Vector valued Thiele–Werner-type osculatory rational interpolants

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Abstract

A method of solving vector-valued osculatory rational interpolation problems is presented by using the Samelson inverse of vectors. The rational interpolants are constructed in the form of Thiele–Werner-type continued fraction expression. Its interpolation structures are analyzed and important characteristic properties are obtained.

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1. Introduction

Wynn [9] observed that the Samelson inverse may be used to solve the question of rational interpolation of vectors. Graves–Morris [3], and Roberts and Graves-Morris [5], at first solved vector-valued rational interpolation problem by using Thiele-type continued fraction and applied this method to the model analysis of vibrating structures. Gu [1,2] generalized the definition of Samelson inverse to the case of matrices, and applied it to deal with the problems of rational interpolation of matrices. As compared with scalar osculatory rational interpolation problem [4,6,8], vector-valued osculatory rational interpolants (VORIs) are more difficult to solve, where partial numerators and partial denominators are vector polynomials and polynomials, respectively. Until recently, few results for VORIs have been known.

A flexible continued fraction structure called Thiele–Werner-type fraction was first proposed by Werner [7] to construct scalar rational interpolant, which has the advantage to handle degenerating

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Thiele-type rational interpolant. In the paper we generalize the scalar Thiele–Werner-type fraction to the case of vector-valued data and apply this structure to VORIs. The interpolation structure of vector-valued quantities given on a set of distinct interpolation points is discussed, and its characteristic properties are obtained. By the Thiele–Werner-type structure, different expression of an interpolant on the same data may be constructed, which is different from the method of vector-valued Thiele-type interpolant defined uniquely. In the same way, some basic principles should be provided as in [3].

2. Thiele–Werner-type VORIs

A vector inverse denoted \( \tilde{v}^{-1} \), called a Samelson inverse, is defined by

\[
\tilde{v}^{-1} = \tilde{v}^* / |\tilde{v}|^2
\]

for any nonnull vector \( \tilde{v} \) in a complex finite-dimensional linear space \( \mathcal{C}^d \), where the star denotes complex conjugate.

A vector \( \tilde{v} \) and its inverse \( \tilde{v}^{-1} \) have the desirable properties that

\[
(\tilde{v}^{-1})^{-1} = \tilde{v}, \quad \tilde{v} \cdot \tilde{v}^{-1} = 1.
\]

Furthermore, given any two nonnull vector \( \tilde{a}, \tilde{b} \) and nonnull \( c \in \mathcal{C} \), if \( \tilde{a} = c/\tilde{b} \), then it holds \( \tilde{b} = c/\tilde{a} \).

Note that the Samelson inverse \( \tilde{v}^{-1} \) is uniquely specified by (1), in no sense is it a unique inverse of \( \tilde{v} \). We consider the data set \( \{(x_i, \tilde{v}^{(j)}_i), i = 0, 1, \ldots, n, j = 0, 1, \ldots, m_i\} \) of vectors \( \tilde{v}^{(j)}_i \in \mathcal{C}^d \) at distinct interpolation points \( x_i \in \mathcal{R} \), where \( \tilde{v}^{(j)}_i \) denotes \( j \)th derivative information up to \( m_i \)th at interpolation point \( x_i \). Let us find an osculatory rational interpolation function \( \tilde{R}(x) = \tilde{N}(x)/D(x) \), such that

\[
\tilde{R}^{(k)}(x_i) = \left. \frac{d^k \{\tilde{N}(x)/D(x)\}}{dx^k} \right|_{x=x_i}
= \tilde{v}^{(k)}_i, \quad k = 0, 1, \ldots, m_i, \quad i = 0, 1, \ldots, n,
\]

where \( \tilde{N}(x) \) is a vector polynomial, and \( D(x) \) is a real polynomial.

We divide the set of interpolation points \( \{x_0, x_1, \ldots, x_n\} \) into \( t + 1 \) subsets:

\[
\{x_0, x_1, \ldots, x_{h_0}\}, \ldots, \{x_{h_s}, x_{h_s+1}, \ldots, x_{h_t}\}, \ldots, \{x_{h_t}, x_{h_t+1}, \ldots, x_n\}.
\]

The subsets may be achieved by reordering the interpolation points if necessary. Obviously, we have

\[
\sum_{s=0}^t (k_s - h_s + 1) = n + 1.
\]

Let us consider the following rational function with Thiele–Werner-type continued fraction formation:

\[
\tilde{R}(x) = \tilde{p}_0(x) + \frac{w_0(x)}{\tilde{p}_1(x)} + \cdots + \frac{w_1(x)}{\tilde{p}_2(x)} + \cdots + \frac{w_{t-1}(x)}{\tilde{p}_t(x)},
\]

where the polynomials

\[
w_s(x) = \prod_{i=h_s}^{k_s} (x - x_i)^{m_i+1}, \quad s = 0, 1, \ldots, t - 1
\]
are derived from the subset points \{x_{h_s}, x_{h_s+1}, \ldots, x_{h_t}\}, and \(\tilde{p}_s(x)\), \(s = 0, 1, \ldots, t\), are vector polynomials. By a tail-to-head evaluation using (1), the continued fraction (3) may be reduced to the fraction form \(\tilde{N}(x)/D(x)\). Let \(\tilde{R}_t(x) = \tilde{p}_t(x)\), for \(s = t - 1, \ldots, 1, 0\), define
\[
\tilde{R}_s(x) = \tilde{p}_s(x) + \frac{w_s(x)}{\tilde{p}_{s+1}(x)} + \frac{w_{s+1}(x)}{\tilde{p}_{s+2}(x)} + \cdots + \frac{w_{t-1}(x)}{\tilde{p}_t(x)},
\]
then we have \(\tilde{R}(x) = \tilde{R}_0(x)\).

To get a VORI, the vector polynomials \(\tilde{p}_s(x)\) \((s = 0, 1, \ldots, t)\) must be computed so that the continued fraction (3) satisfies the interpolating equality (2). It is easy to find that the solving of \(\tilde{p}_s(x)\) is only related to the former interpolation points \(x_0, x_1, \ldots, x_{h_s}\). By the definition of (4), it holds
\[
\tilde{R}_s(x) = \tilde{p}_s(x) + \frac{w_s(x)}{\tilde{R}_{s+1}(x)}.
\]

Then we have
\[
\tilde{R}_{s+1}(x) = \frac{w_s(x)}{\tilde{R}_s(x) - \tilde{p}_s(x)}, \quad s = 0, 1, \ldots, t-1.
\]

If we have known \(\tilde{p}_s(x)\) and \(\tilde{R}_s(x)\), then from \(\tilde{R}_{s+1}(x)\), we may compute \(\tilde{p}_{s+1}(x)\) by interpolating \(\tilde{R}_{s+1}(x)\) on points \(x_{h_{s+1}}, x_{h_{s+1}+1}, \ldots, x_{h_{s+1}}\).

By above equality, if \(\tilde{R}_s(x)\) is replaced by vector-valued Newton–Hermite interpolating polynomial \(\tilde{y}_s(x)\) which is interpolated at the points \(x_{h_{s}}, x_{h_{s}+1}, \ldots, n\), then it holds
\[
\tilde{R}_{s+1}^{(k)}(x_i) = \tilde{y}_{s+1}^{(k)}(x_i), \quad k = 0, \ldots, m_i, \quad i = h_{s+1}, h_{s+1} + 1, \ldots, n.
\]

So the computation of \(\tilde{R}_{s+1}^{(k)}(x_i)\) and \(\tilde{p}_{s}(x)\) can be processed provided with the initial values \(\tilde{y}_0(x)\) and \(\tilde{p}_0(x)\). A complete depiction of the computation of \(\tilde{p}_s(x)\) \((s = 0, 1, \ldots, t)\) is given as follows. \(\tilde{p}_s(x)\) \((s = 0, 1, \ldots, t)\) is defined a vector valued Newton–Hermite interpolating polynomial which interpolates \(\tilde{y}_s(x)\) on the points \(x_{h_{s}}, x_{h_{s}+1}, \ldots, x_{k_s}\), such that
\[
\tilde{p}_{s}^{(k)}(x_i) = \tilde{y}_{s}^{(k)}(x_i), \quad k = 0, \ldots, m_i, \quad i = h_s, \ldots, k_s
\]
and the data for each stage of the iterative construction of the fraction are obtained by
\[
\tilde{y}_{s+1}^{(k)}(x_i) = \frac{d^k}{dx^k} \left( \frac{w_s(x)}{\tilde{y}_{s}(x) - \tilde{p}_{s}(x)} \right) \bigg|_{x=x_i}, \quad k = 0, \ldots, m_i,
\]
\[
i = h_{s+1}, h_{s+1} + 1, \ldots, n, \quad s = 0, 1, \ldots, t - 1,
\]
where \(\tilde{y}_s(x)\), \(s=0,1,\ldots,t-1\), are vector valued Newton–Hermite interpolating polynomials satisfying above interpolating equality with
\[
\tilde{y}_{0}^{(k)}(x_i) = \tilde{y}_{0}^{(k)}(x_i), \quad k = 0, 1, \ldots, m_i, \quad i = 0, 1, \ldots, n.
\]
Thus, if all \(\tilde{p}_s(x)\), \(s = 1, 2, \ldots, t\), are nonnull vector polynomials and
\[
\tilde{R}_s(x) \neq 0, \quad i = h_s, h_s + 1, \ldots, k_s, \quad s = 1, 2, \ldots, t,
\]
then the fraction function \(\tilde{R}(x)\) constructed by (3) is a solution to VORIs.
Because the choice of \( w_s(x) \) \((s = 0, 1, \ldots, t - 1)\) is arbitrary, there are various solutions to choose from the Thiele–Werner-type structure. Note that, if each \( \tilde{p}_s(x) \) only interpolates \( \tilde{y}_s(x) \) on one point \( x_s \) and each \( m_i \) in (2) is zero, then the Thiele–Werner-type rational interpolation becomes the Thiele-type rational interpolation.

Let \( g(x) = \{ g_i(x), \ i = 1, 2, \ldots, d \} \) be a finite-dimensional vector polynomial. The degree of \( g(x) \) is defined as \( \deg g(x) = \max \{ \deg g_i(x), \ i = 1, 2, \ldots, d \} \).

**Definition.** If vector-valued rational function \( \tilde{R}(x) = \tilde{N}(x)/D(x) \) satisfying

(i) \( D(x) \) is real polynomial, \( \deg D(x) = 2k \), \( \deg \tilde{N}(x) = l \),
(ii) \( D(x) \parallel \tilde{N}(x) \|^2 \),
(iii) \( \tilde{R}^{(k)}(x_i) = \tilde{v}_l^{(k)}, \ k = 0, \ldots, m; \ i = 0, 1, \ldots, n, \)

then \( \tilde{R}(x) \) is defined as \( \lceil l/2k \rceil \) type VORIs.

### 3. Characteristic properties of VORIs

**Theorem 1.** Let \( \tilde{R}(x) \) be a vector valued rational interpolant as in (3). Define \( \tilde{R}_j(x) \) as in (4) with a tail-to-head evaluation using (1) understood. Then for \( j = t - 1, \ldots, 1, 0 \), a vector polynomial \( \tilde{N}_j(x) \) and a real scalar polynomial \( D_j(x) \) exist such that

(i) \( \tilde{R}_j(x) = \tilde{N}_j(x)/D_j(x) \),
(ii) \( D_j(x) \geq 0 \),
(iii) \( D_j(x) \parallel \tilde{N}_j(x) \|^2 \). \( \tag{5} \)

**Proof.** Note that the fraction \( \tilde{R}_j(x) \) may be constructed, recursively,
for \( i = t, t - 1, \ldots, j + 1 \),

\[
\tilde{R}_{i-1}(x) = \tilde{p}_{i-1}(x) + \frac{w_{i-1}(x)}{\tilde{R}_i(x)}. \quad \tag{6}
\]

Consider the following algorithm for the construction of \( \tilde{N}_j(x) \), \( D_j(x) \) and \( \tilde{R}_j(x) \):

(i) **Initialization:** \( \tilde{N}_t(x) = \tilde{p}_t(x), \ D_t(x) = 1. \)
(ii) **Recursion:** For \( j = t, \ldots, 1 \), define

\[
D_{j-1}(x) = |\tilde{N}_j(x)|^2/D_j(x), \quad \tag{7}
\]

\[
\tilde{N}_{j-1}(x) = D_{j-1}(x)\tilde{p}_{j-1}(x) + w_{j-1}(x)\tilde{N}_j(x)^* \quad \tag{8}
\]

(iii) **Termination:**

\[
\tilde{R}_j(x) = \tilde{N}_j(x)/D_j(x), \quad j = t, t - 1, \ldots, 0.
\]
Note that (5) holds for \( j = t \). Next, make the inductive hypothesis that they hold for \( j = k \), so that \( \tilde{N}_k(x) \) and \( D_k(x) \) are polynomial for which

\[
D_k(x) | \tilde{N}_k(x)^2.
\]

From (8),

\[
|\tilde{N}_{k-1}(x)|^2 = |D_{k-1}(x)|^2 |\tilde{p}_{k-1}(x)|^2 + |w_{k-1}(x)|^2 |\tilde{N}_k(x)|^2 \\
+ 2w_{k-1}(x)D_{k-1}(x)\text{Re}[\tilde{p}_{k-1}(x) \cdot \tilde{N}_k(x)].
\]

From (7) and (10), we find that

\[
D_{k-1}(x) | \tilde{N}_{k-1}(x)|^2.
\]

In initialization, choosing \( D_j(x) = 1 \), from definition (7), we get that \( D_j(x) \geq 0 \) for \( x \) real. □

In the following, we discuss the characteristic property about the degree of the numerator and denominator of the continued fraction (3). It is crucial to understand the structure of vector valued Thiele–Werner-type rational interpolants. To avoid the degeneracy cases, we suppose that there is no cancellation of the leading terms in the right-hand side of (8).

we define a functional sequence \( \{\delta_i\}_{i=0,1,...,t} \) acting on \( \{\tilde{p}_i(x)\} \), respectively, which each functional \( \delta_i \) is established by relying on the latter functional \( \delta_{i+1} \), with

\[
\delta_i(\tilde{p}_i(x)) = \text{deg } \tilde{p}_i(x)
\]

and

\[
\delta_{i-1}(\tilde{p}_{i-1}(x)) = \begin{cases} 
\text{deg } w_{i-1}(x) & \text{if } \delta_{i-1}(\tilde{p}_{i-1}(x)) = 0, \\
\text{deg } \tilde{p}_{i-1}(x) & \text{if } \delta_{i-1}(\tilde{p}_{i-1}(x)) \neq 0, 
\end{cases} \quad i = 1, \ldots, t.
\]

**Theorem 2.** Let \( \tilde{R}(x) \) be a vector valued rational interpolant as in (3). Define \( \tilde{R}_{t-k}(x) = \tilde{N}_{t-k}(x)/D_{t-k}(x) \) as in (4) by a tail-to-head rationalization using (1) and \( \tilde{p}_i \neq 0, \quad i = 0, 1, \ldots, t \). Then for \( k = 0, 1, \ldots, t \), it holds

\[
\text{deg } \tilde{N}_{t-k}(x) = \delta_{t-k}(\tilde{p}_{t-k}(x)) + 2 \sum_{i=0}^{k-1} \delta_{t-i}(\tilde{p}_{t-i}(x)),
\]

\[
\text{deg } D_{t-k}(x) = 2 \sum_{i=0}^{k-1} \delta_{t-i}(\tilde{p}_{t-i}(x)).
\]

**Proof.** For the case of \( k = 0 \), the initialization shows that

\[
\text{deg } \tilde{N}_t(x) = \text{deg } \tilde{p}_t(x), \quad \text{deg } D_t(x) = 0.
\]

Next, make the inductive hypothesis that (13) hold “up to \( k \).” From (8), we have

\[
\text{deg } \tilde{N}_{t-k-1}(x) = \text{deg } \tilde{p}_{t-k-1}(x) + 2\delta_{t-k}(\tilde{p}_{t-k}(x)) + 2 \sum_{i=0}^{k-1} \delta_{t-i}(\tilde{p}_{t-i}(x))
\]

(14)
or
\[
\deg \tilde{N}_{t-k-1}(x) = \deg \bar{p}_{t-k-1}(x) + 1 + \delta_{t-k}(\bar{p}_{t-k}(x)) + 2 \sum_{i=0}^{k-1} \delta_{t-i}(\bar{p}_{t-i}(x)). \tag{15}
\]

We now discuss the choice of \((14)\) and \((15)\) under all possible cases based on the definition \((12)\).

**Case 1:** If \(\deg \bar{p}_{t-k+1}(x) = \deg \bar{p}_{t-k}(x)\),

(I) When \(\deg \bar{p}_{t-k+1}(x) = 0\), then \(\deg \bar{p}_{t-k}(x) = \deg w_{t-k}(x)\), we choose \((14)\).

(II) When \(\deg \bar{p}_{t-k+1}(x) \neq 0\), then \(\deg \bar{p}_{t-k}(x) = \deg \bar{p}_{t-k}(x)\). Now, if \(\deg \bar{p}_{t-k}(x) = 0\), then

\[
\delta_{t-k-1}(\bar{p}_{t-k-1}(x)) = \deg w_{t-k-1}(x) = \deg \bar{p}_{t-k-1}(x) + 1,
\]
we choose \((15)\); if \(\deg \bar{p}_{t-k} \neq 0\), then

\[
\delta_{t-k-1}(\bar{p}_{t-k-1}(x)) = \deg \bar{p}_{t-k-1}(x),
\]
we choose \((14)\).

**Case 2:** If \(\delta_{t-k+1}(\bar{p}_{t-k+1}(x)) = \deg w_{t-k+1}(x) \neq 0\), the deduction is continued by turning to (II).

Unless there is cancellation of the leading terms in the right-hand side of \((8)\), in terms of above discussion, we deduce that the \(\deg \tilde{N}_{t-k-1}(x)\) can always be expressed as follows:

\[
\deg \tilde{N}_{t-k-1}(x) = \delta_{t-k-1}(\bar{p}_{t-k-1}(x)) + 2 \sum_{i=0}^{k} \delta_{t-i}(\bar{p}_{t-i}(x)).
\]

On the other hand,

\[
\deg \tilde{N}_{t-k-1}(x) = 2\deg \tilde{N}_{t-k}(x) - \deg D_{t-k}(x)
\]
\[
= 2\delta_{t-k}(\bar{p}_{t-k}(x)) + 2 \sum_{i=0}^{k-1} \delta_{t-i}(\bar{p}_{t-i}(x))
\]
\[
= 2 \sum_{i=0}^{k} \delta_{t-i}(\bar{p}_{t-i}(x)).
\]

Thus \((13)\) hold for \(k \to k + 1\), and therefore for \(k = 0, 1, \ldots, t\). \(\Box\)

In terms of Theorem 2, the following corollaries may be obtained easily.

**Corollary 1.** Let \(\tilde{R}(x) = \tilde{N}(x)/D(x)\) be a vector-valued rational interpolant as in \((3)\). Then it holds

\[
\deg \tilde{N}(x) = \delta_0(\bar{p}_0(x)) + 2 \sum_{i=1}^{l} \delta_i(\bar{p}_i(x)),
\]
\[
\deg D(x) = 2 \sum_{i=1}^{l} \delta_i(\bar{p}_i(x)).
\]

\(\tag{16}\)
Corollary 2. Define $\tilde{R}_0(x) = \tilde{N}_{t-k}(x)/D_{t-k}(x)$ as in (4) and $m_i \neq 0$, $i = 0, 1, \ldots, n$ in (2). Then for $k = 0, 1, \ldots, t$, it holds

$$\deg \tilde{N}_{t-k}(x) = \deg \tilde{p}_{t-k}(x) + 2 \sum_{i=0}^{k-1} \deg \tilde{p}_{t-i}(x),$$

$$\deg D_{t-k}(x) = 2 \sum_{i=0}^{k-1} \deg \tilde{p}_{t-i}(x).$$

Corollary 3. Let the interpolation points be $\pi = \{x_i, i = 0, 1, \ldots, n, x_i \in \mathbb{R} \}$ and let the corresponding interpolated vectors be

$$\phi = \{\tilde{v}_i^{(j)}, j = 0, 1, i = 0, 1, \ldots, n, \tilde{v}_i^{(j)} \in \mathbb{C}^d \}.$$ 

Define the VORIs as the following form:

$$\tilde{R}_0(x) = \tilde{p}_0(x) + \frac{(x - x_0)^2}{\tilde{p}_1(x)} + \frac{(x - x_1)^2}{\tilde{p}_2(x)} + \cdots + \frac{(x - x_{n-1})^2}{\tilde{p}_n(x)},$$

(17)

then $\tilde{R}_0(x)$ is of the type $[2n + 1/2n]$.

The continued fraction (17) may be seen as the Thiele-type osculatory rational interpolant. If we choose different $w_s(x)$, $s = 0, 1, \ldots, t$ in (3), then we may construct different osculatory rational solution. From all of these solutions, the possible types are listed in Theorem 3. By using Theorem 2, the proof of Theorem 3 can be obtained easily by means of mathematical induction.

Theorem 3. Let $S = \sum_{i=0}^{n} (m_i + 1) - 1$ and let $S \geq 2$. Then the osculatory rational functions as in (3) are normally of the type:

$$[S/2k], \quad k = 0, 1, \ldots, \left[\frac{S}{2}\right],$$

$$\left[S + l/2 \left[\frac{S + l - k}{2}\right]\right], \quad l = 1, \ldots, S - 2 - k.$$ (18)

In the case of $m_i = 0$ ($i = 0, 1, \ldots, n$), all the types as (18) can be achieved. In other case, $w_s(x)$ have power factor, some types listed above may not be achieved. For instance, given the data set $\{(x_i, \tilde{v}_i^{(j)}), i = 0, 1, 2, 3, j = 0, 1\}$, the Thiele–Werner-type osculatory rational interpolants are normally not of the type [8/8], [10/10], [12/12].

Theorem 4. Assume that $S = \sum_{i=0}^{n} (m_i + 1) - 1$ and $S \geq 2$. If any two VORIs $\tilde{R}(x)$ and $\tilde{r}(x)$ satisfy the following conditions:

(i) $\tilde{R}^{(k)}(x_i) = \tilde{r}^{(k)}(x_i) = \tilde{v}^{(k)}_i$, $k = 0, 1, \ldots, m_i$, $i = 0, 1, \ldots, n$,
(ii) $\tilde{R}(x)$ and $\tilde{r}(x)$ are of the same type:

$$[S/2k], \quad k = 0, 1, \ldots, \left\lfloor \frac{S}{2} \right\rfloor.$$ 

Then $\tilde{R}(x)$ is equal to $\tilde{r}(x)$.

**Proof.** Let $\tilde{R}(x)$ and $\tilde{r}(x)$ be reduced forms:

$$\tilde{R}(x) = \tilde{P}(x)/Q(x), \quad \tilde{r}(x) = \tilde{p}(x)/q(x)$$

with real polynomials $Q(x)$ and $q(x)$ satisfying

$$Q(x)||\tilde{P}(x)|^2, \quad q(x)||\tilde{p}(x)|^2,$$

$$\deg Q = 2k - f, \quad \deg \tilde{P} \leq S + l - f,$$

$$\deg q = 2k - g, \quad \deg \tilde{p} \leq S + l - g,$$

$$l, f, g \geq 0, \quad 0 \leq k \leq \left\lfloor \frac{S}{2} \right\rfloor.$$

Let $t(x)$ be the greatest common factor of $Q(x)$ and $q(x)$. $t(x)$ is necessarily real. Define polynomials $q_r(x)$ and $Q_r(x)$ by

$$Q(x) = t(x)Q_r(x), \quad q(x) = t(x)q_r(x), \quad (19)$$

so that $Q_r(x)$ and $q_r(x)$ have no nontrivial common factor.

Define $\tilde{T}(x)$ by

$$\tilde{T}(x) = \tilde{P}(x)q_r(x) - \tilde{p}(x)Q_r(x), \quad (20)$$

which holds

$$\tilde{T}^{(k)}(x_i) = 0, \quad k = 0, 1, \ldots, m_i, \quad i = 0, 1, \ldots, n.$$ 

Define $w(x)$ by

$$w(x) = (x - x_0)^{m_0+1}(x - x_1)^{m_1+1} \cdots (x - x_n)^{m_n+1}$$

and a vector polynomial $\tilde{V}(x)$ by

$$\tilde{V}(x) = w(x)\tilde{P}(x). \quad (21)$$

Then

$$|\tilde{T}(x)|^2 = \tilde{T}(x) \cdot \tilde{T}^*(x)$$

$$= |\tilde{P}(x)|^2 q_r(x)^2 + |\tilde{p}(x)|^2 Q_r(x)^2 - 2q_r(x)Q_r(x)\text{Re}(\tilde{p}(x) \cdot \tilde{P}^*(x)).$$
Then we find that 

\[ Q_r(x)\|\tilde{T}(x)\|^2, \quad q_r(x)\|\tilde{T}(x)\|^2. \]

Because the interpolation vectors are finite, we deduce that 

\[ Q_r(x)\|\tilde{V}(x)\|^2, \quad q_r(x)\|\tilde{V}(x)\|^2. \]

Unless \( \tilde{V}(x) \equiv 0 \), it follows that

\[ 2 \deg \tilde{V}(x) \geq \deg Q_r(x) + \deg q_r(x). \] (22)

The following inequalities follow from (19)–(22):

\[
\begin{align*}
\deg Q_r &= 2k - f - \deg t, \quad \deg q_r = 2k - g - \deg t, \\
\deg T &\leq S + l + 2k - f - g - \deg t, \\
\deg |\tilde{V}|^2 &\leq 2(l + 2k - f - g - \deg t - 1), \\
\deg |\tilde{V}|^2 &\geq 4k - f - g - 2\deg t.
\end{align*}
\]

Because \( 2(l + 2k - f - g - \deg t - 1) - (4k - f - g - 2\deg t) = 2l - f - g - 2 \), it shows a contradiction when \( f + g > 2l - 2 \), and so \( \tilde{V}(x) \equiv 0 \). It shows \( \tilde{T}(x) \equiv 0 \) in (20) if only \( l = 0 \). So the result on uniqueness holds.

4. Example

In this section, we give two examples of computing VORIs. For each example, we adopt two forms of Thiele–Werner fraction but get identical results. It is easily understood from the uniqueness Theorem 4.

Example 1. Given the points \( x_0 = 0, \ x_1 = 1 \), to find a rational function \( \tilde{R}(x) \), such that 

\[
\begin{align*}
\tilde{R}(x_0) &= (1, 1), \quad \tilde{R}^{(1)}(x_0) = (2, 0), \quad \tilde{R}(x_1) = (2, 1), \quad \tilde{R}^{(1)}(x_1) = (0, 1).
\end{align*}
\]

Solution. Let the rational function have the following form:

\[ \tilde{R}(x) = \tilde{p}_0(x) + \frac{x^2}{\tilde{p}_1(x)}. \]

We get

\[
\begin{align*}
\tilde{y}_0(x) &= (-x^2 + 2x + 1, x^3 - x^2 + 1), \\
\tilde{p}_0(x) &= (1, 1) + x(2, 0) = (1 + 2x, 1),
\end{align*}
\]
\[
\tilde{y}_1(x) = \frac{x^2}{\tilde{y}_0(x) - \tilde{p}_0(x)} = \frac{x^2}{(-x^2, x^3 - x^2)} \\
= \frac{1}{x^2 - 2x + 2} (-1, x - 1),
\]
\[
\tilde{p}^{(1)}_1(x) = \frac{1}{(x^2 - 2x + 2)^2} (2x - 2, -x^2 + 2x),
\]
\[
\tilde{y}_1(x_1) = (-1, 0), \quad \tilde{y}^{(1)}_1(x_1) = (0, 1),
\]
\[
\tilde{p}_1(x) = (-1, 0) + (x - 1)(0, 1) = (-1, x - 1).
\]

Thus we find that
\[
\tilde{R}(x) = (1 + 2x, 1) + \frac{x^2}{(-1, x - 1)}
\]
\[
= (1 + 2x, 1) + \frac{(-x^2, x^3 - x^2)}{x^2 - 2x + 2}
\]
\[
= \frac{1}{x^2 - 2x + 2} (2x^3 - 4x^2 + 2x + 2, x^3 - 2x + 2).
\]

Let the rational function define as the following form:
\[
\tilde{R}(x) = \tilde{p}_0(x) + \frac{(x - 1)^2}{\tilde{p}_1(x)},
\]
then it holds \( \tilde{p}_0(x) = (2, x) \), \( \tilde{p}_1(x) = \frac{1}{2}(x - 1, 1) \). So the rational function is
\[
\tilde{R}(x) = (2, x) + \frac{(x - 1)^2}{\frac{1}{2}(x - 1, 1)}
\]
\[
= \frac{1}{x^2 - 2x + 2} (2x^3 - 4x^2 + 2x + 2, x^3 - 2x + 2).
\]

**Example 2.** Let the points be \( x_0 = 0 \), \( x_1 = 1 \) and find a rational function \( \tilde{R}(x) \), such that
\( \tilde{R}(x_0) = (1, 0) \), \( \tilde{R}^{(1)}(x_0) = (1, 1) \), \( \tilde{R}(x_1) = (0, 1) \), \( \tilde{R}^{(1)}(x_1) = (1, 0) \).

**Solution.** Let the rational function have the following form
\[
\tilde{R}(x) = \tilde{p}_0(x) + \frac{x^2}{\tilde{p}_1(x)}.
\]
We have
\[
\tilde{y}_0(x) = (4x^3 - 6x^2 + x + 1, -x^3 + x^2 + x),
\]
\[
\tilde{p}_0(x) = (1, 0) + x(1, 1) = (1 + x, x),
\]
\[
\tilde{y}_1(x) = \frac{x^2}{\tilde{y}_0(x) - \tilde{p}_0(x)} = \frac{x^2}{(4x^3 - 6x^2, -x^3 + x^2)}
\]
\[
= \frac{1}{(4x - 6, -x + 1)},
\]
\[
\tilde{y}_1(x_1) = (-1/2, 0), \quad \tilde{y}_1^{(1)}(x_1) = (-1, -1/4),
\]
\[
\tilde{p}_1(x) = (-1/2, 0) + (x - 1)(-1, -1/4) = \frac{1}{4}(-4x + 2, -x + 1).
\]

So we get
\[
\tilde{R}(x) = (1 + x, x) \frac{x^2}{\frac{1}{4}(-4x + 2, -x + 1)}
\]
\[
= \frac{(x^3 + 7x^2 - 13x + 5, 13x^3 - 14x^2 + 5x)}{17x^2 - 18x + 5}.
\]

Let the rational function be the following form:
\[
\tilde{R}(x) = \tilde{p}_0(x) + \frac{(x - 1)^2}{\tilde{p}_1(x)},
\]
then \(\tilde{p}_0(x) = (x - 1, 1), \quad \tilde{p}_1(x) = \frac{1}{25}(-16x + 10, 13x - 5)\). So the rational function is
\[
\tilde{R}(x) = (x - 1, 1) + \frac{(x - 1)^2}{\frac{1}{25}(-16x + 10, 13x - 5)}
\]
\[
= \frac{(x^3 + 7x^2 - 13x + 5, 13x^3 - 14x^2 + 5x)}{17x^2 - 18x + 5}.
\]

5. Conclusion

In this paper, we give a method of solving vector-valued osculatory rational interpolation problems. This method is based on vector-valued Thiele–Werner-type continued fractions. The structure is employed to construct various rational interpolants. By the structure, it is also flexible to construct a rational interpolation of vectors with many forms, which is available in choosing a form of less rational process. The characteristic properties on Thiele–Werner continued fraction are established. The possible types of the Thiele–Werner continued fraction are discussed, which give us a scope of choosing different degree to construct vector numerators and scalar denominators. In the whole discussion, we ignore the degeneracy cases, which are known to plague all rational interpolation schemes. We choose all interpolation nodes \(x_i\) to be real, in the hope that it can be extended to the complex variable case.

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References