Journal of Algebra 322 (2009) 1979–1999

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Crystal bases for quantum affine algebras and Young walls

Seok-Jin Kang [a,1], Hyeonmi Lee [b,*,2]

[a] Department of Mathematical Sciences, Seoul National University, Seoul 151-747, Republic of Korea
[b] Department of Mathematics and Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Republic of Korea

ARTICLE INFO

Article history:
Received 14 August 2008
Available online 9 July 2009
Communicated by Efim Zelmanov

Keywords:
Crystal
Reduced Young wall
Quantum affine algebra
Slice
Splitting of blocks

ABSTRACT

We provide a unified approach to the Young wall description of crystal graphs for arbitrary level irreducible highest weight representations over classical quantum affine algebras. The crystal graph is realized as the affine crystal consisting of all reduced Young walls built on a ground-state wall.

1. Introduction

The crystal basis theory for integrable modules over quantum groups $U_q(\mathfrak{g})$ associated with a symmetrizable Kac–Moody algebra $\mathfrak{g}$ was introduced by Kashiwara [11,12]. It was given in a combinatorial language and an equivalent theory called the canonical basis theory, given in a more geometric language, was developed by Lusztig [18]. A crystal basis of a $U_q(\mathfrak{g})$-module can be viewed as its basis at $q = 0$, and it is given a structure of colored oriented graph, called the *crystal graph*, with arrows defined by the Kashiwara operators. The crystal graphs have many nice combinatorial features reflecting the internal structure of integrable modules over quantum groups.

In [19], Misra and Miwa constructed the crystal bases for basic representations of $U_q(A_n^{(1)})$ using the Fock space representation, which is built on colored Young diagrams. Their idea was extended to construct crystal bases for irreducible highest weight $U_q(A_n^{(1)})$-modules of arbitrary level [3]. The crystal graphs constructed in [3,19] can be parameterized by certain paths which arise naturally in the theory of solvable lattice models. Motivated by this observation, the theory of perfect crystals

* Corresponding author.
E-mail addresses: sjkang@math.snu.ac.kr (S.-J. Kang), hyeonmi@hanyang.ac.kr (H. Lee).

1 Supported in part by KRF Grant # 2007-341-C00001.
2 Supported by the research fund of Hanyang University (HY-2008-N).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.06.010

for general quantum affine algebras was developed and a realization of crystal graphs for irreducible highest weight modules of arbitrary levels over quantum affine algebras in terms of *paths* was given in [5,6].

The notion of level-1 Young walls [4], which is combinatorial in nature, is a development of [3,19] in another direction. The Young walls consist of colored blocks of various shapes and can be viewed as generalizations of colored Young diagrams. A new description of the crystal graph for basic representations for the classical quantum affine algebras was given as the set of level-1 reduced Young walls [2,4]. The work was used to give crystal graphs for finite classical types [7], and developed further to construct crystal graphs of higher level irreducible highest weight modules for classical quantum affine algebras excluding the $D_n^{(1)}$ type [10]. Under those constructions, the crystal graph of a level-l irreducible highest weight module is described as l layers of level-1 Young walls and this is done using only the generic whole blocks. The constructions for each type were made in an ad hoc and case by case manner.

The remaining $D_n^{(1)}$ type was dealt by [16], in a different manner. In the current work, we extend the idea for $D_n^{(1)}$ type used in [16] to all other classical quantum affine algebra types. We provide a Young wall description of the crystal graphs for arbitrary level irreducible highest weight representations over the classical affine types $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, and $D_{n+1}^{(2)}$ in a unified manner. The notions of *splitting of blocks*, *pre-slices*, and *slices* play important roles in our Young wall construction. A *level-l Young wall* ($l \geqslant 1$) will be a concatenating level-l slices which are *split forms* of pre-slices, rather than l layers of level-1 Young walls, and even the *broken halves* of (generic) whole blocks will be used in the construction. These notions enable us to provide a unified approach to the construction of crystal graphs for arbitrary level irreducible highest weight representations for all classical quantum affine algebras. The crystal graphs are realized as affine crystals consisting of all reduced Young walls built on certain ground-state walls. The character of an irreducible highest weight representation can easily be computed by counting the number of colored blocks in reduced Young walls that have been added to the ground-state wall during their constructions. One may view our result as a development of the *Young tableaux* theory of Kashiwara and Nakashima [13], in the affine direction.

The construction of the crystal graph for arbitrary level irreducible highest weight $U_q(A_n^{(1)})$-modules, given in [3,19], can be recovered from our Young wall construction. The $A_n^{(1)}$-case is the simplest of all cases, with no appearance of the splitting.

As one immediate application of this work, recently, using the result of the current paper, a realization of $\mathcal{B}(\infty)$, the crystal basis of the negative part $U_q^-(\mathfrak{g})$ of a quantum group, was given for classical affine types [17]. We also expect to be able to construct the arbitrary level Fock space representations of all classical quantum affine algebras using combinatorics of our Young walls. This should lead to a generalized Lascoux–Leclerc–Thibon algorithm for computing the global bases of arbitrary level irreducible highest weight representations. It will be an extension of the results [8,9,15], giving a recursive algorithm for computing the global bases of the basic representations for classical quantum affine algebras.

Our paper is organized as follows. In Section 2, we fix notation. Section 3 is devoted to introducing the new notion of splitting of blocks, level-l pre-slices, and level-l slices and also their combinatorics. Note that the patterns for stacking blocks, the notions of splitting of blocks and slices are different from the ones given in [10]. Here, some actions on the set of level-l slices are also defined. In Section 4, we show that the actions we have defined carry over to sets of certain equivalence classes of level-l slices, and we give an explicit correspondence between these sets and the perfect crystals of [6,14]. In the next section, we define the notions of proper Young walls, reduced Young walls, and ground-state walls. A level-l Young wall is defined to be a concatenation of level-l slices. We give an affine crystal structure on the set $\mathcal{F}(\lambda)$ of all proper Young walls, built on a ground-state wall Y_λ, where λ is a dominant integral weight of level-l. The action of Kashiwara operators on the set is given explicitly in terms of combinatorics of Young walls, using the actions defined on the set of level-l slices. The subcrystal $\mathcal{Y}(\lambda)$ consisting of reduced Young walls is stable under the action of Kashiwara operators and is shown to a description for the crystal graph $\mathcal{B}(\lambda)$, through an isomorphism with the path description of [5].

2. Quantum affine algebras

We will follow the general notations given in [1].

Let $(A, P^\vee, P, \Pi^\vee, \Pi)$ be an affine Cartan datum of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, or $D_{n+1}^{(2)}$. We denote by $I = \{0, 1, \ldots, n\}$ the index set for the simple roots, $A = (a_{ij})_{i,j\in I}$ the affine generalized Cartan matrix, $P^\vee = (\bigoplus_{i\in I} \mathbf{Z}h_i) \oplus \mathbf{Z}d$ the dual weight lattice, $\mathfrak{h} = \mathbf{C} \otimes_{\mathbf{Z}} P^\vee$ the Cartan subalgebra, $P = (\bigoplus_{i\in I} \mathbf{Z}\Lambda_i) \oplus \mathbf{Z}\delta$ (and $\bar{P} = \bigoplus_{i\in I} \mathbf{Z}\Lambda_i$) the weight lattice (resp. classical weight lattice), $\Pi^\vee = \{h_i \mid i \in I\}$ the set of simple coroots and $\Pi = \{\alpha_i \mid i \in I\}$ the set of simple roots. We also denote by δ the null root, Λ_i $(i \in I)$ the fundamental weights and P^+ the set of affine dominant integral weights.

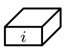
Let $U_q(\mathfrak{g})$ be the quantum affine algebra associated with $(A, P^\vee, P, \Pi^\vee, \Pi)$ and let $e_i, f_i, K_i^{\pm 1}$ $(i \in I)$, q^d be the generators of $U_q(\mathfrak{g})$. The subalgebra $U_q'(\mathfrak{g})$ generated by $e_i, f_i, K_i^{\pm 1}$ $(i \in I)$ is also called the quantum affine algebra.

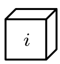
We will not repeat the basic facts such as tensor product rule on general crystal basis theory. We just recall that for a dominant integral weight $\lambda \in P^+$, the crystal graph $\mathcal{B}(\lambda)$ of the irreducible highest weight module $V(\lambda)$ over $U_q(\mathfrak{g})$ will be referred to as the ***irreducible highest weight crystal***. The explicit construction of irreducible highest weight crystal $\mathcal{B}(\lambda)$ is one of the central problems in crystal basis theory.

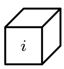
3. Splitting of blocks and slices

In this section, we will introduce the notions of splitting of blocks and slices.

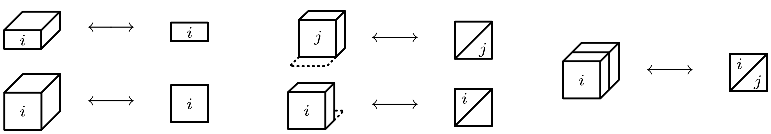
The basic ingredient of our discussion will be the following colored ***blocks*** of three different shapes.


: half-unit height, unit width, unit depth.


: unit height, unit width, half-unit depth.


: unit height, unit width, unit depth.

The coloring of a block will be given differently according to the types of blocks and the types of quantum affine algebras. For simplicity, we will use the following notation.



That is, we use just the frontal view for most blocks. For half-unit depth blocks, we use triangles to cope with their co-presence inside a single unit cube.

Definition 3.1.

(1) If $\mathfrak{g} \neq C_n^{(1)}$, a ***level-1 slice*** is defined to be a set of finitely many blocks stacked in one column of unit depth following the patterns given in Fig. 1.

(2) If $\mathfrak{g} = C_n^{(1)}$, such a set of blocks will be called a ***level-$\frac{1}{2}$ slice***.

In stacking the blocks, no block can be placed on top of a block of half-unit depth. As we can see in the figure, the blocks are stacked in a repeating pattern. Roughly speaking, the first complete cycle of each of the repeating patterns is symmetric with respect to the n-block, except in the $A_n^{(1)}$ case. For cases other than $A_n^{(1)}$, we say that an i-block is a ***supporting*** (or ***covering***) ***i-block*** if it lies in the bottom half (resp. upper half) of such a cycle. An i-block that appears only once in a cycle is regarded

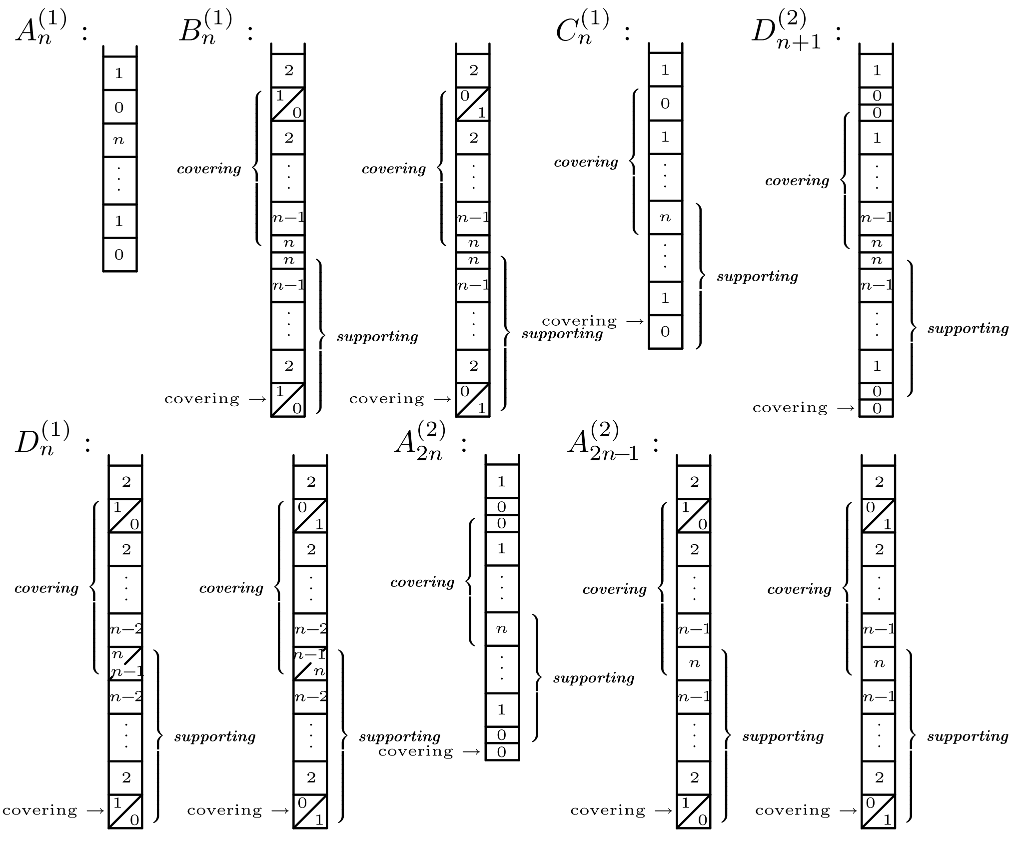


Fig. 1. Patterns for building slices.

as both a supporting and a covering block. The notions of covering or supporting are not defined in the $A_n^{(1)}$ case.

An ***i*-slot** is the top of a level-1 (or level-$\frac{1}{2}$) slice where we may add an i-block. The notions of ***supporting i-slots*** and ***covering i-slots*** are defined in a similar manner.

A ***δ-set*** is a set of blocks (and its cyclic variations) that form one cycle of the stacking patterns. For a level-1 or level-$\frac{1}{2}$ slice c, we define $c+\delta$ (and $c-\delta$) to be a level-1 or level-$\frac{1}{2}$ slice obtained from c by adding (resp. removing) a δ-set. For level-1 (or level-$\frac{1}{2}$) slices c and c', we write $c \subset c'$ if c is part of c'. For example, we have $c-\delta \subset c \subset c+\delta$.

Definition 3.2.

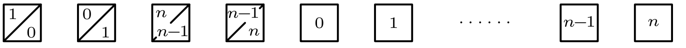
(1) If $\mathfrak{g} \neq C_n^{(1)}$, ***a level-l pre-slice*** is defined to be an ordered l-tuple $C=(c_1,\dots,c_l)$ of level-1 slices such that $c_1 \subset c_2 \subset \cdots \subset c_l \subset c_1+\delta$.
(2) If $\mathfrak{g} = C_n^{(1)}$, ***a level-l pre-slice*** is defined to be an ordered $2l$-tuple $C=(c_1,\dots,c_{2l})$ of level-$\frac{1}{2}$ slices such that $c_1 \subset c_2 \subset \cdots \subset c_{2l} \subset c_1+\delta$.
(3) The level-1 (or level-$\frac{1}{2}$) slice c_i in C is called the ***ith layer*** of C.

Remark 3.3.

(1) For those quantum affine algebras that allow two stacking patterns, we choose only one pattern in building a level-l pre-slice. Still, two different level-l pre-slices can be made from two different stacking patterns.

(2) A level-l pre-slice can be visualized as the set of l columns (or $2l$ columns) with the ith layer placed in front of $(i+1)$th layer.

Next, we explain the notion of splitting of (whole) blocks. By a whole block, we mean a unit cube or a gluing of two half-unit depth blocks as is shown in the following picture:



The first two (and the next two) whole blocks all be referred to as $0|1$-blocks (resp. $(n-1)|n$-blocks). Thus, when we deal with whole blocks, we may choose their colors among $I \cup \{0|1, (n-1)|n\}$. Similarly, we may consider whole i-slots for $I \cup \{0|1, (n-1)|n\}$.

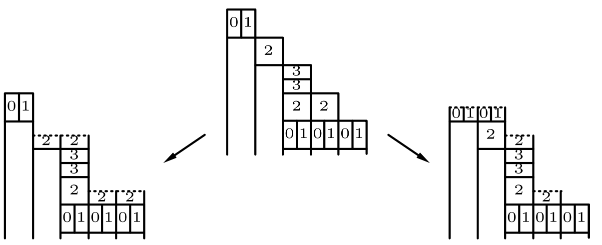
Note that in a given pre-slice, there can be at most two heights in which a covering (or supporting) i-block may appear as the top of one layer. Similarly, there may be at most two heights in which a covering (or a supporting) i-slot may appear.

Definition 3.4. Fix $i \in I \cup \{0|1, (n-1)|n\}$. Suppose that there is a layer whose top is a covering (or supporting) whole i-block and another layer whose top is a supporting (resp. covering) whole i-slot. Choose the covering (resp. supporting) i-block lying in the fore-most layer among the ones with the higher height and the supporting (resp. covering) i-slot lying in the rear-most layer among the ones with the lower height.

To ***split a whole i-block*** means to break off the top half of the chosen covering (resp. supporting) i-block and to place it in the chosen supporting (resp. covering) i-slot. A ***split form*** of a pre-slice is a result obtained by splitting all the whole blocks that can be split. In the $A_n^{(1)}$ case, the split form of any pre-slice is itself.

Note that a pre-slice may have several different split forms.

Example 3.5. This example shows two different split forms of a single pre-slice for $B_3^{(1)}$-type. Neither of the split results allows further splitting.



The lower left figure shows the splitting of a covering 2-block and a supporting 2-block. The lower right figure shows the splitting of a supporting 2-block and a $0|1$-block from the same pre-slice.

Definition 3.6. Fix a pattern for building pre-slices. A ***level-l slice*** is a split form of a level-l pre-slice. We denote by $\mathcal{S}^{(l)}$ the set of all level-l slices built on a fixed pattern.

Remark 3.7.

(1) The notions of i-*slot*, δ-*set*, and *layer*, defined for pre-slices, naturally carry over to those of slices. Some care must be exercised, however. For example, δ-sets should allow for halves of blocks to add up to a δ, and we should now consider halves of i-slots.

(2) We would like to remind the readers that in a level-l (pre-)slice, the top of a layer is regarded as an i-slot if it is an i-slot when the layer is viewed a level-1 slice.

We now define the action of Kashiwara operators on the set $\mathcal{S}^{(l)}$ of level-l slices. Let C be a level-l slice and fix an index $i \in I$. Below, those blocks which are at the top of each level-1 slice (or level-$\frac{1}{2}$ slice for $C_n^{(1)}$) making up the level-l slice C will be referred to as the top blocks of C.

Case 1. For the $A_n^{(1)}$ type, the actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by (E1)–(E2) and (F1)–(F2), respectively.

(E1) If C contains some i-blocks as top blocks, then remove the fore-most i-block among the ones with the higher height.
(E2) If C contains no i-blocks as top blocks, we define $\tilde{e}_i C = 0$.
(F1) If C contains some i-slots, then we place an i-block in the rear-most i-slot among the ones with the lower height.
(F2) If C contains no i-slots, we define $\tilde{f}_i C = 0$.

Case 2. Suppose that $i \neq 0, n$ and that the i-block is a unit cube. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by (E1)–(E4) and (F1)–(F4), respectively.

(E1) If C contains both covering whole i-blocks and supporting whole i-blocks as top blocks, then remove the upper half i-block from the fore-most covering whole i-block among the ones with the higher height and another upper half i-block from the fore-most supporting whole i-block among the ones with the higher height.
(E2) If C contains whole i-blocks as top blocks and all of them are of the same type, then remove the upper half i-block from the fore-front whole i-block among the ones with the higher height. This would create a lower half i-block. Consider the (lower) half i-blocks having the same type as this new one. We then remove the fore-most half i-block among the ones with the higher height.
(E3) If C contains no whole i-blocks as top blocks, but does contain some half i-blocks, then the number of covering half i-blocks and that of supporting half i-blocks must be the same. We remove the fore-most covering half i-block and the fore-most supporting half i-block among the ones with the higher height.
(E4) If C contains no i-blocks as top blocks, we define $\tilde{e}_i C = 0$.
(F1) If C contains both covering whole i-slots and supporting whole i-slots, then we place a half i-block in the rear-most covering whole i-slot among the ones with the lower height and another half i-block in the rear-most supporting whole i-slot among the ones with the lower height.
(F2) If C contains whole i-slots and all of them are at the same type, then we place a half i-block in the rear-most whole i-slot among the ones with the lower height. This would create an (upper) half i-slot. Consider the (upper) half i-slots having the same type as this new one. We then place another half i-block in the rear-most (upper) half i-slots among the ones with the lower height.
(F3) If C contains no whole i-slots, but does contain some half i-slots, then the number of covering half i-slots and that of supporting half i-slots must be the same. We place a half i-block in the rear-most covering half i-slots and another half i-block in the rear-most supporting half i-slots among the ones with the lower height.
(F4) If C contains no i-slots, we define $\tilde{f}_i C = 0$.

Case 3. Suppose that $i = 0, n$ and that the i-block is a unit cube. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by (E1)–(E3) and (F1)–(F3), respectively.

(E1) If C contains at least two whole i-blocks as top blocks, first remove an upper half i-block from the fore-most whole i-block among the ones with the higher height. Then, do the same once more on the resulting set of blocks. If C contains one whole i-block as top blocks, then remove the upper half i-block from the whole i-block and remove the fore-most (lower) half i-block among the ones with the higher height.
(E2) If C contains no whole i-blocks as top blocks, but does contain some half i-blocks, then the number of half i-blocks must be even. We first remove the fore-most half i-block among the ones with the higher height. Then, do the same once more on the resulting set of blocks.

(E3) If C contains no i-blocks as top blocks, we define $\tilde{e}_i C = 0$.
(F1) If C contains at least two whole i-slots, we first place a half i-block in the rear-most whole i-slot among the ones with the lower height. Then, do the same once more on the resulting set of blocks. If C contains one whole i-slot, then we place a half i-block in the whole i-slot and another half i-block in the rear-most (upper) half i-slot among the ones with the lower height.
(F2) If C contains no whole i-slots, but does contain some half i-slots, then the number of half i-slots must be even. We first place a half i-block in the rear-most half i-slot among the ones with the lower height. Then, do the same once more on the resulting set of blocks.
(F3) If C contains no i-slots, we define $\tilde{f}_i C = 0$.

Case 4. Suppose that $i = 0$ (or $i = n$) and that the i-block is of half-height. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by (E1)–(E3) and (F1)–(F3), respectively.

(E1) If C contains supporting i-blocks (resp. covering i-blocks) as top blocks, then we remove the fore-most supporting (resp. covering) i-block among the ones with the higher height.
(E2) If C contains i-blocks as top blocks and all of them are covering blocks (resp. supporting blocks), then remove the fore-most covering (resp. supporting) i-block among the ones with the higher height.
(E3) If C contains no i-blocks as top blocks, then $\tilde{e}_i C = 0$.
(F1) If C contains covering i-slots (resp. supporting i-slots), then we place an i-block in the rear-most covering (resp. supporting) i-slot among the ones with the lower height.
(F2) If C contains i-slots and all of them are supporting slots (resp. covering slots), then we place an i-block in the rear-most supporting (resp. covering) i-slot among the ones with the lower height.
(F3) If C contains no i-slot, the we define $\tilde{f}_i C = 0$.

Case 5. Suppose that $i = 0, 1$ (or $n-1$, n) and that the i-block is of half-depth. ***Un-split*** all half $0|1$-blocks (resp. $(n-1)|n$-blocks) in C to obtain C'.

(E1) If C' contains i-blocks as top blocks, remove the fore-most i-block among the ones with the higher height to obtain C''. We split all $0|1$-blocks (resp. $(n-1)|n$-blocks) in C'' to obtain $\tilde{e}_i C$.
(E2) If C' contains no i-block as top blocks, we define $\tilde{e}_i C = 0$.
(F1) If C' contains i-slots, place an i-block in the rear-most i-slot among the ones with the lower height to obtain C''. We split all $0|1$-blocks (resp. $(n-1)|n$-blocks) in C'' to obtain $\tilde{f}_i C$.
(F2) If C' contains no i-slot, we define $\tilde{f}_i C = 0$.

Remark 3.8. In a slice, unlike the case for other whole blocks, the result of un-splitting all $0|1$ and $(n-1)|n$-blocks, mentioned in the above actions of Kashiwara operators for $i = 0, 1, n-1, n$, is unique.

In Fig. 2, we illustrate $\tilde{f}_i$ actions on slices. The first one shows $\tilde{f}_2$ actions on slices of $B_3^{(1)}$ type, the second one shows $\tilde{f}_2$ actions for the $A_4^{(2)}$ type, and the last one shows an $\tilde{f}_0$ action for the $B_3^{(1)}$ type.

4. Equivalence classes of slices and perfect crystals

We will define a classical crystal structure on the set $\mathcal{C}^{(l)}$ of equivalence classes of level-l slices and show that the result is isomorphic to the level-l perfect crystal $\mathcal{B}^{(l)}$ constructed for each of the classical quantum affine algebras, in [6].

Let $C = (c_1, \ldots, c_l)$ be a level-l slice. We define the slices $C \pm \delta$ to be:

$$C + \delta = (c_2, \ldots, c_l, c_1 + \delta), \qquad C - \delta = (c_l - \delta, c_1, \ldots, c_{l-1}). \tag{4.1}$$

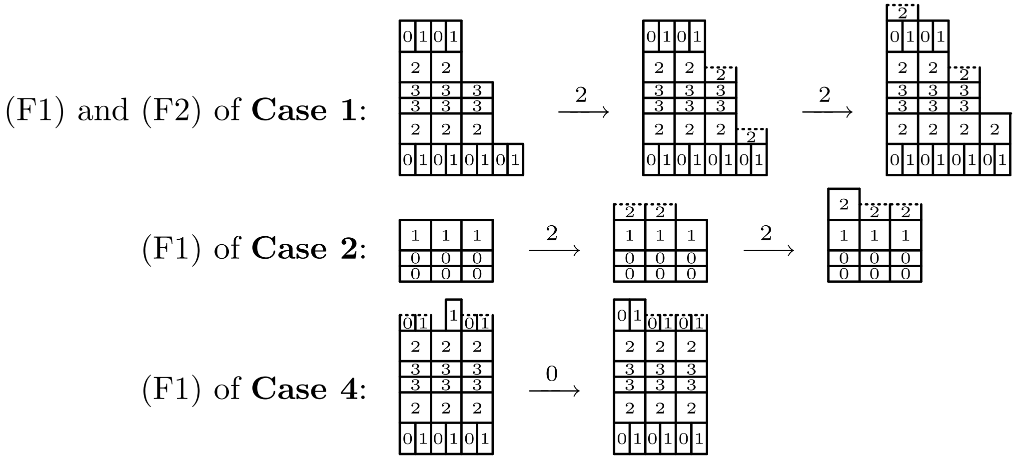


Fig. 2. $\tilde{f}_i$ actions on slices.

We say that two slices C and C' are ***related***, denoted by $C \sim C'$, if one of them may be obtained from the other by adding finitely many δ's. Let

$$\mathcal{C}^{(l)} = \mathcal{S}^{(l)}/\sim \tag{4.2}$$

be the set of equivalence classes of level-l slices under this relation. For the equivalence class containing a level-l slice C, we will use the same symbol C. Since the map $C (\in \mathcal{S}^{(l)}) \mapsto C + \delta$ commutes with the action of Kashiwara operators, we may define the induced Kashiwara operators on $\mathcal{C}^{(l)}$. We also define

$$\begin{aligned} \varphi_i(C) &= \max\{k \mid \tilde{f}_i^k C \in \mathcal{C}^{(l)}\}, \\ \varepsilon_i(C) &= \max\{k \mid \tilde{e}_i^k C \in \mathcal{C}^{(l)}\}, \\ \overline{\mathrm{wt}}(C) &= \sum_i \bigl(\varphi_i(C) - \varepsilon_i(C)\bigr)\Lambda_i. \end{aligned} \tag{4.3}$$

Then one can verify in a straightforward manner that the set $\mathcal{C}^{(l)}$, together with the induced Kashiwara operators and the maps φ_i, ε_i $(i \in I)$, $\overline{\mathrm{wt}}$ becomes a $U_q'(\mathfrak{g})$-crystal.

We now list the explicit sets for the perfect crystals $\mathcal{B}^{(l)}$. We refer readers to [6,14] for the maps $\overline{\mathrm{wt}}$, $\tilde{e}_i$, $\tilde{f}_i$, φ_i, and ε_i $(i \in I)$ defining the crystal structure on them.

- $A_n^{(1)}$ $(n \geqslant 1)$:

$$\mathcal{B}^{(l)} = \left\{ (x_0, x_1, \ldots, x_n) \;\middle|\; x_i \in \mathbf{Z}_{\geqslant 0},\ \sum_{i=0}^{n} x_i = l \right\},$$

- $B_n^{(1)}$ $(n \geqslant 3)$:

$$\mathcal{B}^{(l)} = \left\{ \bigl(x_1, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_1\bigr) \;\middle|\; x_0 = 0 \text{ or } 1,\ x_i, \bar{x}_i \in \mathbf{Z}_{\geqslant 0},\ x_0 + \sum_{i=1}^{n} (x_i + \bar{x}_i) = l \right\},$$

- $C_n^{(1)}$ $(n \geqslant 2)$:

$$\mathcal{B}^{(l)} = \left\{ (x_1, \dots, x_n | \bar{x}_n, \dots, \bar{x}_1) \;\middle|\; x_i, \bar{x}_i \in \mathbf{Z}_{\geqslant 0},\ 2l \geqslant \sum_{i=1}^{n} (x_i + \bar{x}_i) \in 2\mathbf{Z} \right\},$$

- $D_n^{(1)}$ $(n \geqslant 3)$:

$$\mathcal{B}^{(l)} = \left\{ (x_1, \dots, x_n | \bar{x}_n, \dots, \bar{x}_1) \;\middle|\; x_n = 0 \text{ or } \bar{x}_n = 0,\ x_i, \bar{x}_i \in \mathbf{Z}_{\geqslant 0},\ \sum_{i=1}^{n} (x_i + \bar{x}_i) = l \right\},$$

- $A_{2n}^{(2)}$ $(n \geqslant 1)$:

$$\mathcal{B}^{(l)} = \left\{ (x_1, \dots, x_n | \bar{x}_n, \dots, \bar{x}_1) \;\middle|\; x_i, \bar{x}_i \in \mathbf{Z}_{\geqslant 0},\ \sum_{i=1}^{n} (x_i + \bar{x}_i) \leqslant l \right\},$$

- $A_{2n-1}^{(2)}$ $(n \geqslant 3)$:

$$\mathcal{B}^{(l)} = \left\{ (x_1, \dots, x_n | \bar{x}_n, \dots, \bar{x}_1) \;\middle|\; x_i, \bar{x}_i \in \mathbf{Z}_{\geqslant 0},\ \sum_{i=1}^{n} (x_i + \bar{x}_i) = l \right\},$$

- $D_{n+1}^{(2)}$ $(n \geqslant 2)$:

$$\mathcal{B}^{(l)} = \left\{ \big(x_1, \dots, x_n | x_0 | \bar{x}_n, \dots, \bar{x}_1\big) \;\middle|\; x_0 = 0 \text{ or } 1,\ x_i, \bar{x}_i \in \mathbf{Z}_{\geqslant 0},\ x_0 + \sum_{i=1}^{n} (x_i + \bar{x}_i) \leqslant l \right\}.$$

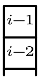
Recall that a typical level-l slice is a split form of a level-l pre-slice. Hence the top of each layer of a level-l slice may be a usual block or a broken half of a block.

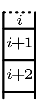
We will classify the layers of level-l slices into several types depending on the shape of their top parts, and present any element of the set $\mathcal{C}^{(l)}$ in terms of the numbers of the layers of each type. Note that each element of $\mathcal{C}^{(l)}$ is determined by these numbers of layers. We then construct a *canonical bijection* $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ and its inverse $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$, which will turn out to be crystal isomorphisms. We will use the symbols s_i, $\bar{s}_i$, t_i, $\bar{t}_i$ to denote the types of layers, classified according to its top parts.

(1) $\boldsymbol{A_n^{(1)}}$ **case:** Let C be an element of $\mathcal{C}^{(l)}$. If we write the number of layers in C with i-slots $(0 \leqslant i \leqslant n)$ by y_i, then $\sum_{i=0}^{n} y_i = l$. We define the map $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ by $C \mapsto (y_0, y_1, \dots, y_n)$ and the map $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$ by $(x_0, x_1, \dots, x_n) \mapsto C$, where $y_i = x_i$ for all $0 \leqslant i \leqslant n$.

(2) $\boldsymbol{A_{2n}^{(2)}}$ **case:** We first list all types of layers classified according to their top parts.
- s_0: supporting 0-slot (= covering 0-block),
- s_i: supporting i-slot $(1 \leqslant i \leqslant n)$,
- $\bar{s}_i$: covering i-block $(1 \leqslant i \leqslant n)$,
- t_i: half of a supporting i-block $(1 \leqslant i \leqslant n)$,
- $\bar{t}_i$: half of a covering i-block $(1 \leqslant i \leqslant n)$.

For example, the layers of type s_i and $\bar{t}_i$ have the following form

$s_i =$ 

$\bar{t}_i =$ 

Let C be an element of $\mathcal{C}^{(l)}$. Note that the number of layers of type t_i must be the same as that of layers of type $\bar{t}_i$. We denote

- u_0: the number of layers of type s_0 in C,
- y_i: the number of layers of type s_i in C $(1 \leqslant i \leqslant n)$,
- $\bar{y}_i$: the number of layers of type $\bar{s}_i$ in C $(1 \leqslant i \leqslant n)$,
- z_i: the number of layers of type t_i in C $(1 \leqslant i \leqslant n)$ $(=$ the number of layers of type $\bar{t}_i$ in $C)$.

These have the properties $z_n \in 2\mathbf{Z}_{\geqslant 0}$, $y_i\bar{y}_i = 0$, and $u_0 + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=1}^{n-1} z_i + z_n = l$. We define the map $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ by

$$\phi(C) = (x_1, \dots, x_n|\bar{x}_n, \dots, \bar{x}_1),$$

where $x_i = y_i + z_i$, $\bar{x}_i = \bar{y}_i + z_i$ $(1 \leqslant i \leqslant n-1)$, $x_n = y_n + \frac{1}{2}z_n$, $\bar{x}_n = \bar{y}_n + \frac{1}{2}z_n$. We also define the map ψ by

$$\psi(x_1, \dots, x_n|\bar{x}_n, \dots, \bar{x}_1) = C,$$

where $u_0 = l - \sum_{i=1}^{n}(x_i + \bar{x}_i)$, $z_n = 2\min\{x_n, \bar{x}_n\}$, $z_i = \min\{x_i, \bar{x}_i\}$ $(1 \leqslant i \leqslant n-1)$, $y_i = \max\{0, x_i - \bar{x}_i\}$, and $\bar{y}_i = \max\{0, \bar{x}_i - x_i\}$ $(1 \leqslant i \leqslant n)$.

(3) $\boldsymbol{D_{n+1}^{(2)}}$ **case:** The types of layers are as follows.

- s_0: covering 0-block $(=$ supporting 0-slot),
- s_i: supporting i-slot $(1 \leqslant i \leqslant n)$,
- $\bar{s}_i$: covering i-block $(1 \leqslant i \leqslant n)$,
- t_i: half of a supporting i-block $(1 \leqslant i \leqslant n-1)$,
- $\bar{t}_i$: half of a covering i-block $(1 \leqslant i \leqslant n-1)$,
- t_n: supporting n-block $(=$ covering n-slot).

For any $C \in \mathcal{C}^{(l)}$, we write

- u_0: the number of layers of type s_0 in C,
- y_i: the number of layers of type s_i in C $(1 \leqslant i \leqslant n)$,
- $\bar{y}_i$: the number of layers of type $\bar{s}_i$ in C $(1 \leqslant i \leqslant n)$,
- z_i: the number of layers of type t_i in C $(1 \leqslant i \leqslant n-1)$ $(=$ the number of layers of type $\bar{t}_i)$,
- z_n: the number of layers of type t_n in C.

They have the properties $y_i\bar{y}_i = 0$ and $u_0 + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=1}^{n-1} z_i + z_n = l$. We define the map $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ by $\phi(C) = (x_1, \dots, x_n|x_0|\bar{x}_n, \dots, \bar{x}_1)$, where

$$x_0 = \begin{cases} 0 & \text{if } z_n \text{ is even}, \\ 1 & \text{if } z_n \text{ is odd}, \end{cases} \qquad x_n = \begin{cases} y_n + z_n & \text{if } z_n \text{ is even}, \\ y_n + z_n - 1 & \text{if } z_n \text{ is odd}, \end{cases}$$
$$x_i = y_i + z_i, \qquad \bar{x}_i = \bar{y}_i + z_i \quad (1 \leqslant i \leqslant n-1),$$
$$\bar{x}_n = \begin{cases} \bar{y}_n + z_n & \text{if } z_n \text{ is even}, \\ \bar{y}_n + z_n - 1 & \text{if } z_n \text{ is odd}, \end{cases}$$

and the map $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$ by $\psi(x_1, \dots, x_n|x_0|\bar{x}_n, \dots, \bar{x}_1) = C$, where

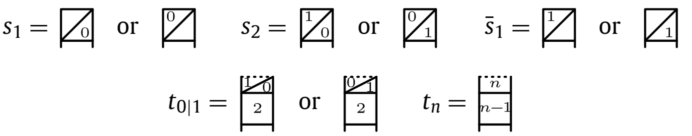
$$u_0 = l - x_0 - \sum_{i=1}^{n}(x_i + \bar{x}_i), \qquad z_n = 2\min\{x_n, \bar{x}_n\} + x_0,$$
$$y_i = \max\{0, x_i - \bar{x}_i\}, \qquad \bar{y}_i = \max\{0, \bar{x}_i - x_i\} \quad (1 \leqslant i \leqslant n),$$
$$z_i = \min\{x_i, \bar{x}_i\} \quad (1 \leqslant i \leqslant n-1).$$

(4) $\boldsymbol{A_{2n-1}^{(2)}}$ **case:** The types of layers are as follows.

- s_1: 1-slot with a 0-block,

- $\bar{s}_1$: 0-slot with a 1-block,
- s_i: supporting i-slot ($2 \leqslant i \leqslant n$),
- $\bar{s}_i$: covering i-block ($2 \leqslant i \leqslant n$),
- t_i: half of a supporting i-block ($2 \leqslant i \leqslant n-1$),
- $\bar{t}_i$: half of a covering i-block ($2 \leqslant i \leqslant n-1$),
- t_n: half of an n-block,
- $t_{0|1}$: half of a 0|1-block.

For example, we have



For any $C \in \mathcal{C}^{(l)}$, we write
- u_0: the number of layers of type $t_{0|1}$,
- y_i: the number of layers of type s_i ($1 \leqslant i \leqslant n$),
- $\bar{y}_i$: the number of layers of type $\bar{s}_i$ ($1 \leqslant i \leqslant n$),
- z_i: the number of layers of type t_i ($2 \leqslant i \leqslant n-1$) (= the number of layers of type $\bar{t}_i$),
- z_n: the number of layers of type t_n.

They have the properties $u_0, z_n \in 2\mathbf{Z}_{\geqslant 0}$, $y_i \bar{y}_i = 0$, and $u_0 + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=2}^{n-1} z_i + z_n = l$. We define the map $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ by

$$\phi(C) = (x_1, \ldots, x_n | \bar{x}_n, \ldots, \bar{x}_1),$$

where $x_1 = y_1 + \frac{1}{2}u_0$, $\bar{x}_1 = \bar{y}_1 + \frac{1}{2}u_0$, $x_n = y_n + \frac{1}{2}z_n$, $\bar{x}_n = \bar{y}_n + \frac{1}{2}z_n$ $x_i = y_i + z_i$, $\bar{x}_i = \bar{y}_i + z_i$ ($2 \leqslant i \leqslant n-1$), and the map $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$ by $\psi(x_1, \ldots, x_n | \bar{x}_n, \ldots, \bar{x}_1) = C$, where

$$u_0 = 2\min\{x_1, \bar{x}_1\}, \qquad z_n = 2\min\{x_n, \bar{x}_n\},$$

$$y_i = \max\{0, x_i - \bar{x}_i\}, \qquad \bar{y}_i = \max\{0, \bar{x}_i - x_i\} \quad (1 \leqslant i \leqslant n),$$

$$z_i = \min\{x_i, \bar{x}_i\} \quad (2 \leqslant i \leqslant n-1).$$

(5) $\boldsymbol{D_n^{(1)}}$ **case:** The types of layers are as follows.
- s_1: 1-slot with a 0-block,
- $\bar{s}_1$: 0-slot with a 1-block,
- s_i: supporting i-slot ($2 \leqslant i \leqslant n-2$),
- $\bar{s}_i$: covering i-block ($2 \leqslant i \leqslant n-2$),
- s_{n-1}: $(n-1)$ and n-slots,
- $\bar{s}_{n-1}$: $(n-1)$ and n-blocks,
- s_n: n-slot with an $(n-1)$-block,
- $\bar{s}_n$: $(n-1)$-slot with an n-block,
- t_i: half of a supporting i-block ($2 \leqslant i \leqslant n-2$),
- $\bar{t}_i$: half of a covering i-block ($2 \leqslant i \leqslant n-2$),
- $t_{0|1}$: half of a 0|1-block,
- $t_{(n-1)|n}$: half of a $(n-1)|n$-block.

For any $C \in \mathcal{C}^{(l)}$, we denote
- u_0: the number of layers of type $t_{0|1}$,
- w_0: the number of layers of type $t_{(n-1)|n}$,
- y_i: the number of layers of type s_i ($1 \leqslant i \leqslant n$),
- $\bar{y}_i$: the number of layers of type $\bar{s}_i$ ($1 \leqslant i \leqslant n$),

- z_i: the number of layers of type t_i $(2 \leqslant i \leqslant n-2)$ (= the number of layers of type $\bar{t}_i$).

Then $u_0, w_0 \in 2\mathbf{Z}_{\geqslant 0}$, $y_i \bar{y}_i = 0$, and $u_0 + w_0 + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=2}^{n-2} z_i = l$. We define the map $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ by

$$\phi(C) = (x_1, \dots, x_n | \bar{x}_n, \dots, \bar{x}_1),$$

where $x_1 = y_1 + \frac{1}{2}u_0$, $\bar{x}_1 = \bar{y}_1 + \frac{1}{2}u_0$, $x_n = y_n$, $\bar{x}_n = \bar{y}_n$, $x_i = y_i + z_i$, $\bar{x}_i = \bar{y}_i + z_i$ $(2 \leqslant i \leqslant n-2)$, $x_{n-1} = y_{n-1} + \frac{1}{2}w_0$, $\bar{x}_{n-1} = \bar{y}_{n-1} + \frac{1}{2}w_0$, and the map $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$ by $\psi(x_1, \dots, x_n | \bar{x}_n, \dots, \bar{x}_1) = C$, where

$$u_0 = 2\min\{x_1, \bar{x}_1\}, \qquad w_0 = 2\min\{x_{n-1}, \bar{x}_{n-1}\},$$

$$y_i = \max\{0, x_i - \bar{x}_i\}, \qquad \bar{y}_i = \max\{0, \bar{x}_i - x_i\} \quad (1 \leqslant i \leqslant n-1),$$

$$y_n = x_n, \qquad \bar{y}_n = \bar{x}_n, \qquad z_i = \min\{x_i, \bar{x}_i\} \quad (2 \leqslant i \leqslant n-2).$$

(6) $\boldsymbol{B_n^{(1)}}$ **case:** The types of layers are as follows.

- s_1: 1-slot with a 0-block,
- $\bar{s}_1$: 0-slot with a 1-block,
- s_i: supporting i-slot $(2 \leqslant i \leqslant n)$,
- $\bar{s}_i$: covering i-block $(2 \leqslant i \leqslant n)$,
- t_i: half of a supporting i-block $(2 \leqslant i \leqslant n-1)$,
- $\bar{t}_i$: half of a covering i-block $(2 \leqslant i \leqslant n-1)$,
- t_n: supporting n-block (= covering n-slot),
- $t_{0|1}$: half of a $0|1$-block.

For any $C \in \mathcal{C}^{(l)}$, we write

- u_0: the number of layers of type $t_{0|1}$ in C,
- z_n: the number of layers of type t_n in C,
- y_i: the number of layers of type s_i in C $(1 \leqslant i \leqslant n)$,
- $\bar{y}_i$: the number of layers of type $\bar{s}_i$ in C $(1 \leqslant i \leqslant n)$,
- z_i: the number of layers of type t_i in C $(2 \leqslant i \leqslant n-1)$ (= the number of layers of type $\bar{t}_i$ in C).

Then $u_0 \in 2\mathbf{Z}_{\geqslant 0}$, $y_i \bar{y}_i = 0$, and $u_0 + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=2}^{n-1} z_i + z_n = l$. We define the map $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ by

$$\phi(C) = (x_1, \dots, x_n | x_0 | \bar{x}_n, \dots, \bar{x}_1),$$

where $x_1 = y_1 + \frac{1}{2}u_0$, $\bar{x}_1 = \bar{y}_1 + \frac{1}{2}u_0$, $x_i = y_i + z_i$, $\bar{x}_i = \bar{y}_i + z_i$ $(2 \leqslant i \leqslant n-1)$, $x_n = y_n + 2[\frac{z_n}{2}]$, $\bar{x}_n = \bar{y}_n + 2[\frac{z_n}{2}]$, $x_0 = 0$ if z_n is even, $x_0 = 1$ if z_n is odd, and the map $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$ by $\psi(x_1, \dots, x_n | x_0 | \bar{x}_n, \dots, \bar{x}_1) = C$, where

$$u_0 = 2\min\{x_1, \bar{x}_1\}, \qquad z_n = 2\min\{x_n, \bar{x}_n\} + x_0,$$

$$y_i = \max\{0, x_i - \bar{x}_i\}, \qquad \bar{y}_i = \max\{0, \bar{x}_i - x_i\} \quad (1 \leqslant i \leqslant n),$$

$$z_i = \min\{x_i, \bar{x}_i\} \quad (2 \leqslant i \leqslant n-1).$$

(7) $\boldsymbol{C_n^{(1)}}$ **case:** The types of layers are as follows.

- s_i: supporting i-slot,
- $\bar{s}_i$: covering i-block $(1 \leqslant i \leqslant n)$,
- t_i: half of a supporting i-block $(1 \leqslant i \leqslant n-1)$,
- $\bar{t}_i$: half of a covering i-block $(1 \leqslant i \leqslant n-1)$,
- t_0: half of a 0-block,
- t_n: half of a n-block.

For any $C \in \mathcal{C}^{(l)}$, we write

- z_0: the number of layers of type t_0,
- z_n: the number of layers of type t_n,
- y_i: the number of layers of type s_i $(1 \leqslant i \leqslant n)$,
- $\bar{y}_i$: the number of layers of type $\bar{s}_i$ $(1 \leqslant i \leqslant n)$,
- z_i: the number of layers of type t_i $(1 \leqslant i \leqslant n-1)$ (= the number of layers of type $\bar{t}_i$).

Then $z_0, z_n \in 2\mathbf{Z}_{\geqslant 0}$, $y_i \bar{y}_i = 0$, and $z_0 + z_n + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=1}^{n-1} z_i = 2l$. We define the map $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ by

$$\phi(C) = (x_1, \ldots, x_n | \bar{x}_n, \ldots, \bar{x}_1),$$

where $x_i = y_i + z_i$, $\bar{x}_i = \bar{y}_i + z_i$ $(1 \leqslant i \leqslant n-1)$, $x_n = y_n + \frac{1}{2} z_n$, $\bar{x}_n = \bar{y}_n + \frac{1}{2} z_n$, and the map $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$ by $\psi(x_1, \ldots, x_n | \bar{x}_n, \ldots, \bar{x}_1) = C$, where

$$z_0 = 2l - \sum_{i=1}^{n}(x_1 + \bar{x}_i), \qquad z_n = 2\min\{x_n, \bar{x}_n\},$$

$$y_i = \max\{0, x_i - \bar{x}_i\}, \qquad \bar{y}_i = \max\{0, \bar{x}_i - x_i\} \quad (1 \leqslant i \leqslant n),$$

$$z_i = \min\{x_i, \bar{x}_i\} \quad (1 \leqslant i \leqslant n-1).$$

It is now easy to verify that ϕ and ψ are inverse to each other. Also it is easy to see that the maps $\overline{\mathrm{wt}}$, φ_i, and ε_i are preserved under ϕ and that the action of the Kashiwara operators commute with ϕ.

For instance, let us consider the $B_n^{(1)}$ type and $2 \leqslant i \leqslant n-1$, that is, the i-block is a unit cube. We denote the element C of $\mathcal{C}^{(l)}$, with layers as in the above discussion by an ordered tuple $\lceil u_0 | y_1, \ldots, y_n, \bar{y}_n, \ldots, \bar{y}_1 | z_2, \ldots, z_n \rfloor$. Depending on which of the rules (F1), (F2), or (F3) for the actions of Kashiwara operators is applicable, the following situations arise.

(F1): We have $y_{i+1}(=0) < \bar{y}_{i+1}$ and $y_i > 0$, and

$$\begin{aligned}\tilde{f}_i C = \lceil u_0 | y_1, \ldots, y_{i-1}, y_i - 1, y_{i+1}, \ldots, \bar{y}_{i+2}, \bar{y}_{i+1} - 1, \bar{y}_i, \ldots, \bar{y}_1 \\ | z_2, \ldots, z_{i-1}, z_i + 1, z_{i+1}, \ldots, z_n \rfloor.\end{aligned}$$

(F2): One of the following is true.

- We have $y_{i+1} \geqslant \bar{y}_{i+1}(=0)$ and $y_i > 0$, and

$$\tilde{f}_i C = \lceil u_0 | y_1, \ldots, y_{i-1}, y_i - 1, y_{i+1} + 1, y_{i+2}, \ldots, \bar{y}_1 | z_2, \ldots, z_n \rfloor.$$

- We have $y_{i+1}(=0) < \bar{y}_{i+1}$ and $y_i = 0$, and

$$\tilde{f}_i C = \lceil u_0 | y_1, \ldots, \bar{y}_{i+2}, \bar{y}_{i+1} - 1, \bar{y}_i + 1, \bar{y}_{i-1}, \ldots, \bar{y}_1 | z_2, \ldots, z_n \rfloor.$$

(F3): We have $y_{i+1} \geqslant \bar{y}_{i+1}(=0)$ and $y_i = 0$, and

$$\begin{aligned}\tilde{f}_i C = \lceil u_0 | y_1, \ldots, y_i, y_{i+1} + 1, y_{i+2}, \ldots, \bar{y}_{i+1}, \bar{y}_i + 1, \bar{y}_{i-1}, \ldots, \bar{y}_1 \\ | z_2, \ldots, z_{i-1}, z_i - 1, z_{i+1}, \ldots, z_n \rfloor.\end{aligned}$$

Recall that $\phi(C) = (x_1, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_1)$, where $x_1 = y_1 + \frac{1}{2} u_0$, $\bar{x}_1 = \bar{y}_1 + \frac{1}{2} u_0$, $x_i = y_i + z_i$, $\bar{x}_i = \bar{y}_i + z_i$ $(2 \leqslant i \leqslant n-1)$, $x_n = y_n + 2[\frac{z_n}{2}]$, $\bar{x}_n = \bar{y}_n + 2[\frac{z_n}{2}]$, $x_0 = 0$ if z_n is even, and $x_0 = 1$ if z_n is odd. Now,

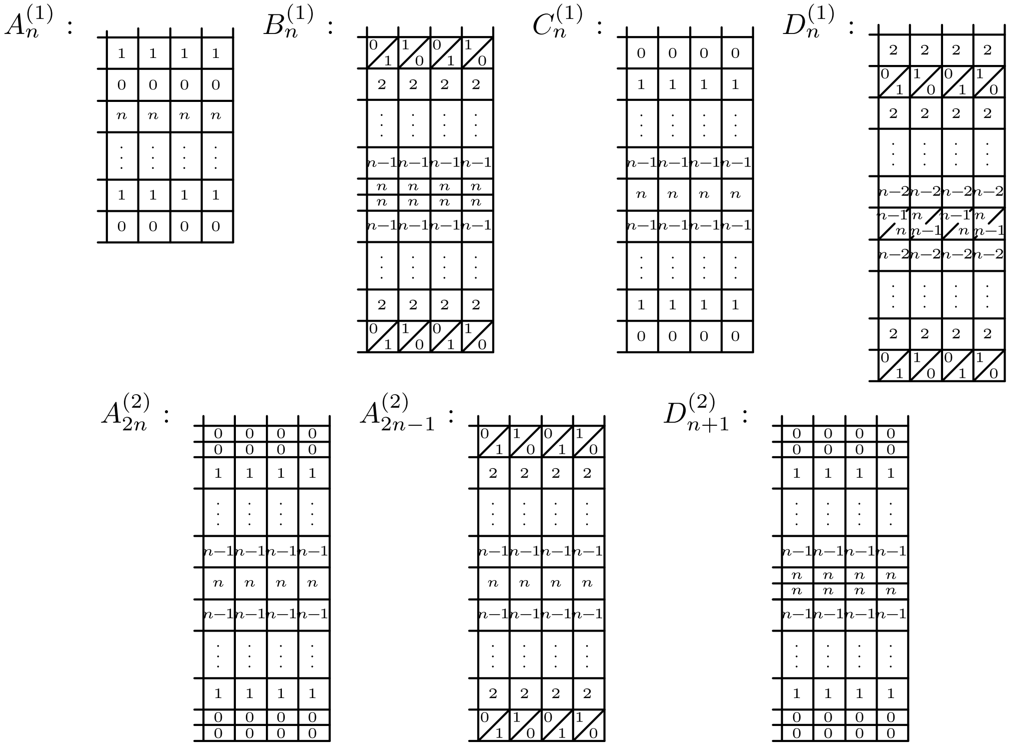


Fig. 3. Patterns for building Young walls.

- when $y_{i+1}(=0) < \bar{y}_{i+1}$,

$$\phi(\tilde{f}_i C) = (x_1, \dots, x_n | x_0 | \bar{x}_n, \dots, \bar{x}_{i+2}, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \bar{x}_{i-1}, \dots, \bar{x}_1),$$

- and when $y_{i+1} \geqslant \bar{y}_{i+1}(=0)$,

$$\phi(\tilde{f}_i C) = (x_1, \dots, x_{i-1}, x_i - 1, x_{i+1} + 1, x_{i+2}, \dots, x_n | x_0 | \bar{x}_n, \dots, \bar{x}_1),$$

which is equal to $\tilde{f}_i(\phi(C))$, as desired.

To summarize, we obtained a new description of level-l perfect crystals $\mathcal{B}^{(l)}$ as the set of equivalence classes of level-l slices.

Theorem 4.1. *For all classical quantum affine algebras, there is a crystal isomorphism $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ given by above formulas.*

5. Higher level Young walls and crystal $\mathcal{B}(\lambda)$

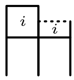
In this section, we will define the notion of (arbitrary level) proper Young walls, reduced Young walls, ground-state walls, etc., and give a realization of arbitrary level irreducible highest weight crystals in terms of reduced Young walls. The patterns for building Young walls are given in Fig. 3.

Definition 5.1. A ***level-l Young wall*** $Y = (Y(k))_{k=0}^{\infty}$ is a concatenation of level-l slices $Y(k)$, extending infinitely to the left, satisfying the following conditions.

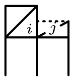
(1) It is concatenated following the pattern given in Fig. 3.
(2) At each ith layer of Y, that is, the concatenation of ith layers of the slices $Y(k)$, there is no free space to the right of any block (or broken half-block).

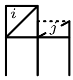
Remark 5.2.

(1) In most cases, it is easy to judge whether there is a free space to the right of a given block (or broken half-block). In addition, the following nontrivial cases will be considered as having a free space to the right of a given block (or broken half-block).
- The left is a whole block and the right is a broken half of a whole i-block.

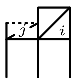


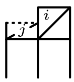
- The left is a single block of half-unit depth and the right is the *upper* broken half of a whole j-block.



or


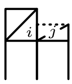
Here, $i = 0, 1$ and $j = 0|1$ or $i = n-1, n$ and $j = (n-1)|n$. Note that the color of the broken half on the right will depend on i.
- The right is a single block of half-unit depth and the left is the *lower* broken half of a whole j-block.

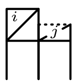


or


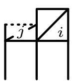
Here, $i = 0, 1$ and $j = 0|1$ or $i = n-1, n$ and $j = (n-1)|n$.

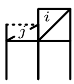
(2) However, the following cases will be considered as having no free space to the right of a given block (or broken half-block).



and


with $i = 0, 1$ and $j = 0|1$ or $i = n–1, n$ and $j = (n-1)|n$ and where the right is a lower half.



and


with $i = 0, 1$ and $j = 0|1$ or $i = n-1, n$ and $j = (n-1)|n$ and where the left is an upper half.

Definition 5.3.

(1) A ***full column*** is a layer of a level-l slice whose height is an integer and whose top is of unit depth.
(2) In all classical cases excluding $A_n^{(1)}$, a level-l Young wall Y is said to be ***proper*** if for each layer of Y, none of the full columns have the same height. In the $A_n^{(1)}$ case, every level-l Young wall is said to be ***proper***.

(3) A column in a level-l proper Young wall is said to contain a ***removable*** δ if one may remove a δ-set from that column and still obtain a proper Young wall.
(4) A level-l proper Young wall is said to be ***reduced*** if none of its columns contain a removable δ.

Let $\lambda = \sum_{i\in I} a_i \Lambda_i$ be a dominant integral weight of level-l so that

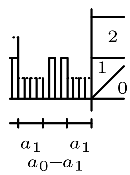
$$\begin{cases} l = a_0 + a_1 + \cdots + a_n & A_n^{(1)}, C_n^{(1)} \text{ cases},\\ l = a_0 + a_1 + 2(a_2 + \cdots + a_{n-1}) + a_n & B_n^{(1)} \text{ case},\\ l = a_0 + a_1 + 2(a_2 + \cdots + a_{n-2}) + a_{n-1} + a_n & D_n^{(1)} \text{ case},\\ l = a_0 + 2(a_1 + \cdots + a_n) & A_{2n}^{(2)} \text{ case},\\ l = a_0 + a_1 + 2(a_2 + \cdots + a_n) & A_{2n-1}^{(2)} \text{ case},\\ l = a_0 + 2(a_1 + \cdots + a_{n-1}) + a_n & D_{n+1}^{(2)} \text{ case}. \end{cases} \tag{5.1}$$

We would like to define the ***ground-state wall*** Y_λ ***of weight*** λ. It will be constructed as a level-l reduced Young wall. In Fig. 4, we have drawn a part of Y_λ. In the $A_{2n-1}^{(2)}$ and $B_n^{(1)}$ cases (or $D_n^{(1)}$ case) depending on whether $a_0 \geqslant a_1$ or $a_0 \leqslant a_1$ (resp. $a_0 \geqslant a_1$ or $a_0 \leqslant a_1$, and $a_{n-1} \geqslant a_n$ or $a_{n-1} \leqslant a_n$), there are two (resp. four) different forms of the ground-state wall. We will just draw the case when $a_0 \leqslant a_1$ (resp. $a_0 \leqslant a_1$ and $a_{n-1} \geqslant a_n$).

Recall that a Young wall is a concatenation of slices. We have drawn the left-side-views of the first two slices that make up the Young wall. The actual ground-state wall should extend infinitely to the left, repeating the same pattern. At the right end, we have drawn the pattern for stacking the blocks, so as to show the color of the blocks placed at each height.

For $A_{2n-1}^{(2)}$ and $B_n^{(1)}$ (or $D_n^{(1)}$) cases, the pattern on the right is just for the even ith columns. As given by the figures at the beginning of this section, the odd columns will be stacked in a pattern with 0, 1 (resp. 0, 1, and $n-1$, n) exchanged. Since the outlines of the first two slices given above (actually, all slices) are exactly the same, this means that the even and odd columns are identical except for the exchange of 0 with 1 (resp. 0 with 1, and $n-1$ with n). Below the 0th slice, we have written down how many layers of each shape should be used.

When $a_0 \geqslant a_1$,



should appear in the first few layers of the 0th column. Notice that in addition to the exchange of a_0 with a_1 from the full diagram above, the position of half-unit depth blocks appearing in the middle has shifted so that they are now 1-blocks instead of the 0-blocks used in the full diagram.

A level-l proper Young wall obtained by adding finitely many blocks to the ground-state wall Y_λ is said to have been ***built on*** Y_λ. We denote by $\mathcal{F}(\lambda)$ (and $\mathcal{Y}(\lambda)$) the set of all proper (resp. reduced) Young walls built on Y_λ.

Let Y be a level-l proper Young wall built on Y_λ and let C be a column of Y. Recall that a column C is a level-l slice and that for each $i \in I$, $\varphi_i(C)$ (resp. $\varepsilon_i(C)$) is the largest integer $k \geqslant 0$ such that $\tilde{f}_i^k(C) \neq 0$ (resp. $\tilde{e}_i^k(C) \neq 0$).

We now define the action of Kashiwara operators $\tilde{f}_i$, $\tilde{e}_i$ $(i \in I)$ on Y as follows.

(1) For each column C of Y, write $\varepsilon_i(C)$-many $-$'s followed by $\varphi_i(C)$-many $+$'s under C. This sequence is called the ***i-signature of*** C.

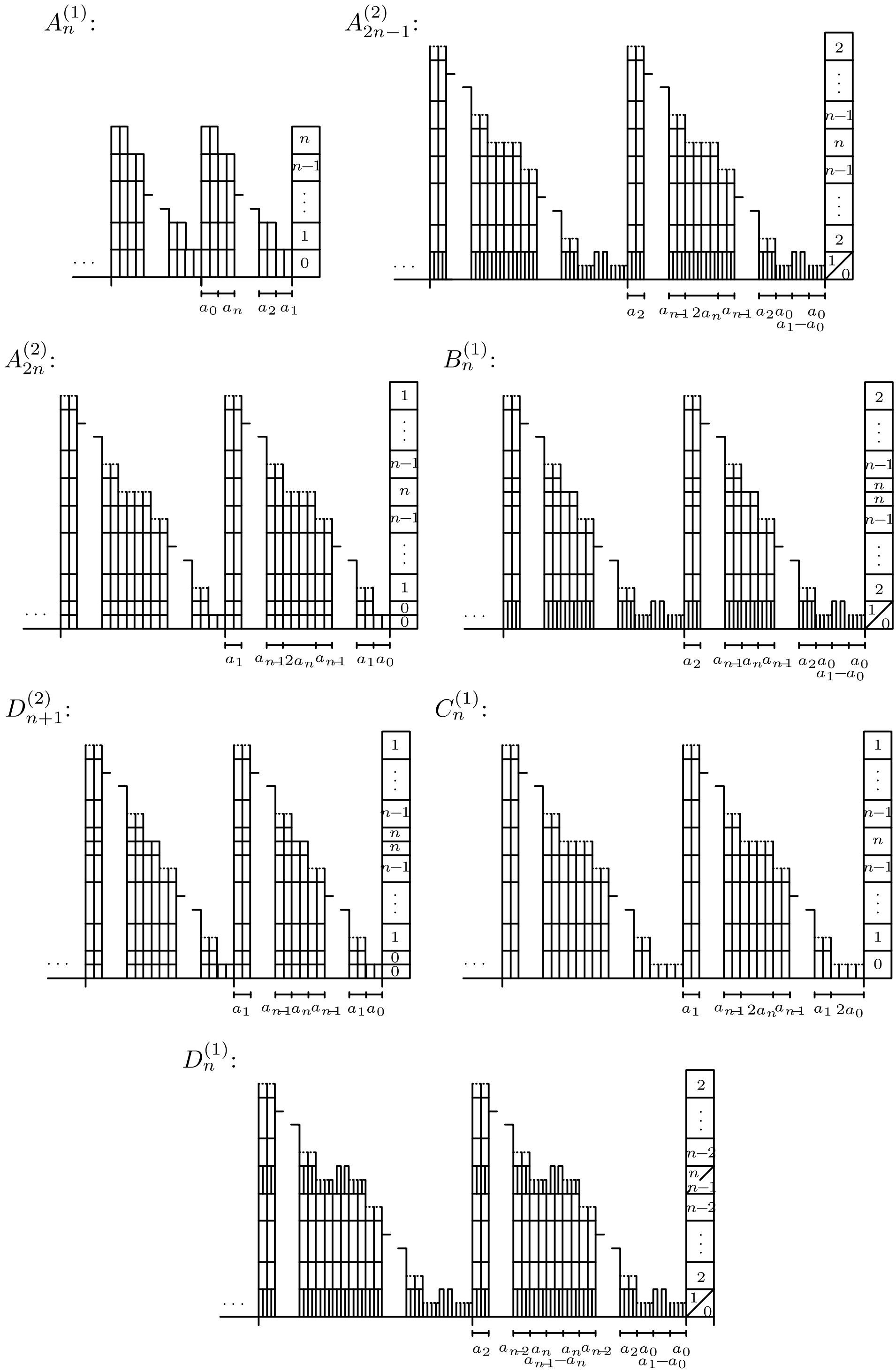


Fig. 4. Ground-state walls.

(2) From this sequence of $-$'s and $+$'s, we cancel out each $(+,-)$-pair to obtain a sequence of $-$'s followed by $+$'s (reading from left to right). This sequence is called the ***i*-signature of** Y.
(3) We define $\tilde{f}_i Y$ to be the proper Young wall obtained from Y by replacing the column C corresponding the leftmost $+$ in the i-signature of Y with the column $\tilde{f}_i C$.
(4) We define $\tilde{e}_i Y$ to be the proper Young wall obtained from Y by replacing the column C corresponding the rightmost $-$ in the i-signature of Y with the column $\tilde{e}_i C$.
(5) If there is no $+$ (or $-$) in the i-signature of Y, we define $\tilde{f}_i Y = 0$ (resp. $\tilde{e}_i Y = 0$).

We need to show that the action of Kashiwara operators on $\mathcal{F}(\lambda)$ is well defined. We will just deal with the $\tilde{f}_i$ operator.

Since a Young wall extends infinitely to the left, it is not immediately clear as to whether there exists a *leftmost* $+$. That is, it is not clear whether the number of zeros in the i-signature of Y is finite. Let us briefly comment on this here.

Since only finitely many blocks were added to the ground-state wall in building Y, the wall will eventually become identical to the ground-state wall at some point, as it proceeds to the left. Thus it suffices to check if the ground-state walls give finite signatures. This one may do easily with each of the explicit ground-state walls.

Now, we fix some notations. Denote by C the column corresponding to the leftmost $+$ in the i-signature of a proper Young wall Y. The column sitting to the right of column C will be denoted by C'.

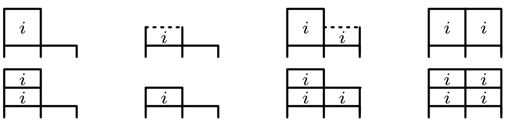
Suppose that $\tilde{f}_i Y$ is not a proper Young wall. In fact, it could be that $\tilde{f}_i Y$ is not even a Young wall. In such a case, the following statement would be true.

- There is a free space to the right of some block or half-block in some layer of the column of $\tilde{f}_i Y$, that corresponds to the column C of Y.

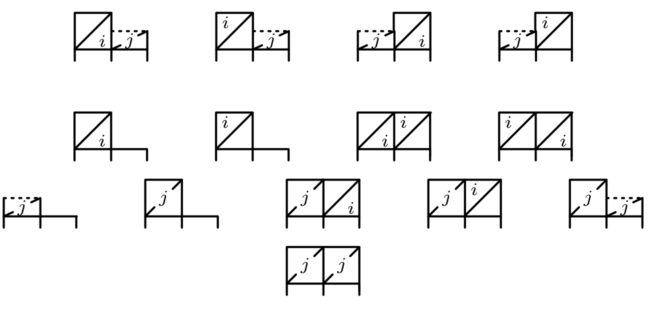
If the result is a Young wall, but just not proper, then the following statement would be true.

- The columns of $\tilde{f}_i Y$ corresponding to the columns C and C' of Y contain a layer in which the tops are of unit depth and of the same integer height.

For the index i corresponding to a unit cube or a half-height block, the following lists all possible nontrivial forms for $\tilde{f}_i Y$ that satisfy one of the above two statements.



For the indices corresponding to blocks of half-unit depth; that is, for $i = 0, 1$ or $n-1, n$, the following lists (almost) all possible forms for $\tilde{f}_i Y$ that satisfy one of the above two statements.



Here, $j = 0|1$ or $(n-1)|n$. As mentioned in Remark 5.2, the right (or left) of the first (resp. last) two diagrams in the first row is the upper (resp. lower) broken half of a whole j-block.

In each of these cases, it is possible to obtain one of the following three conclusions.

(1) There is a free space to the right of some block or half-block in some layer of the column C of Y.
(2) The columns C and C' of the proper Young wall Y contain a layer in which the tops are of unit depth and of the same integer height.
(3) $\varphi_i(C) \leqslant \varepsilon_i(C')$.

The first of these conclusions violates the assumption that we started out with a Young wall Y. The second conclusion is in violation of the properness of Y, in all algebra types excluding $A_n^{(1)}$. As for the third, since C is the column corresponding to the leftmost $+$ in the i-signature of Y we must have $\varphi_i(C) > \varepsilon_i(C')$. In the $A_n^{(1)}$ type, we can only arrive at either conclusion (1) or conclusion (3). Each of these conclusions brings us to a contradiction, and hence the resulting $\tilde{f}_i Y$ must have been a proper Young wall.

We define the maps $\mathrm{wt}: \mathcal{F}(\lambda) \to P$, $\varphi_i, \varepsilon_i : \mathcal{F}(\lambda) \to \mathbf{Z}$ by setting

$$\begin{aligned} \mathrm{wt}(Y) &= \lambda - \sum_{i=0}^{n} k_i \alpha_i, \\ \varphi_i(Y) &= \text{the number of } +\text{'s in the } i\text{-signature of } Y, \\ \varepsilon_i(Y) &= \text{the number of } -\text{'s in the } i\text{-signature of } Y, \end{aligned} \tag{5.2}$$

where k_i is the number of i-blocks that have been added to Y_λ.

Remark 5.4. We have seen that the i-signatures are always finite. Hence it makes sense to count the number of $+$'s and $-$'s in the signature.

Now it is straightforward to verify that the following theorem holds.

Theorem 5.5. *The set $\mathcal{F}(\lambda)$ of all level-l proper Young walls built on Y_λ, together with the maps $\tilde{e}_i$, $\tilde{f}_i$, ε_i, φ_i ($i \in I$), and* wt, *forms a $U_q(\mathfrak{g})$-crystal.*

Finally, we give a new realization of arbitrary level irreducible highest weight crystals $\mathcal{B}(\lambda)$ in terms of reduced Young walls.

First, recall the path realization of irreducible highest weight crystals for quantum affine algebras that use perfect crystals [5]. For any $\lambda \in \bar{P}^+$ of level-l, a ***λ-path*** in $\mathcal{B}^{(l)}$ is a sequence

$$\mathbf{p} = \bigotimes_{k=0}^{\infty} \mathbf{p}(k) = \cdots \otimes \mathbf{p}(k+1) \otimes \mathbf{p}(k) \otimes \cdots \otimes \mathbf{p}(1) \otimes \mathbf{p}(0)$$

in $\mathcal{B}^{(l)}$ such that $\mathbf{p}(k)$ is identical to the kth entry $\mathbf{p}_\lambda(k)$ of the ***ground-state path*** $\mathbf{p}_\lambda$ of weight λ, for all $k \gg 0$. The set $\mathcal{P}(\lambda) = \mathcal{P}(\lambda, \mathcal{B}^{(l)})$ of all λ-paths in $\mathcal{B}^{(l)}$ is given a crystal structure by the tensor product rule, and this gives the path realization of the irreducible highest weight crystal $\mathcal{B}(\lambda)$ [5]:

$$\mathcal{B}(\lambda) \xrightarrow{\sim} \mathcal{P}(\lambda), \qquad u_\lambda \mapsto \mathbf{p}_\lambda.$$

Now, it suffices to show that there is a crystal isomorphism $\Phi : \mathcal{Y}(\lambda) \xrightarrow{\sim} \mathcal{P}(\lambda)$.

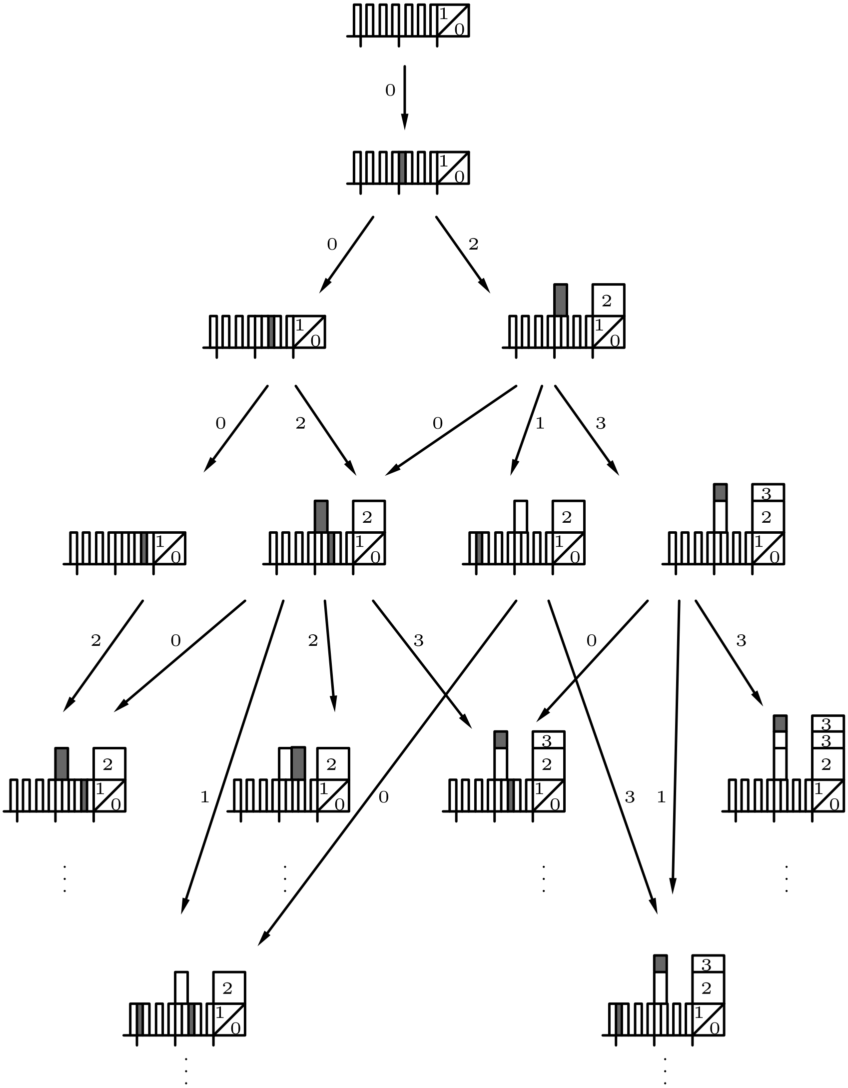


Fig. 5. The crystal $\mathcal{Y}(3\Lambda_0)$ for $B_3^{(1)}$.

We define the map $\Phi : \mathcal{Y}(\lambda) \to \mathcal{P}(\lambda)$ as follows. Given a reduced Young wall $Y = (Y(k))_{k=0}^{\infty}$ in $\mathcal{Y}(\lambda)$, consider the crystal isomorphism $\phi : \mathcal{C}^{(l)} \xrightarrow{\sim} \mathcal{B}^{(l)}$ given in Theorem 4.1, and define $\Phi(Y)$ to be

$$\Phi(Y) = \bigotimes_{k=0}^{\infty} \phi\big(Y(k)\big). \tag{5.3}$$

Then we have $\Phi(Y_\lambda) = \mathbf{p}_\lambda$.

Conversely, to each λ-path $\mathbf{p} = \bigotimes_{k=0}^{\infty} \mathbf{p}(k)$, by removing an appropriate number of δ's, one can easily see that there exists a unique reduced Young wall $Y = (Y(k))_{k=0}^{\infty}$ such that $\phi^{-1}(\mathbf{p}(k)) = Y(k)$ for all $k \geqslant 0$. Hence Φ is a bijection.

Moreover, by the same argument used in the proof of Theorem 6.1 in [16] for $D_n^{(1)}$ type, we can show that $\mathcal{Y}(\lambda)$ is a subcrystal of $\mathcal{F}(\lambda)$ and the map Φ commutes with the Kashiwara operators. Therefore we obtain our main result.

Theorem 5.6. *We have a $U_q(\mathfrak{g})$-crystal isomorphism*

$$\mathcal{Y}(\lambda) \xrightarrow{\sim} \mathcal{P}(\lambda) \xrightarrow{\sim} \mathcal{B}(\lambda). \tag{5.4}$$

In Fig. 5, we illustrate the top part of the affine crystal $\mathcal{Y}(3\Lambda_0)$ for $B_3^{(1)}$. The shaded part denotes the change through the action of $\tilde{f}_i$ $(i \in I)$.

References

[1] J. Hong, S.-J. Kang, Introduction to Quantum Groups and Crystal Bases, Grad. Stud. Math., vol. 42, Amer. Math. Soc., Providence, RI, 2002.
[2] J. Hong, S.-J. Kang, H. Lee, Young wall realization of crystal graphs for $U_q(C_n^{(1)})$, Comm. Math. Phys. 244 (1) (2004) 111–131.
[3] M. Jimbo, K.C. Misra, T. Miwa, M. Okado, Combinatorics of representations of $U_q(\widehat{\mathfrak{sl}}(n))$ at $q=0$, Comm. Math. Phys. 136 (3) (1991) 543–566.
[4] S.-J. Kang, Crystal bases for quantum affine algebras and combinatorics of Young walls, Proc. London Math. Soc. (3) 86 (2003) 29–69.
[5] S.-J. Kang, M. Kashiwara, K.C. Misra, T. Miwa, T. Nakashima, A. Nakayashiki, Affine crystals and vertex models, Internat. J. Modern Phys. A 1A (1992) 449–484.
[6] S.-J. Kang, M. Kashiwara, K.C. Misra, T. Miwa, T. Nakashima, A. Nakayashiki, Perfect crystals of quantum affine Lie algebras, Duke Math. J. 68 (3) (1992) 499–607.
[7] S.-J. Kang, J.-A. Kim, H. Lee, D.-U. Shin, Young wall realization of crystal bases for classical Lie algebras, Trans. Amer. Math. Soc. 356 (6) (2004) 2349–2378.
[8] S.-J. Kang, J.-H. Kwon, Quantum affine algebras, combinatorics of Young walls, and global bases, Electron. Res. Announc. Amer. Math. Soc. 8 (2002) 35–46 (electronic).
[9] S.-J. Kang, J.-H. Kwon, Fock space representations of quantum affine algebras and generalized Lascoux–Leclerc–Thibon algorithm, J. Korean Math. Soc. 45 (4) (2008) 1135–1202.
[10] S.-J. Kang, H. Lee, Higher level affine crystals and Young Walls, Algebr. Represent. Theory 9 (6) (2006) 593–632.
[11] M. Kashiwara, Crystalizing the q-analogue of universal enveloping algebras, Comm. Math. Phys. 133 (1) (1990) 249–260.
[12] M. Kashiwara, On crystal bases of the Q-analogue of universal enveloping algebras, Duke Math. J. 63 (2) (1991) 465–516.
[13] M. Kashiwara, T. Nakashima, Crystal graphs for representations of the q-analogue of classical Lie algebras, J. Algebra 165 (2) (1994) 295–345.
[14] A. Kuniba, K.C. Misra, M. Okado, T. Takagi, J. Uchiyama, Crystals for Demazure modules of classical affine Lie algebras, J. Algebra 208 (1) (1998) 185–215.
[15] A. Lascoux, B. Leclerc, J.-Y. Thibon, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, Comm. Math. Phys. 181 (1) (1996) 205–263.
[16] H. Lee, Crystals of type $D_n^{(1)}$ and Young walls, Trans. Amer. Math. Soc. 358 (4) (2006) 1847–1867.
[17] H. Lee, Crystal $\mathcal{B}(\infty)$ for quantum classical affine algebras; A Young wall description, A presentation given at the workshop "Combinatorics, Representation Theory and Related Topics", RIMS, Kyoto University, Oct. 2006, preprint.
[18] G. Lusztig, Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. 3 (2) (1990) 447–498.
[19] K. Misra, T. Miwa, Crystal base for the basic representation of $U_q(\widehat{\mathfrak{sl}}(n))$, Comm. Math. Phys. 134 (1) (1990) 79–88.