left(y^{1/\alpha}\right) \geq 0 \quad \text{for} \quad y \neq 0, \tag{2.31}
\]

\[
\liminf_{n \to \infty} \sum_{j=n-\tau}^{n-1} Q_1(j) > \left( \frac{\tau}{\tau + 1} \right)^{\tau+1}. \tag{2.32}
\]

\[
(J_2) \quad \int_{\pm0} du f(u^{1/\alpha}) < \infty, \tag{2.33}
\]

\[
\sum_{j=n-\tau}^{\infty} Q_1(j) = \infty, \tag{2.34}
\]

where \( \{Q_1(n)\} \) is as in Theorem 2.7.

**Theorem 2.8.** Let conditions (i)–(iv) and (2.10) hold and assume that there exists a sequence \( \{\eta(n)\} \) such that \( g(n) \geq \eta(n) \geq n \) for \( n \geq n_0 \in \mathbb{N} \). If the half-linear equation
\[
\Delta \left( \frac{1}{a_3(n)} \left( \Delta y(n) \right)^{\alpha_3} \right) + Q_2(n) f\left(y^{1/\alpha_2}(n)\right) = 0 \tag{2.35}
\]
is oscillatory, where
\[
Q_2(n) = q(n) f\left( \left( \sum_{j=\eta(n)}^{g(n)-1} a_1^{1/\alpha_1}(j) \right) \left( \sum_{j=n}^{\eta(n)-1} a_2^{1/\alpha_2}(j) \right)^{1/\alpha_1} \right).
\]
then Eq. (E1) has no solution of type \( B_3 \).

**Proof.** Let \( \{x(n)\} \) be an eventually positive solution of Eq. (E1) of type \( B_3 \). Then there exists an \( n_0 \in \mathbb{N} \) such that (2.2) holds for \( n \geq n_0 \). Now for \( t - 1 \geq s \geq n_0 \), we have
\[
x(t) = x(s) + \sum_{j=s}^{t-1} a_1^{1/\alpha_1}(j) L_1^{1/\alpha_1} x(j),
\]
or

\[ x(t) \geq \left( \sum_{j=s}^{t-1} a_j^{1/\alpha_1}(j) \right) L_1^{1/\alpha_1} x(s). \]

Replacing \( t \) and \( s \) by \( g(n) \) and \( \eta(n) \), respectively, we have

\[ x[g(n)] \geq \left( \sum_{j=\eta(n)}^{g(n)-1} a_j^{1/\alpha_1}(j) \right) L_1^{1/\alpha_1} x[\eta(n)] \quad \text{for } n \geq n_1 \geq n_0. \]  

(2.36)

Similarly, one can easily obtain

\[ L_1 x[\eta(n)] \geq \left( \sum_{i=n}^{\eta(n)-1} a_i^{1/\alpha_2}(i) \right) L_2^{1/\alpha_2} x(n) \quad \text{for } n \geq n_1. \]

(2.37)

Combining (2.36) and (2.37), we have

\[ x[g(n)] \geq \left( \sum_{j=\eta(n)}^{g(n)-1} a_j^{1/\alpha_1}(j) \right) \left( \sum_{i=n}^{\eta(n)-1} a_i^{1/\alpha_2}(i) \right)^{1/\alpha_1} L_1^{1/\alpha_1} L_2^{1/\alpha_2} x(n) \quad \text{for } n \geq n_2 \geq n_1. \]

(2.38)

Using (2.10) and (2.38) in Eq. (E1) and letting \( z(n) = L_2 x(n) \) for \( n \geq n_2 \), we have

\[ \Delta \left( \frac{1}{a_3(n)} (\Delta z(n))^{\alpha_3} \right) + Q_2(n) f \left( z^{1/\alpha_2}(n) \right) \leq 0 \quad \text{for } n \geq n_2. \]

By applying Lemma 2.1, we see that Eq. (2.35) has an eventually positive solution, which contradicts the hypotheses and completes the proof. \( \square \)

**Theorem 2.9.** Let conditions (i)–(iv) and (2.18) hold and assume that

\[ g(n) = n + \tau + 1, \quad \tau \geq 0 \text{ is a real number}, \]  

(2.39)

and

\[ \sum_{j=n}^{\infty} q(j) < \infty. \]  

(2.40)

If

\[ \sum_{j=n-\tau}^{\infty} a_1(n) \sum_{j=n-\tau}^{n-1} a_2(j) \sum_{i=j-\tau-1}^{j-1} a_3(i) \sum_{k=i+\tau+1}^{\infty} q(k) \left( \sum_{k=i+\tau+1}^{\infty} q(k) \right)^{1/\alpha_3} \left( \sum_{k=i+\tau+1}^{\infty} q(k) \right)^{1/\alpha_2} \left( \sum_{k=i+\tau+1}^{\infty} q(k) \right)^{1/\alpha_1} = \infty, \]  

(2.41)

then Eq. (E1) has no solution of type \( B_3 \).

**Proof.** Let \( \{ x(n) \} \) be an eventually positive solution of Eq. (E1) of type \( B_3 \). There exists an \( n_0 \in \mathbb{N} \) such that (2.2) holds for \( n \geq n_0 \). Summing Eq. (E1) from \( n \geq n_0 \) to \( m-1 \geq n \) and letting \( m \to \infty \), we get

\[ L_3 x(n) \geq \sum_{j=n}^{\infty} q(j) f(\max(j+\tau+1)) \geq \left( \sum_{j=n+\tau+1}^{\infty} q(j) \right) f(\max(n+2\tau+2)), \]
or
\[ \Delta L_2 x(n) \geq \left( a_3(n) \sum_{j=n+\tau+1}^\infty q(j) \right)^{1/\alpha_3} f^{1/\alpha_3}(x[n + 2\tau + 2]). \]

Summing the above inequality from \( n - \tau - 1 > n_0 \) to \( n - 1 \), we have
\[ \Delta L_1 x(n) \geq \left( a_2(n) \sum_{s=n-\tau-1}^{n-1} \left( a_3(s) \sum_{j=s+\tau+1}^\infty q(j) \right)^{1/\alpha_3} \right)^{1/\alpha_2} f^{1/\alpha_2}(x[n + \tau + 1]) \]
for \( n \geq n_1 \geq n_0 \).

Once again, summing the above inequality from \( n - \tau \geq n_0 \) to \( n - 1 \), we get for all \( n \geq n_2 \geq n_1 \)
\[ \Delta x(n) \geq \left( a_1(n) \sum_{j=n-\tau}^{n-1} \left( a_2(j) \sum_{i=j-\tau-1}^{j-1} \left( a_3(i) \sum_{k=i+\tau+1}^\infty q(k) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \right)^{1/\alpha_1} f^{1/\alpha}(x[n + 1]). \]
\[ (2.42) \]
Dividing both sides of inequality (2.42) by \( f^{1/\alpha}(x[n + 1]) \), summing both sides of the resulting inequality from \( n_2 \) to \( n \) we have
\[ \sum_{s=n_2}^{x(n+1)} \frac{\Delta x(s)}{f^{1/\alpha}(x[s + 1])} \leq \int_{x(n_2)}^{x(n+1)} \frac{dy}{f^{1/\alpha}(y)} \leq \int_{x(n_2)}^{\infty} \frac{dy}{f^{1/\alpha}(y)} = \int_{x(n_2)}^{\infty} \frac{dy}{F^{1/\alpha_1}(y)} < \infty. \]

Accordingly, we arrive at the desired contradiction. This completes the proof. \( \square \)

2.3. Oscillation criteria

Next, we shall combine the obtained results for the nonexistence of solutions of Eq. (E1) of types \( B_1 \) and \( B_3 \) and establish some oscillation criteria for Eq. (E1).

First, by combining Theorems 2.4 and 2.6 we have:

**Theorem 2.10.** Let conditions (i)–(iv), (2.10) and (2.11) hold. If
\[ \sum_{n=0}^{\infty} q(n) f \left( g(n) \sum_{m=n_0 \geq 0} ^{g(n)-1} \sum_{j=n_0 \geq 0} ^{m-1} a_1(m) \sum_{j=n_0 \geq 0} ^{m-1} a_2(j) \right)^{1/\alpha_1} \]
\[ = \infty \] (2.43)

and
\[ \sum_{s=n} ^{\infty} a_2(n) \sum_{s=n} ^{\infty} a_3(s) \sum_{j=s} ^{\infty} q(j) \right)^{1/\alpha_2} \]
\[ = \infty, \] (2.44)

then Eq. (E1) is oscillatory.

Next, by combining Theorems 2.2 and 2.7 we get:
Theorem 2.11. Let conditions (i)–(iv), (2.9)–(2.11) hold. If the first order delay equations
\[
\Delta y(n) + q(n) f \left( \sum_{s=n_0}^{n-\tau} \left( \sum_{u=n_0}^{s-1} \left( \sum_{j=n_0}^{u-1} a_1^{1/\alpha_1} (j) \right)^{1/\alpha_1} \right)^{1/\alpha_2} \right) f(y^{1/\alpha}[n-\tau]) = 0
\]
and
\[
\Delta z(n) + \left( \sum_{s=n_0}^{\infty} \left( \sum_{j=s}^{\infty} q(j) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \times f^{1/\alpha_3} (z^{1/\alpha}[n-\tau]) = 0
\]
are both oscillatory, then Eq. (\(E_1\)) is oscillatory.

We note that we can combine Corollaries 2.1 and 2.2 and obtain some sufficient conditions for the oscillation of Eq. (\(E_1\)). Here we omit the details.

Finally, we combine Theorems 2.1 and 2.8 and obtain

Theorem 2.12. Let conditions (i)–(iv), (2.10) and (2.11) hold and assume that there exists a sequence \(\{\eta(n)\}\) such that \(g(n) \geq \eta(n) \geq n\) for \(n \geq n_0 \in \mathbb{N}\). If the half-linear equations
\[
\Delta \left( \frac{1}{a_1(n)} (\Delta y(n))^{\alpha_1} \right) + \left( \sum_{s=n}^{\infty} \left( \sum_{j=s}^{\infty} q(j) \right)^{1/\alpha_3} \right)^{1/\alpha_2} \times f^{1/\alpha_3} (y^{[g(n)]}) = 0
\]
and
\[
\Delta \left( \frac{1}{a_3(n)} (\Delta z(n))^{\alpha_3} \right) + q(n) f \left( \left( \sum_{j=\eta(n)}^{g(n)-1} a_1^{1/\alpha_1} (j) \right)^{1/\alpha_1} \left( \sum_{i=n}^{\eta(n)-1} a_2^{1/\alpha_2} (i) \right)^{1/\alpha_2} \right) f(z^{[g(n)]}) = 0
\]
are oscillatory, then Eq. (\(E_1\)) is oscillatory.

We note that the literature is filled with oscillation results for equations of type (2.47) and (2.48) and accordingly these oscillation results, become oscillation criteria for Eq. (\(E_1\)). The details are left to the reader.

As an illustrative example, we consider a special case of equation (\(E_1\)), namely, the equation
\[
\Delta (\Delta x(n))^{\alpha_1}, \quad q(n)x^{\beta}[g(n)] = 0,
\]
where \(\beta\) is the ratio of two positive odd integers.

From Theorems 2.10–2.12 one can easily see that Eq. (2.49) is oscillatory if one of the following set of conditions holds:

\((O_1)\) Condition (2.11),
\[
\sum_{m=n_0}^{\infty} q(n) m^{1/\alpha_1} = \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \left( \sum_{j=s}^{\infty} q(j) \right)^{1/\alpha_1} \left( \sum_{i=n}^{\eta(n)-1} a_2^{1/\alpha_2} (i) \right)^{1/\alpha_2} = \infty.
\]
(O2) Conditions (2.9) and (2.11) hold and both first order delay equations
\[ \Delta y(n) + q(n)\left( \sum_{s=n_0}^{n-\tau} \left( \sum_{j=n_0}^{s-1} j^{\frac{1}{\alpha_2}} \right)^{\frac{1}{\alpha_1}} \right) y^{\frac{\beta}{\alpha}} [n - \tau] = 0 \]
and
\[ \Delta z(n) + (n - \tau + 1) \left( \sum_{s=n}^{\infty} \left( \sum_{j=s}^{\infty} q(j) \right)^{\frac{1}{\alpha_3}} \right)^{\frac{1}{\alpha_2}} z^{\frac{\beta}{\alpha}} [n - \tau] = 0 \]
are oscillatory.

(O3) Condition (2.11) holds and assume that there exists a sequence \( \{\eta(n)\} \) such that \( g(n) \geq \eta(n) \geq n \) for \( n \geq n_0 \in \mathbb{N} \) and both the second order half-linear equations
\[ \Delta \left( \Delta y(n) \right)^{\alpha_1} + \left( \sum_{s=n}^{\infty} \left( \sum_{j=s}^{\infty} q(j) \right)^{\frac{1}{\alpha_3}} \right)^{\frac{1}{\alpha_2}} y^{\frac{\beta}{\alpha} \alpha_3} [g(n)] = 0 \]
and
\[ \Delta \left( \Delta z(n) \right)^{\alpha_3} + q(n) (g(n) - \eta(n))^{\frac{\beta}{\alpha_1}} \left( \eta(n) - n \right)^{\frac{\beta}{\alpha_1}} z^{\frac{\beta}{\alpha_3}} \alpha_2 (n) = 0 \]
are oscillatory.

For a special case of Eq. (2.49), namely, the equation
\[ \Delta^4 x(n) + q(n)x[n - \tau + 1] = 0, \quad (2.50) \]
by applying condition (O2), we see that this equation is oscillatory if either both the first order delay difference equations
\[ \Delta y(n) + q(n)\left( \sum_{s=n_0}^{n-\tau} \sum_{j=n_0}^{s-1} j \right) y[n - \tau] = 0 \]
and
\[ \Delta z(n) + (n - \tau + 1) \left( \sum_{s=n}^{\infty} \sum_{j=s}^{\infty} q(j) \right) z[n - \tau] = 0 \]
are oscillatory, or, the delay equation
\[ \Delta x(n) + P(n)x[n - \tau] = 0 \]
is oscillatory, where
\[ P(n) = \min \left\{ q(n) \left( \sum_{s=n_0}^{n-\tau} \sum_{j=n_0}^{s-1} j \right), (n - \tau + 1) \left( \sum_{s=n}^{\infty} \sum_{j=s}^{\infty} q(j) \right) \right\}. \]

3. Properties of solutions of equation (E_{-1})

For Eq. (E_{-1}) we have the following three cases of the signs of \( L_i x(n) \), \( i = 0, 1, 2, 3, 4 \), for a nonoscillatory solution \( \{x(n)\} \):
\((-1)^i x(n) L_i x(n) > 0, \; i = 1, 2, 3, \) and \(x(n) L_4 x(n) \geq 0\) eventually, \(\quad (3.1)\)
\[x(n) L_i x(n) > 0, \; i = 1, 2, \; x(n) L_3 x(n) < 0\] and
\(x(n) L_4 x(n) \geq 0\) eventually, \(\quad (3.2)\)
\[x(n) L_i x(n) > 0, \; i = 1, 2, 3, \) and \(x(n) L_4 x(n) \geq 0\) eventually. \(\quad (3.3)\)

We shall say that the solution \(\{x(n)\}\) is of type \(B_0, B_2\) and \(B_4\) if it satisfy respectively (3.1), (3.2) and (3.3).

3.1. Nonexistence of solutions of type \(B_2\)

**Theorem 3.1.** Let conditions (i)--(iv) and (2.10) hold. If
\[
\sum_{j=0}^{g(n)-1} q(j) f\left(\sum_{s=n_2}^{g(n)-1} a_1^{1/\alpha_1}(s)\right) = \infty,
\] (3.4)
then Eq. \((E_{-1})\) has no solution of type \(B_2\).

**Proof.** Let \(\{x(n)\}\) be an eventually positive solution of Eq. \((E_{-1})\) of type \(B_2\). There exists an \(n_0 \in \mathbb{N}\) such that (3.2) holds for \(n \geq n_0\). Next, there exist a constant \(b > 0\) and \(n_1 \geq n_0\) such that
\[L_1 x(n) = \frac{1}{a_1(n)} (\Delta x(n))^{\alpha_1} \geq b,
\]
or
\[\Delta x(n) \geq b^{1/\alpha_1} a_1^{1/\alpha_1}(n) \quad \text{for} \; n \geq n_1.
\]
Summing the above inequality from \(n_2\) to \(g(n) - 1 > n_1\), we get
\[x[g(n)] \geq b^{1/\alpha_1} \sum_{s=n_2}^{g(n)-1} a_1^{1/\alpha_1}(s)\] (3.5)

Using (3.5) and (2.10) in Eq. \((E_{-1})\) we have
\[L_4 x(n) \geq f\left(b^{1/\alpha_1}\right) q(n) f\left(\sum_{s=n_2}^{g(n)-1} a_1^{1/\alpha_1}(s)\right) \quad \text{for} \; n \geq n_2.
\]
Summing the above inequality from \(n_2\) to \(n - 1\) we obtain
\[
\infty > -L_3 x(n_2) \geq L_3 x(n) - L_3 x(n_2)
\]
\[
\geq f\left(b^{1/\alpha_1}\right) \sum_{j=n_2}^{n-1} q(j) f\left(\sum_{s=n_2}^{g(j)-1} a_1^{1/\alpha_1}(s)\right) \rightarrow \infty \quad \text{as} \; n \rightarrow \infty,
\]
contradicting condition (3.14). This completes the proof. \(\square\)

**Theorem 3.2.** Let conditions (i)--(iv), (2.9) and (2.10) hold. If all bounded solutions of the half-linear difference equation
\[
\Delta\left(\frac{1}{a_3(n)} (\Delta y(n))^{\alpha_3}\right) - q(n) f\left(\sum_{s=n_0}^{n-\tau} a_1(s) \sum_{j=n_0}^{s-1} a_2^{1/\alpha_2}(j)\right) f\left(\frac{1}{a_1(n)} y[n - \tau]\right) = 0
\] (3.6)
are oscillatory, then Eq. \((E_{-1})\) has no solution of type \(B_2\).
Theorem 3.3. Let conditions (i)–(iv), (2.9) and (2.10) hold and let $\tau_1 > 0$ be a constant such that $\tau_1 < \tau$. If the first order delay difference equation

$$
\Delta z(n) + q(n) f \left( \sum_{j=n_0}^{n-\tau} \left( a_1(j) \sum_{s=n_0}^{j-1} a_2^{1/\alpha_2}(s) \right)^{1/\alpha_1} \right) f \left( \sum_{j=n-\tau}^{n-\tau_1} a_3^{1/\alpha_3}(j) \right)^{1/(\alpha_1 \alpha_2)} x(n-\tau_1) = 0
$$

is oscillatory, then Eq. (E−1) has no solution of type $B_2$.

Proof. Let $\{x(n)\}$ be an eventually positive solution of Eq. (E−1) of type $B_2$. There exists an $n_0 \in \mathbb{N}$ such that (3.2) holds for $n \geq n_0$. Now

$$
L_1 x(n) = L_1 x(n_0) + \sum_{s=n_0}^{n-1} a_2^{1/\alpha_2}(s) L_2^{1/\alpha_2} x(s).
$$

Using the fact that $\{L_2 x(n)\}$ is a nonincreasing sequence for $n \geq n_0$, we get

$$
L_1 x(n) \geq \left( \sum_{s=n_0}^{n-1} a_2^{1/\alpha_2}(s) \right) L_2^{1/\alpha_2} x(n),
$$

or

$$
\Delta x(n) \geq \left( a_1(n) \sum_{s=n_0}^{n-1} a_2^{1/\alpha_2}(s) \right)^{1/\alpha_1} L_2^{1/\alpha_2} x(n) \quad \text{for } n \geq n_0.
$$

Summing the above inequality from $n_1 \geq n_0$ to $n - \tau > n_1$ and letting $y(n) = L_2 x(n)$, $n \geq n_1$, we have

$$
x[n - \tau + 1] \geq \sum_{j=n_1}^{n-\tau} \left( a_1(j) \sum_{s=n_0}^{j-1} a_2^{1/\alpha_2}(s) \right)^{1/\alpha_1} y^{1/\alpha_2}[n - \tau] \quad \text{for } n \geq n_1. \quad (3.7)
$$

Using (3.7) and (2.10) in Eq. (E−1) we have

$$
\Delta \left( \frac{1}{a_3(n)} (\Delta y(n))^{\alpha_3} \right) \geq q(n) f \left( \sum_{j=n_1}^{n-\tau} \left( a_1(j) \sum_{s=n_0}^{j-1} a_2^{1/\alpha_2}(s) \right)^{1/\alpha_1} \right) f \left( y^{1/\alpha_2}[n - \tau] \right) \quad \text{for } n \geq n_1.
$$

Since $\{L_2 x(n)\}$ is positive and nonincreasing sequence, so bounded, using Lemma 2.2, we see that Eq. (3.6) has an eventually positive and bounded solution, which contradicts the hypothesis and completes the proof. \qed
or, since \( \{L_3x(n)\} \) is nondecreasing sequence, we have
\[
y(k) = L_2x(k) \geq -L_3^{1/\alpha_3}x(m - 1) \sum_{j=k}^{m-1} a_3^{1/\alpha_3}(j) \quad \text{for } m - 1 \geq k \geq n_1.
\]
Replacing \( k \) and \( m \) by \( n - \tau \) and \( n - \tau_1 + 1 \), respectively, we find
\[
y[n - \tau] \geq -L_3^{1/\alpha_3}x[n - \tau_1] \sum_{j=n-\tau}^{n-\tau_1} a_3^{1/\alpha_3}(j) \quad \text{for } n \geq n_2 \geq n_1.
\]
Using (3.9) in (3.7), we have for all \( n \geq n_2 \)
\[
x[n - \tau + 1]
\geq \sum_{j=n_1}^{n-\tau} \left( a_1(j) \sum_{s=n_0}^{j-1} a_2^{1/\alpha_2}(s) \right)^{1/\alpha_1} \left( \sum_{j=n-\tau}^{n-\tau_1} a_3^{1/\alpha_3}(j) \right)^{1/(\alpha_1 \alpha_2)} \left( -L_3^{1/\alpha}x[n - \tau_1] \right).
\]
Using (2.10) and (3.10) in Eq. (E\(_{-1}\)) and letting \( z(n) = -L_3x(n) > 0 \) for \( n \geq n_2 \), we obtain
\[
\Delta z(n) + q(n) f \left( \sum_{j=n_1}^{n-\tau} \left( a_1(j) \sum_{s=n_0}^{j-1} a_2^{1/\alpha_2}(s) \right)^{1/\alpha_1} \left( \sum_{j=n-\tau}^{n-\tau_1} a_3^{1/\alpha_3}(j) \right)^{1/(\alpha_1 \alpha_2)} \right)
\times f \left( z^{1/\alpha}[n - \tau_1] \right) \leq 0.
\]
By Lemma 2.3, Eq. (3.8) has an eventually positive solution which contradicts the hypothesis and completes the proof. \( \square \)

**Theorem 3.4.** Let conditions (i)–(iv), (2.9)–(2.11) hold. If the first order delay equation
\[
\Delta z(n) + \left( a_3(n) \sum_{s=n_0}^{\infty} q(s) \right)^{1/\alpha_3} f^{1/\alpha_3} \left( \sum_{j=n_0}^{n-\tau} \left( a_1(j) \sum_{s=n_0}^{j-1} a_2^{1/\alpha_2}(s) \right)^{1/\alpha_1} \right)
\times f^{1/\alpha_3} \left( z^{\frac{1}{\alpha_3}}[n - \tau] \right) = 0
\]
is oscillatory, then Eq. (E\(_{-1}\)) has no solution of type \( B_2 \).

**Proof.** Let \( \{x(n)\} \) be an eventually positive solution of Eq. (E\(_{-1}\)) of type \( B_2 \). There exists an \( n_0 \in \mathbb{N} \) such that (3.2) holds for \( n \geq n_0 \). Summing Eq. (E\(_{-1}\)) from \( n \) to \( m - 1 \geq n \) we have
\[
L_3x(m) - L_3x(n) = \sum_{j=n}^{m-1} q(j) f(x[j - \tau + 1]),
\]
and since \( \{L_3x(m)\} \) is nonpositive and nondecreasing sequence, i.e., \( \lim_{m \to \infty} L_3x(m) = \lambda \leq 0 \), by letting \( m \to \infty \) we get
\[
-L_3x(n) \geq \sum_{j=n}^{\infty} q(j) f(x[j - \tau + 1]) \geq f(x[n - \tau + 1]) \sum_{j=n}^{\infty} q(j),
\]
or
\[
-\Delta L_2x(n) \geq \left( a_3(n) \sum_{j=n}^{\infty} q(j) \right)^{1/\alpha_3} f^{1/\alpha_3}(x[n - \tau + 1]) \quad \text{for } n \geq n_0.
\]
As in the proof of Theorem 3.2, we obtain (3.7) for \( n \geq n_1 \), so that by using (2.10) and (3.7) in inequality (3.12) and letting \( L_2x(n) = y(n) \) for \( n \geq n_1 \), we obtain

\[
\Delta y(n) + \left( a_3(n) \sum_{j=n}^{\infty} q(j) \right)^{1/\alpha_3} f^{1/\alpha_3} \left( \sum_{j=n_1}^{n-\tau} \left( a_1(j) \sum_{s=n_0}^{j-1} a_2^{1/\alpha_2}(s) \right)^{1/\alpha_1} \right) \times f^{1/\alpha_3} \left( y^{1/\alpha_2} [n - \tau] \right) \leq 0.
\]

The rest of the proof is similar to that of Theorem 3.3 and hence omitted.  

3.2. Nonexistence of solutions of type \( B_4 \)

Next, we shall present some results for the nonexistence of solution of type \( B_4 \) for Eq. (E−1).

**Theorem 3.5.** Let conditions (i)–(iv) and (2.10) hold, \( g(n) > n + 1 \) for \( n \geq n_0 \in \mathbb{N} \). Moreover, assume that there exist nondecreasing sequences \( \{\rho(n)\} \) and \( \{\xi(n)\} \) such that \( g(n) > \rho(n) > \xi(n) > n + 1 \) for \( n \geq n_0 \). If all unbounded solutions of the second order half-linear equation

\[
\Delta \left( \frac{1}{a_3(n)} \left( \Delta y(n) \right)^{\alpha_3} \right) - q(n) f \left( \left( \sum_{j=\rho(n)}^{g(n)-1} a_1^{1/\alpha_1}(j) \right) \left( \sum_{s=\xi(n)}^{\rho(n)-1} a_2^{1/\alpha_2}(s) \right)^{1/\alpha_1} \right) \times f \left( y^{1/\alpha_2} [\xi(n)] \right) = 0,
\]

are oscillatory, then Eq. (E−1) has no solution of type \( B_4 \).

**Proof.** Let \( \{x(n)\} \) be an eventually positive solution of Eq. (E−1) of type \( B_4 \). There exists an \( n_0 \in \mathbb{N} \) such that (3.3) holds for \( n \geq n_0 \). Now,

\[
x(\sigma) = x(\tau) + \sum_{s=\tau}^{\sigma-1} \Delta x(s) = x(\tau) + \sum_{s=\tau}^{\sigma-1} a_1^{1/\alpha_1}(s)L_1^{1/\alpha_1}x(s)
\geq \left( \sum_{s=\tau}^{\sigma-1} a_1^{1/\alpha_1}(s) \right) L_1^{1/\alpha_1}x(\tau) \quad \text{for} \quad \sigma \geq \tau \geq n_0.
\]

Setting \( \sigma = g(n) \), \( \tau = \rho(n) \) in (3.14), we have

\[
x[g(n)] \geq \left( \sum_{s=\rho(n)}^{g(n)-1} a_1^{1/\alpha_1}(s) \right) L_1^{1/\alpha_1}x[\rho(n)] \quad \text{for} \quad n \geq n_1 \geq n_0.
\]

Similarly, one can easily find

\[
L_1x[\rho(n)] \geq \left( \sum_{j=\xi(n)}^{\rho(n)-1} a_2^{1/\alpha_2}(j) \right) L_2^{1/\alpha_2}x[\xi(n)] \quad \text{for} \quad n \geq n_1.
\]

Combining (3.15) and (3.16), we get

\[
x[g(n)] \geq \left( \sum_{s=\rho(n)}^{g(n)-1} a_1^{1/\alpha_1}(s) \right) \left( \sum_{j=\xi(n)}^{\rho(n)-1} a_2^{1/\alpha_2}(j) \right)^{1/\alpha_1} L_2^{1/\alpha_2}x[\xi(n)] \quad \text{for} \quad n \geq n_1.
\]

\[]
Using (2.10) and (3.17) in Eq. (E−1) and letting $y(n) = L_2 x(n)$ for $n \geq n_1$, we obtain
\[
\Delta \left( \frac{1}{a_3(n)} (\Delta y(n))^{\alpha_3} \right) \\
\geq q(n) f \left( \sum_{s=\rho(n)}^{g(n)-1} a_1^{1/\alpha_1}(s) \right) f \left( \left( \sum_{j=\xi(n)}^{\rho(n)-1} a_2^{1/\alpha_2}(j) \right)^{1/\alpha_1} \right) \left( \frac{1}{\alpha_1} \right) f \left( \sum_{t=\tau(n)}^{\xi(n)+1} a_3^{1/\alpha_3}(t) \left[ a_1^{1/\alpha_1}(s) \right] \right) \\
\geq q(n) f \left( \sum_{s=\rho(n)}^{g(n)-1} a_1^{1/\alpha_1}(s) \right) f \left( \left( \sum_{j=\xi(n)}^{\rho(n)-1} a_2^{1/\alpha_2}(j) \right)^{1/\alpha_1} \right) \left( \frac{1}{\alpha_1} \right) f \left( \sum_{t=\tau(n)}^{\xi(n)+1} a_3^{1/\alpha_3}(t) \left[ a_1^{1/\alpha_1}(s) \right] \right).
\] (3.18)

By Lemma 2.2, since $L_2 x(n)$ is an unbounded positive solution of inequality (3.18), we see that Eq. (3.13) has an eventually unbounded positive solution, which contradicts the hypotheses and completes the proof. \(\square\)

**Theorem 3.6.** Let conditions (i)–(iv) and (2.10) hold and $g(n) > n + 1$ for $n \geq n_0 \in \mathbb{N}$. Moreover, assume that there exist a nondecreasing sequence $\{\rho(n)\}$ and $\tau_1, \tau_2 \in \mathbb{N}$ such that $g(n) > \rho(n) > n + \tau_2 > n + \tau_1 > n + 1$ for $n \geq n_0$. If the first order advanced equation
\[
\Delta z(n) - q(n) f \left( \sum_{s=\rho(n)}^{g(n)-1} a_1^{1/\alpha_1}(s) \right) f \left( \left( \sum_{j=\xi(n)}^{\rho(n)-1} a_2^{1/\alpha_2}(j) \right)^{1/\alpha_1} \right) \\
\times f \left( \sum_{t=\tau(n)}^{\xi(n)+1} a_3^{1/\alpha_3}(t) \left[ a_1^{1/\alpha_1}(s) \right] \right) \left( \frac{1}{\alpha_1} \right) f \left( \sum_{t=\tau(n)}^{\xi(n)+1} a_3^{1/\alpha_3}(t) \left[ a_1^{1/\alpha_1}(s) \right] \right) = 0
\] (3.19)
is oscillatory, then Eq. (E−1) has no solution of type $B_4$.

**Proof.** Let $\{x(n)\}$ be an eventually positive solution of Eq. (E−1) of type $B_4$. As in the proof of Theorem 3.5, we obtain inequality (3.17). Next, we can easily find
\[
L_2 x[n + \tau_2 - 1] \geq \left( \sum_{t=n+\tau_1}^{n+\tau_2-1} a_3^{1/\alpha_3}(t) \right) L_3^{1/\alpha_3} x[n + \tau_1] \quad \text{for} \quad n \geq n_2 \geq n_1.
\] (3.20)

Combining (3.17) and (3.20), for $n \geq n_2$ we have
\[
x[g(n)] \geq \left( \sum_{s=\rho(n)}^{g(n)-1} a_1^{1/\alpha_1}(s) \right) f \left( \left( \sum_{j=\xi(n)}^{\rho(n)-1} a_2^{1/\alpha_2}(j) \right)^{1/\alpha_1} \right) \\
\times f \left( \sum_{t=\tau(n)}^{\xi(n)+1} a_3^{1/\alpha_3}(t) \left[ a_1^{1/\alpha_1}(s) \right] \right) L_3^{1/\alpha_3} x[n + \tau_1].
\] (3.21)

Using (2.10) and (3.21) in Eq. (E−1) and letting $z(n) = L_3 x(n)$ for $n \geq n_2$ we have
\[
\Delta z(n) \geq q(n) f \left( \sum_{s=\rho(n)}^{g(n)-1} a_1^{1/\alpha_1}(s) \right) f \left( \left( \sum_{j=\xi(n)}^{\rho(n)-1} a_2^{1/\alpha_2}(j) \right)^{1/\alpha_1} \right) \\
\times f \left( \sum_{t=\tau(n)}^{\xi(n)+1} a_3^{1/\alpha_3}(t) \left[ a_1^{1/\alpha_1}(s) \right] \right) f \left( \sum_{t=\tau(n)}^{\xi(n)+1} a_3^{1/\alpha_3}(t) \left[ a_1^{1/\alpha_1}(s) \right] \right) = 0
\] (3.19)

By Lemma 2.3, Eq. (3.19) has an eventually positive solution which contradicts the hypotheses and completes the proof. \(\square\)
Corollary 3.1. Let conditions (i)–(iv) and (2.10) hold and \(g(n) > n + 1\) for \(n \geq n_0 \in \mathbb{N}\). Moreover, assume that there exist nondecreasing sequences \(\{\rho(n)\}\) and \(\tau_1, \tau_2 \in \mathbb{N}\) such that \(g(n) > \rho(n) > n + \tau_2 > n + \tau_1 > n + 1\) for \(n \geq n_0\). Equation (E₁) has no solution of type \(B_4\) if one of the following conditions holds:

\[(P_1)\quad \frac{f(u^{\alpha_1 \alpha_2 \alpha_3})}{u} \geq 1 \quad \text{for } u \neq 0 \quad \text{and either} \]

\[
\liminf_{n \to \infty} \sum_{j=n+1}^{n+\tau_1-1} P(j) > \frac{1}{\tau_1} \left( \frac{\tau_1 - 1}{\tau_1} \right)^{\tau_1}, \quad \text{or} \quad \limsup_{n \to \infty} \sum_{j=n+1}^{n+\tau_1-1} P(j) > 1, \quad \text{where}
\]

\[
P(n) = q(n) f \left( \sum_{s=\rho(n)}^{\rho(n)-1} a_1^{1/\alpha_1}(s) f \left( \sum_{j=n+\tau_2}^{\rho(n)-1} a_2^{1/\alpha_2}(j) \right) \right) f \left( \sum_{t=n+\tau_2}^{n+\tau_1-1} a_3^{1/\alpha_3}(t) \right)\frac{1}{a_1^{1/\alpha_1} a_2^{1/\alpha_2}}.
\]

\[(P_2)\quad \frac{u}{f(u^{\alpha_1 \alpha_2 \alpha_3})} \to 0 \quad \text{as } u \to \infty \quad \text{and} \quad \liminf_{n \to \infty} \sum_{j=n+1}^{n+\tau_1-1} P(j) > 0.
\]

\[(P_3)\quad \int \frac{du}{f(u^{\alpha_1 \alpha_2 \alpha_3})} < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} P(j) = \infty.
\]

Theorem 3.7. Let conditions (i)–(iv) and (2.10) hold and \(g(n) > n + 1\) for \(n \geq n_0 \in \mathbb{N}\) and assume that there exist nondecreasing sequences \(\{\rho(n)\}\) and \(\{\xi(n)\}\) such that \(g(n) > \rho(n) > \xi(n) > n + 1\) for \(n \geq n_0\). Equation (E₁) has no solution of type \(B_4\) if one of the following conditions holds:

\[(S_1)\quad \frac{f^{1/\alpha_3}(u^{\alpha_1 \alpha_2})}{u} \geq 1 \quad \text{for } u \neq 0 \quad \text{and} \quad \limsup_{n \to \infty} \sum_{j=n}^{\xi(n)-1} P^*(j) > 1, \quad \text{where}
\]

\[
P^*(m) = a_3^{1/\alpha_3}(m) \left[ \sum_{j=n}^{m-1} q(j) f \left( \sum_{s=\rho(j)}^{g(j)-1} a_1^{1/\alpha_1}(s) \right) f \left( \sum_{i=\xi(j)}^{\rho(j)-1} a_2^{1/\alpha_2}(i) \right) \right]^{1/\alpha_3},
\]

\[m \geq n.
\]

\[(S_2)\quad \frac{u}{f^{1/\alpha_3}(u^{\alpha_1 \alpha_2})} \to 0 \quad \text{as } u \to \infty \quad \text{and} \quad \limsup_{n \to \infty} \sum_{j=n}^{\xi(n)-1} P^*(j) > 0.
\]

Proof. Let \(\{x(n)\}\) be an eventually positive solution of Eq. (E₁) of type \(B_4\). As in the proof of Theorem 3.5, we obtain inequality (3.18) for \(n \geq n_1\). Also, we see that \(y(n) = L_2 x(n) > 0\) and \(\Delta y(n) > 0\) for \(n \geq n_1\).

Summing both sides of (3.18) from \(n \geq n_1\) to \(m - 1 \geq n\), we have
\[
\frac{1}{a_3(m)}(\Delta y(m))^{\alpha_3} \geq \sum_{j=n}^{m-1} q(j) f \left( \sum_{s=\rho(j)}^{g(j)-1} a_1^{1/\alpha_1}(s) \right) f \left( \sum_{i=\xi(j)}^{\rho(j)-1} a_2^{1/\alpha_2}(i) \right)^{1/\alpha_1} f \left( y^{1/\alpha_2} \left[ \xi(j) \right] \right),
\]
which implies for \( m - 1 \geq n \geq n_1 \) that
\[
\Delta y(m) \geq a_3(m) \sum_{j=n}^{m-1} q(j) \left[ a_1^{1/\alpha_1}(s) \right] \times f \left( \sum_{i=\xi(j)}^{\rho(j)-1} a_2^{1/\alpha_2}(i) \right)^{1/\alpha_1} f \left( y^{1/\alpha_2} \left[ \xi(j) \right] \right)^{1/\alpha_3}.
\]
Combining (3.22) with the relation
\[
y(k) \geq y(k) - y(n) = \sum_{m=n}^{k-1} \Delta y(m) \quad \text{for} \quad k - 1 \geq m \geq n \geq n_1
\]
and putting \( k = \xi(n) \), we have
\[
\frac{y[\xi(n)]}{f^{1/\alpha_3} \left( y^{1/\alpha_2} \left[ \xi(n) \right] \right)} \geq \sum_{m=n}^{\xi(n)-1} P^s(m).
\]
Taking \( \text{lim sup} \) of both sides of the above inequalities as \( n \to \infty \) and applying the hypotheses in \((S_1)\) or \((S_2)\) we arrive at the desired contradiction. This completes the proof. \( \square \)

### 3.3. Nonexistence of solutions of type \( B_0 \)

Next, we shall present some criteria for the nonexistence of solution of Eq. \((E_{-1})\) of type \( B_0 \), i.e., the oscillatory behavior of all bounded solutions of Eq. \((E_{-1})\).

**Theorem 3.8.** Let conditions (i)–(iv) and (2.10) hold and \( g(n) < n + 1 \) for \( n \geq n_0 \in \mathbb{N} \). Assume that there exist nondecreasing sequences \( \{\xi(n)\} \) and \( \{\eta(n)\} \) such that \( g(n) < \xi(n) < \eta(n) < n + 1 \) for \( n \geq n_0 \). If all bounded solutions of the second order half-linear difference equation
\[
\Delta \left( \frac{1}{a_3(n)}(\Delta y(n))^{\alpha_3} \right) - q(n) f \left( \sum_{s=g(n)}^{\xi(n)-1} a_1^{1/\alpha_1}(s) \right) f \left( \sum_{j=\xi(n)}^{\eta(n)-1} a_2^{1/\alpha_2}(j) \right)^{1/\alpha_1} \times f \left( y^{1/\alpha_2} \left[ \eta(n) \right] \right) = 0
\]
are oscillatory, then Eq. \((E_{-1})\) has no solution of type \( B_0 \).

**Proof.** Let \( \{x(n)\} \) be an eventually positive solution of Eq. \((E_{-1})\) of type \( B_0 \). There exists \( n_0 \in \mathbb{N} \) such that (3.1) holds for \( n \geq n_0 \). For \( t \geq s \geq n_0 \) we have
\[
x(t) - x(s) = \sum_{j=s}^{t-1} a_1^{1/\alpha_1}(j) L_1^{1/\alpha_1} x(j)
\]
and since for a solution \{x(n)\} of type B_0 sequence \{L_1x(n)\} is nondecreasing, we obtain

\[ x(s) \geq (-L_1^{1/\alpha_1} x(t)) \sum_{j=s}^{t-1} a_1^{1/\alpha_1}(j). \]

Replacing \(s\) and \(t\) in the above inequality by \(g(n)\) and \(\xi(n)\), respectively, we find

\[ x[g(n)] \geq \left( \sum_{j=g(n)}^{\xi(n)-1} a_1^{1/\alpha_1}(j) \right) (-L_1^{1/\alpha_1} x[\xi(n)]) \]

for \(n \geq n_1 \geq n_0\). (3.24)

Similarly, using that sequence \(\{L_2x(n)\}\) is nonincreasing, we find that

\[ -L_1 x[\xi(n)] \geq L_2^{1/\alpha_2} x[\eta(n)] \left( \sum_{j=\xi(n)}^{\eta(n)-1} a_2^{1/\alpha_2}(j) \right) \]

for \(n \geq n_1\). (3.25)

so that from (3.24) we get

\[ x[g(n)] \geq \left( \sum_{j=g(n)}^{\xi(n)-1} a_1^{1/\alpha_1}(j) \right) \left( \sum_{j=\xi(n)}^{\eta(n)-1} a_2^{1/\alpha_2}(j) \right)^{1/\alpha_1} \]

\[ L_2^{1/\alpha_2} x[\eta(n)] \]

for \(n \geq n_1\). (3.26)

Using (3.26) and (2.10) in equation (E_−1) and letting \(y(n) = L_2x(n)\) for \(n \geq n_1\), we have

\[ \Delta \left( \frac{1}{a_3(n)} \right) (\Delta y(n))^\alpha_3 \]

\[ \geq q(n) f \left( \sum_{j=g(n)}^{\xi(n)-1} a_1^{1/\alpha_1}(j) \right) f \left( \sum_{j=\xi(n)}^{\eta(n)-1} a_2^{1/\alpha_2}(j) \right)^{1/\alpha_1} f \left( y^{1/\alpha_2}[\eta(n)] \right) \]

(3.27)

Since \(\{L_2x(n)\}\) is positive and nonincreasing sequence, so bounded, by Lemma 2.2, we see that Eq. (3.23) has an eventually positive and bounded solution. Since this contradicts the assumption of theorem, proof is completed. □

**Theorem 3.9.** Let conditions (i)-(iv) and (2.10) hold and \(g(n) < n + 1\) for \(n \geq n_0 \in \mathbb{N}\). Assume that there exist nondecreasing sequences \(\{\xi(n)\}\) and \(\{\eta(n)\}\) and \(\tau \in \mathbb{N}\) such that \(g(n) < \xi(n) < \eta(n) < n - \tau + 1\) for \(n \geq n_0\). If the first order delay difference equation

\[ \Delta z(n) + q(n) f \left( \sum_{j=g(n)}^{\xi(n)-1} a_1^{1/\alpha_1}(j) \right) f \left( \sum_{j=\xi(n)}^{\eta(n)-1} a_2^{1/\alpha_2}(j) \right)^{1/\alpha_1} \]

\[ \times f \left( \sum_{j=\eta(n)}^{n-\tau} a_3^{1/\alpha_3}(j) \right)^{1/\alpha_1\alpha_2} f \left( z^{1/\alpha}[n - \tau] \right) = 0 \]

(3.28)

is oscillatory, then Eq. (E_−1) has no solution of type B_0.

**Proof.** Let \(\{x(n)\}\) be an eventually positive solution of Eq. (E_−1) of type B_0. As in the proof of Theorem 3.8, we obtain inequality (3.26) for \(n \geq n_1\). Next, we easily find

\[ L_2 x[\eta(n)] \geq -L_3^{1/\alpha_3} x[n - \tau] \left( \sum_{j=\eta(n)}^{n-\tau} a_3^{1/\alpha_3}(j) \right) \]

for \(n \geq n_2 \geq n_1\). (3.29)
Using (3.29) in (3.26) for \( n \geq n_2 \), we have
\[
x[g(n)] \geq -L_1^{1/\alpha_3} x[n - \tau] \left( \sum_{j=g(n)}^{\xi(n)-1} a_1^{1/\alpha_1} (j) \right) \left( \sum_{j=\xi(n)}^{\eta(n)-1} a_2^{1/\alpha_2} (j) \right)^{1/\alpha_1} \times \left( \sum_{j=\eta(n)}^{n-\tau} a_3^{1/\alpha_3} (j) \right)^{1/\alpha_1}.
\]
(3.30)
Using (2.10) and (3.30) in Eq. (E−1), letting \( z(n) = -L_3 x(n) > 0 \) for \( n \geq n_2 \), we obtain
\[
-\Delta z(n) \geq q(n) f \left( \sum_{j=g(n)}^{\xi(n)-1} a_1^{1/\alpha_1} (j) \right) \times f \left( \sum_{j=\eta(n)}^{n-\tau} a_3^{1/\alpha_3} (j) \right)^{1/\alpha_1} f \left( x[n - \tau] \right).
\]
Applying Lemma 2.3, we see that Eq. (3.28) has an eventually positive solution, which contradict the hypothesis and completes the proof.

Corollary 3.2. Let conditions (i)–(iv) and (2.10) hold, \( g(n) < n + 1 \) for \( n \geq n_0 \in \mathbb{N} \) and assume that there exist nondecreasing sequences \( \{\xi(n)\} \) and \( \{\eta(n)\} \) and \( \tau \in \mathbb{N} \) such that \( g(n) < \xi(n) < \eta(n) < n - \tau + 1 \) for \( n \geq n_0 \). Equation (E−1) has no solution of type \( B_0 \) if one of the following conditions holds:

\[ (A_1) \quad \frac{f \left( u^{1/\alpha_1} a_2^{-1} a_3 \right)}{u} \geq 1 \quad \text{for} \quad u \neq 0 \quad \text{and either} \]
\[ \limsup_{n \to \infty} \sum_{s=n-\tau}^{n-1} Q_0(s) > 1, \quad \text{or} \quad \liminf_{n \to \infty} \sum_{s=n-\tau}^{n-1} Q_0(s) > \left( \frac{\tau}{\tau+1} \right)^{\tau+1} \]
\[ Q_0(n) = q(n) f \left( \sum_{s=g(n)}^{\xi(n)-1} a_1^{1/\alpha_1} (s) \right) \times \left( \sum_{s=\eta(n)}^{n-\tau} a_3^{1/\alpha_3} (s) \right)^{1/\alpha_1}. \]

\[ (A_2) \quad \frac{u}{f \left( u^{1/\alpha_1} a_2^{-1} a_3 \right)} \to 0 \quad \text{as} \quad u \to \infty \quad \text{and} \quad \limsup_{n \to \infty} \sum_{s=n-\tau}^{n-1} Q_0(s) > 0. \]

\[ \text{Theorem 3.10. Let conditions (i)–(iv) and (2.10) hold, } g(n) < n + 1 \text{ for } n \geq n_0 \in \mathbb{N} \text{ and assume that there exist nondecreasing sequences } \{\xi(n)\} \text{ and } \{\eta(n)\} \text{ such that } g(n) < \xi(n) < \eta(n) < n + 1 \text{ for } n \geq n_0. \text{ Equation (E−1) has no solution of type } B_0 \text{ if one of the following conditions holds:} \]

\[ (A_3) \quad \int_{\pm 0}^{\infty} \frac{du}{f \left( u^{1/\alpha_1} a_2^{-1} a_3 \right)} < \infty \quad \text{and} \quad \sum_{s=n-\tau}^{\infty} Q_0(s) = \infty. \]
\( f^{1/\alpha_3}(u^{\frac{1}{\alpha_1 \alpha_2}}) \frac{u}{u^{\frac{1}{\alpha_1 \alpha_2}}} \geq 1 \quad \text{for } u \neq 0 \) and \( \limsup_{n \to \infty} \sum_{s=\eta(n)}^{n-1} Q_3(s) > 1 \), where

\[
Q_3(s) = a_3^{1/\alpha_3}(s) \left( \sum_{j=s}^{n-1} q(j) f \left( \sum_{i=g(j)}^{\xi(j)-1} a_1^{1/\alpha_1}(i) \left( \sum_{i=\xi(j)}^{\eta(j)-1} a_2^{1/\alpha_2}(i) \right)^{1/\alpha_1} \right) \right)^{1/\alpha_3}.
\]

\( u^{\frac{1}{\alpha_3}} f^{1/\alpha_3}(u^{\frac{1}{\alpha_1 \alpha_2}}) \to 0 \) as \( u \to \infty \) and \( \limsup_{n \to \infty} \sum_{s=\eta(n)}^{n-1} Q_3(s) > 0 \).

**Proof.** Let \( \{x(n)\} \) be an eventually positive solution of Eq. \((E_{-1})\) of type \( B_0 \). As in the proof of Theorem 3.8, we obtain (3.27) for \( n \geq n_1 \), where \( y(n) = L_2 x(n) > 0 \) and \( \Delta y(n) < 0 \) for \( n \geq n_1 \). Summing (3.27) from \( s \) to \( n-1 \geq s \geq n_1 \), we get

\[
\frac{1}{a_3(s)} (-\Delta y(s))^{\alpha_3} \geq \sum_{j=s}^{n-1} q(j) f \left( \sum_{i=g(j)}^{\xi(j)-1} a_1^{1/\alpha_1}(i) \right) \left( \sum_{i=\xi(j)}^{\eta(j)-1} a_2^{1/\alpha_2}(i) \right)^{1/\alpha_1} f \left( y^{\frac{1}{\alpha_1 \alpha_2}}(\eta(j)) \right),
\]

which implies

\[
-\Delta y(s) \geq a_3^{1/\alpha_3}(s) \left( \sum_{j=s}^{n-1} q(j) f \left( \sum_{i=g(j)}^{\xi(j)-1} a_1^{1/\alpha_1}(i) \right) \left( \sum_{i=\xi(j)}^{\eta(j)-1} a_2^{1/\alpha_2}(i) \right)^{1/\alpha_1} \right) \times f \left( y^{\frac{1}{\alpha_1 \alpha_2}}(\eta(j)) \right)^{1/\alpha_3}.
\]

Summing (3.31) from \( k \) to \( n-1 \geq k \geq n_1 \), using with the relation

\[
y(k) = y(n) + \sum_{s=k}^{n-1} (-\Delta y(s)) \quad \text{for } n-1 \geq k \geq n_1
\]

and putting \( k = \eta(n) \), we have

\[
\frac{y(\eta(n))}{f^{1/\alpha_3}(y^{\frac{1}{\alpha_1 \alpha_2}}(\eta(n)))} \geq \sum_{s=\eta(n)}^{n-1} Q_3(s).
\]

Taking \( \limsup \) of both sides of the above inequality as \( n \to \infty \) and applying the hypotheses in (C1) or (C2) we arrive at the desired contradiction. This completes the proof. \( \square \)

**Remarks 3.1.**

1. We may note that the hypotheses which imply the nonexistence of solutions of Eq. \((E_{-1})\) of type \( B_0 \) are equivalent to that which imply the oscillatory behavior of all bounded solutions of equation \((1, 1; -1)\), while the conditions which are concerned with the nonexistence of solutions of Eq. \((E_{-1})\) of type \( B_2 \) or \( B_4 \) are equivalent to that which are concerned with oscillatory behavior of all unbounded solutions of Eq. \((E_{-1})\).
We may apply the known oscillation criteria for first and second order equations to those appeared in the hypotheses of the results of Section 3 and therefore, establish some sufficient conditions for the oscillation of all bounded and/or unbounded solutions of Eq. (E–1). The details are left to the reader.

Next, we shall apply the obtained results of this section to investigate the oscillatory behavior of the difference equations of the form

\[ L_4 x(n) = q_1(n) f_1 \left( x \left[ g_1(n) \right] \right) + q_2(n) f_2 \left( x \left[ g_2(n) \right] \right), \]  

(3.32)

where the operator \( L_4 \) is as in Eq. (Eδ) with \( a_i(n) \) are as in (i), satisfying (1.3), \( a_i \) (i = 1, 2, 3) are as in (iv),

(ii)’ \{q_i(n)\} and \{g_i(n)\} are real sequences, \( q_i(n) > 0, \Delta g_i(n) \geq 0 \) and \( \lim_{n \to \infty} g_i(n) = \infty, i = 1, 2, \)

(iii)’ \( f_i \in C(\mathbb{R}, \mathbb{R}), xf_i(x) > 0 \) and \( f_i’(x) \geq 0 \) for \( x \neq 0, i = 1, 2. \)

We also assume that functions \( f_i, i = 1, 2, \) satisfy the following condition:

\[-f_i(-xy) \geq f_i(xy) \geq f_i(x)f_i(y), \quad xy > 0 \quad \text{and} \quad i = 1, 2. \]  

(3.33)

Now, we state some results which insure the oscillation of Eq. (3.32). First, by combining Theorems 3.3, 3.9 and 3.6 we have:

**Theorem 3.11.** Let conditions (i), (ii)’, (iii)’, (iv) and (3.33) hold, \( g_1(n) < n + 1 \) and \( g_2(n) > n + 1 \) for \( n \geq n_0 \in \mathbb{N} \) and assume that there exist nondecreasing sequences \( \{\xi_i(n)\}, \{\eta_i(n)\} \) and \( \{\sigma_i(n)\}, i = 1, 2, \) such that

\[ g_1(n) < \xi_1(n) < \eta_1(n) < \sigma_1(n) < n + 1 \quad \text{for} \quad n \geq n_0 \]  

(3.34)

and

\[ g_2(n) > \xi_2(n) > \eta_2(n) > \sigma_2(n) > n + 1 \quad \text{for} \quad n \geq n_0. \]  

(3.35)

If the first order delay difference equations

\[ \Delta z(n) + q_1(n) f_1 \left( \sum_{s=n_0}^{g_1(n)-1} \left( a_1(s) \sum_{j=n_0}^{s-1} a_2^{1/\alpha_2} \left( j \right) \right)^{1/\alpha_1} \right) f_1 \left( \sum_{s=g_1(n)}^{\xi_1(n)-1} a_3^{1/\alpha_3} \left( s \right) \right)^{1/(\alpha_1\alpha_2)} \times f_1 \left( z^{1/\alpha} \left[ \xi_1(n) \right] \right) = 0 \]  

(3.36)

and

\[ \Delta y(n) + q_1(n) f_1 \left( \sum_{s=g_1(n)}^{\xi_1(n)-1} a_1^{1/\alpha_1} \left( s \right) \right) f_1 \left( \sum_{s=\xi_1(n)}^{\eta_1(n)-1} a_2^{1/\alpha_2} \left( s \right) \right)^{1/\alpha_1} \times f_1 \left( y^{1/\alpha} \left[ \sigma_1(n) \right] \right) = 0 \]  

(3.37)

and the first order advanced difference equation
\[ \Delta x(n) - q_2(n) f_2 \left( \sum_{s=\xi_2(n)}^{g_2(n)-1} a_1^{1/\alpha_1}(s) \right) f_2 \left( \sum_{s=\eta_2(n)}^{\varepsilon_2(n)-1} a_2^{1/\alpha_2}(s) \right)^{1/\alpha_1} \]
\[ \times f_2 \left( \sum_{s=\sigma_2(n)}^{g_2(n)-1} a_3^{1/\alpha_3}(s) \right)^{1/(\alpha_1 \alpha_2)^{1/\alpha_3}} f_2 \left( x^{1/\alpha} [\sigma_2(n)] \right) = 0 \]  

are oscillatory, then Eq. (E\textsubscript{1}) is oscillatory.

Next, by combining Theorems 3.2, 3.8 and 3.5 we have:

**Theorem 3.12.** Let conditions (i), (ii)', (iii)', (iv) and (3.33) hold, \( g_1(n) \leq n + 1 \) and \( g_2(n) \geq n + 1 \) for \( n \geq n_0 \in \mathbb{N} \) and assume that there exist nondecreasing sequences \( \{\xi_i(n)\} \) and \( \{\eta_i(n)\} \), \( i = 1, 2 \), such that \( g_1(n) < \xi_1(n) < \eta_1(n) \leq n + 1 \) and \( g_2(n) > \xi_2(n) > \eta_2(n) \geq n + 1 \) for \( n \geq n_0 \). If all bounded solutions of the second order delay half-linear difference equations

\[ \Delta \left( \frac{1}{a_3(n)}(\Delta z(n))^{\alpha_3} \right) = q_1(n) f_1 \left( \sum_{s=g_1(n)}^{\varepsilon_1(n)-1} a_1^{1/\alpha_1}(s) \left( \sum_{j=n_0}^{s-1} a_2^{1/\alpha_2}(j) \right)^{1/\alpha_1} \right) \]
\[ \times f_1 \left( x^{1/\alpha_1} [g_1(n)] \right) = 0 \]  

and

\[ \Delta \left( \frac{1}{a_3(n)}(\Delta y(n))^{\alpha_3} \right) = q_1(n) f_1 \left( \sum_{s=\xi_1(n)}^{\eta_1(n)-1} a_1^{1/\alpha_1}(s) \left( \sum_{s=\xi_1(n)}^{\varepsilon_2(n)-1} a_2^{1/\alpha_2}(s) \right)^{1/\alpha_1} \right) \]
\[ \times f_1 \left( y^{1/\alpha_1} [\eta_1(n)] \right) = 0 \]

and all the unbounded solutions of the advanced half-linear difference equation

\[ \Delta \left( \frac{1}{a_3(n)}(\Delta x(n))^{\alpha_3} \right) = q_2(n) f_2 \left( \sum_{s=\xi_2(n)}^{g_2(n)-1} a_1^{1/\alpha_1}(s) \left( \sum_{s=\eta_2(n)}^{\varepsilon_2(n)-1} a_2^{1/\alpha_2}(s) \right)^{1/\alpha_1} \right) \]
\[ \times f_2 \left( x^{1/\alpha_1} [\eta_2(n)] \right) = 0 \]

are oscillatory, then Eq. (E\textsubscript{1}) is oscillatory.

As an illustrative example, we consider a special case of Eq. (3.32), namely, the equation

\[ \Delta(\Delta x(n))^{\alpha_1} = q_1(n)x^{\beta} [g_1(n)] + q_2(n)x^{\gamma} [g_2(n)] \]  

where \( \beta \) and \( \gamma \) are ratios of positive odd integers.

From Theorems 3.11 and 3.12, one can easily see that Eq. (3.42) is oscillatory if either one of the following conditions holds:

(S\textsubscript{1}) Conditions (3.34) and (3.35) hold and the first order delay difference equation

\[ \Delta y(n) + q_1(n) \Omega_1(n)y^\beta [\sigma_1(n)] = 0, \]

where
\[ \Omega_1(n) = \min \left\{ \left( \sum_{s=n_0 \geq 0}^{g_1(n)-1} s^{1/\alpha_1} \right)^{\beta/\alpha_1} \left( \xi_1(n) - g_1(n) \right)^{\beta/\alpha_1}, \left( \xi_1(n) - g_1(n) \right)^{\beta} (\eta_1(n) - \xi_1(n))^{\beta/\alpha_1} (\sigma_1(n) - \eta_1(n))^{\beta/\alpha_1} \right\}, \]

and the advanced first order difference equation
\[ \Delta z(n) - q_2(n) \Omega_2(n) z^{\gamma/\alpha_1} \left[ \sigma_2(n) \right] = 0, \]
where
\[ \Omega_2(n) = (g_2(n) - \xi_2(n))^{\gamma} (\xi_2(n) - \eta_2(n))^{\gamma/\alpha_1} (\eta_2(n) - \sigma_2(n))^{\gamma/\alpha_1}, \]
are oscillatory.

(S_2) Conditions (3.34) and (3.35) hold with \( \eta_i(t) = \sigma_i(t), \ i = 1, 2, \ n \geq n_0 \) and all bounded solutions of the second order delay half-linear difference equation
\[ \Delta (\Delta y(n))^{\alpha_3} - q_1(n) \Omega_3(n) y^{\beta/\alpha_1 \alpha_2} \left[ g_1(n) \right] = 0, \]
where
\[ \Omega_3(n) = \min \left\{ \left( \sum_{s=n_0 \geq 0}^{g_1(n)-1} s^{1/\alpha_1} \right)^{\beta}, \left( \xi_1(n) - g_1(n) \right)^{\beta} (\eta_1(n) - \xi_1(n))^{\beta/\alpha_1} \right\}, \]
and all unbounded solutions of the second order advanced half-linear difference equation
\[ \Delta (\Delta z(n))^{\alpha_3} - q_2(n) \Omega_4(n) z^{\gamma/\alpha_1 \alpha_2} \left[ \eta_2(n) \right] = 0, \]
where
\[ \Omega_4(n) = (g_2(n) - \xi_2(n))^{\gamma} (\xi_2(n) - \eta_2(n))^{\gamma/\alpha_1}, \]
are oscillatory.

Remark 3.2. We note that the results of this paper are presented in a form which is essentially new and of high degree of generality. From the obtained results one may obtain further sufficient conditions which ensure the oscillation of Eq. (E_δ). Also, we may easily find the conditions under which all solutions of Eq. (E_δ) may tend to zero monotonically or else tend to \( \infty \) monotonically as \( n \to \infty \). The details are left to the reader.

References