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JOURNAL OF
Algebra

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BMW algebras of simply laced type

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Received 5 April 2004

Communicated by Michel Broué

Abstract

It is known that the recently discovered representations of the Artin groups of type A_n, the braid groups, can be constructed via BMW algebras. We introduce similar algebras of type D_n and E_n which also lead to the newly found faithful representations of the Artin groups of the corresponding types. We establish finite dimensionality of these algebras. Moreover, they have ideals I_1 and I_2 with $I_2 \subset I_1$ such that the quotient with respect to I_1 is the Hecke algebra and I_1/I_2 is a module for the corresponding Artin group generalizing the Lawrence–Krammer representation. Finally we give conjectures on the structure, the dimension and parabolic subalgebras of the BMW algebra, as well as on a generalization of deformations to Brauer algebras for simply laced spherical type other than A_n.

1. Introduction

In [7], representations were given for the Artin groups of spherical type which are faithful, following the construction of Krammer for braid groups [13]. (We note that [1] also contains a proof of the faithfulness of this representation for type A_n, and that [9] also

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.12.011

generalizes this result to arbitrary spherical types.) Faithful representations for the Artin groups of type A_n, D_n, and E_m for $m = 6, 7, 8$ were explicitly constructed. Since each Artin group of spherical irreducible type embeds into at least one of these, this shows each is linear. As the representations for type A_n occur in earlier work of Lawrence [14], they are called Lawrence–Krammer representations.

Zinno [18] observed that the Lawrence–Krammer representation of the Artin group of type A_n, the braid groups on $n+1$ braids, factors through the BMW algebra, the Birman–Murakami–Wenzl algebra introduced in [2,15].

In this paper we introduce algebras similar to the BMW algebra for other types. We associate a unique algebra with each simply laced Coxeter diagram M of rank n. Here, simply laced means that M has no multiple bonds. We define the algebras by means of $2n$ generators and five kinds of relations. For each node i of the diagram M we define two generators g_i and e_i with $i = 1, \ldots, n$. If two nodes are connected in the diagram we write $i \sim j$, with i, j the indices of the two nodes, and if they are not connected we write $i \not\sim j$. We let l, x be two indeterminates.

Definition 1. Let M be a simply laced Coxeter diagram of rank n. The BMW algebra of type M is the algebra, denoted by $B(M)$ or just B, with identity element, over $\mathbb{Q}(l, x)$, whose presentation is given on generators g_i and e_i $(i = 1, \ldots, n)$ by the following defining relations:

$$g_i g_j = g_j g_i \quad \text{when } i \not\sim j, \tag{B1}$$

$$g_i g_j g_i = g_j g_i g_j \quad \text{when } i \sim j, \tag{B2}$$

$$m e_i = l\bigl(g_i^2 + m g_i - 1\bigr) \quad \text{for all } i, \tag{D1}$$

$$g_i e_i = l^{-1} e_i \quad \text{for all } i, \tag{R1}$$

$$e_i g_j e_i = l e_i \quad \text{when } i \sim j, \tag{R2}$$

where $m = (l - l^{-1})/(1 - x)$.

The first two relations are the braid relations commonly associated with the Coxeter diagram M. Just as for Artin and Coxeter groups, if M is the disjoint union of two diagrams M_1 and M_2, then B is the direct sum of the two BMW algebras $B(M_1)$ and $B(M_2)$. For the solution of many problems concerning B, this gives an easy reduction to the case of connected diagrams M.

In (D1) the generators e_i are expressed in terms of the g_i and so B is in fact already generated by $g_1, \ldots, g_n$. We shall show below that the g_i are invertible elements in B, so that there is a group homomorphism from the Artin group A of type M to the group $B^\times$ of invertible elements of B sending the ith generator s_i of A to g_i. As we shall see at the end of Section 6, the Lawrence–Krammer representation is a constituent of the regular representation of B. This generalizes Zinno's result [18]. As a consequence of [7], the homomorphism $A \to B^\times$ is injective.

The fact that the BMW algebras of type A_n coincide with those defined by Birman and Wenzl [2] and Murakami [15] is given in Theorem 2.7.

The Lawrence–Krammer representation of the Artin groups is based on two parameters, in [7] denoted by t and r. The two parameters m and l here are related by $m = r - r^{-1}$ and $l = 1/(tr^3)$.

Our first major result is as follows.

Theorem 1.1. *The BMW algebras of simply laced spherical type are finite dimensional.*

The proof is at the end of Section 2. Some information and conjectures about dimensions appear in Section 7.

Let I_1 be the ideal of B generated by all e_i, and let I_2 be the ideal generated by all products $e_i e_j$ for i and j distinct and not connected in M. Then clearly $I_2 \subseteq I_1$. Moreover, it is immediate from the defining relations of B that B/I_1 is the Hecke algebra of type M. The main result of this paper concerns the structure of I_1/I_2.

Let (W, R) be the Coxeter system of type M. We write Φ^+ for the set of positive roots of the Coxeter system of type M. By α_0 we denote its highest root, and by C the set of nodes j in M with $(\alpha_j, \alpha_0) = 0$. In case A_n the type of C is A_{n-2}; in case D_n, it is $\mathrm{A}_1 \times \mathrm{D}_{n-2}$, in case E_n it is A_5, D_6, and E_7 for $n = 6, 7, 8$, respectively. If X is a set of nodes of M, we denote by W_X the parabolic subgroup of W corresponding to X. This means that W_X is the subgroup of W generated by all r_j for $j \in X$.

Theorem 1.2. *Let B be the BMW algebra of type A_n $(n \geqslant 1)$, D_n $(n \geqslant 4)$, or E_n $(n = 6, 7, 8)$. Then B/I_2 is semi-simple over $\mathbb{Q}(l, x)$. Let Z_0 be the Hecke algebra of type C. For each irreducible representation θ of Z_0, there is a corresponding representation Γ_θ of B of dimension $|\Phi^+| \dim(\theta)$ and, up to equivalence, these are the irreducible representations of B occurring in I_1/I_2. In particular, the dimension of I_1/I_2 as a vector space over $\mathbb{Q}(l, x)$ equals $|\Phi^+|^2\, |W_C|$.*

The proof of the theorem consists of two major parts. In Section 5, we provide, for each node i of M, a linear spanning set for I_1/I_2 parametrized by triples consisting of two positive roots and an element of W_C. This shows that $|\Phi^+|^2 |W_C|$ is an upper bound for the dimension of $\dim(I_1/I_2)$. The proof that the same number is a lower bound takes place in Section 6, where the Lawrence–Krammer representation of A, studied in [7], is generalized to a representation of the same dimension as before, viz. $|\Phi^+|$, but now over the non-commutative ring of scalars Z_0. Up to a field extension of the scalars, Z_0 is well known to be isomorphic to the group algebra of W_C, so $\dim(Z_0) = |W_C|$.

In the final section, we discuss how the results might carry over to I_2 and for I_r with $r \geqslant 3$. We give a conjecture for the dimension of the BMW algebras of types D_n $(n \geqslant 4)$ and E_n $(n = 6, 7, 8)$. In the theory of Coxeter groups and Artin groups, there is a notion of standard parabolic subgroups. These are subgroups generated by a subset J of the nodes of M and have the special property that they are Coxeter, respectively, Artin groups of type $M|_J$. We expect that, at least for spherical M, the subalgebra of B generated by the g_j for $j \in J$ will be isomorphic to the BMW algebra of type $M|_J$. For type A_n, the Brauer algebra, cf. [4], is obtained as a deformation of the BMW algebra. We conjecture that a similar deformation exists for the spherical simply laced types, in which the 'pictures', forming the monomial basis of the Brauer algebra, are indexed by a combinatorial generalization of the abovementioned triples. As a consequence of Theorem 1.2, these conjectures hold for the quotient algebra B/I_2. We also discuss possible extensions to other spherical types.

The properties of Artin groups needed for the study of our algebras, are mentioned in Section 3. The subsequent section contains a discussion of ideals. We begin however by studying direct consequences of the defining relations.

2. Preliminaries

For the duration of this section, we let M be a simply laced Coxeter diagram of rank n, and we let B be the BMW algebra of type M over $\mathbb{Q}(l, x)$.

The following proposition collects several identities that are useful for the proof of the finite dimensionality of B, Theorem 1.1. Recall that m is related to x and l via

$$m = \left(l - l^{-1}\right)/(1 - x). \tag{1}$$

Proposition 2.1. *For each node i of M, the element g_i is invertible in B and the following identities hold*:

$$e_i g_i = l^{-1} e_i, \tag{2}$$

$$g_i^{-1} = g_i + m - m e_i, \tag{3}$$

$$g_i^2 = 1 - m g_i + m l^{-1} e_i, \tag{4}$$

$$e_i^2 = x e_i. \tag{5}$$

Proof. By (D1), e_i is a polynomial in g_i, so g_i and e_i commute, so (2) is equivalent to (R1).

From (D1) we obtain the expression $g_i^2 + m g_i - m l^{-1} e_i = 1$. Application of (R1) to the third monomial on the left-hand side gives $g_i(g_i + m - m e_i) = 1$. So g_i^{-1} exists and is equal to $g_i + m - m e_i$. This establishes (3).

Also by (D1), the element g_i^2 can be rewritten to a linear combination of g_i, e_i and 1, which leads to (4).

As for (5), using (D1) and (R1), we find

$$e_i^2 = e_i l m^{-1}\left(g_i^2 + m g_i - 1\right) = l m^{-1}\left(l^{-2} e_i + m l^{-1} e_i - e_i\right) = x e_i. \qquad \square$$

Remark 2.2. (i) There is an anti-involution on B determined by

$$g_{i_1} \cdots g_{i_q} \mapsto g_{i_q} \cdots g_{i_1}$$

on products of generators g_i of B. We denote this anti-involution by $x \mapsto x^{\mathrm{op}}$.

(ii) The inverse of g_i can be used for a different definition of the e_i, namely

$$e_i = 1 + m^{-1}\left(g_i - g_i^{-1}\right) \quad \text{for all } i.$$

(iii) By (5), the element $x^{-1} e_i$ is an idempotent of B for each node i of M.

The braid relation (B2) for i and j adjacent nodes of M can be seen as a way to rewrite an occurrence iji of indices into jij. It turns out that there are more of these relations in the algebra, with some e's involved.

Proposition 2.3. *The following identities hold for* $i \sim j$:

$$g_j g_i e_j = e_i g_j g_i = e_i e_j, \tag{6}$$

$$\begin{aligned} g_j e_i g_j &= g_i^{-1} e_j g_i^{-1} \\ &= g_i e_j g_i + m(e_j g_i - e_i g_j + g_i e_j - g_j e_i) + m^2(e_j - e_i), \end{aligned} \tag{7}$$

$$e_j e_i g_j = e_j g_i^{-1} = e_j g_i + m(e_j - e_j e_i), \tag{8}$$

$$g_j e_i e_j = g_i^{-1} e_j = g_i e_j + m(e_j - e_i e_j), \tag{9}$$

$$e_i e_j e_i = e_i. \tag{10}$$

Proof. By (D1) and (B2),

$$\begin{aligned} g_j g_i e_j &= g_j g_i \bigl(lm^{-1}\bigl(g_j^2 + m g_j - 1\bigr)\bigr) = lm^{-1}(g_i g_j g_i g_j + m g_i g_j g_i - g_j g_i) \\ &= lm^{-1}\bigl(g_i^2 g_j g_i + m g_i g_j g_i - g_j g_i\bigr) = lm^{-1}\bigl(g_i^2 + m g_i - 1\bigr) g_j g_i \\ &= e_i g_j g_i, \end{aligned}$$

proving the first equality in (6).

We next prove

$$e_i g_j^n g_i e_j e_i = e_i g_j^{n-1} e_i \quad \text{for } n \in \mathbb{N},\ n \geqslant 1. \tag{11}$$

Indeed, by (B2), (R1), (R2), and the first identity of (6), which we have just established,

$$e_i g_j^n g_i e_j e_i = e_i g_j^{n-1}(e_i g_j g_i) e_i = e_i g_j^{n-1} e_i g_j (g_i e_i) = l^{-1} e_i g_j^{n-1} e_i g_j e_i = e_i g_j^{n-1} e_i.$$

The following relation is very useful for determining relations between the e_i.

$$e_i e_j g_i e_j e_i = \bigl(l + m^{-1}\bigr) e_i - m^{-1} e_i e_j e_i. \tag{12}$$

To verify it, we start rewriting one factor e_j by means of (D1), and then use (11) with $n = 2$ and $n = 1$ as well as (R1) and (R2):

$$\begin{aligned} e_i e_j g_i e_j e_i &= e_i \bigl(lm^{-1}\bigl(g_j^2 + m g_j - 1\bigr)\bigr) g_i e_j e_i = lm^{-1}\bigl(l e_i + m x e_i - l^{-1} e_i e_j e_i\bigr) \\ &= \bigl(l + m^{-1}\bigr) e_i - m^{-1} e_i e_j e_i. \end{aligned}$$

We next show (10). Multiplying (R2) for e_j by the left and by the right with e_i, we find $e_i e_j g_i e_j e_i = l e_i e_j e_i$. Using (12) we obtain $(l + m^{-1}) e_i - m^{-1} e_i e_j e_i = l e_i e_j e_i$, whence $(l + m^{-1}) e_i e_j e_i = (l + m^{-1}) e_i$. As $lm \neq -1$, we find $e_i e_j e_i = e_i$. This proves (10).

In order to prove the second equality of (6), we expand $g_i g_j e_i$ by substituting the relation (10). We find

$$g_i g_j e_i = g_i g_j e_i e_j e_i = e_j g_i g_j e_j e_i = l^{-1} e_j g_i e_j e_i = e_j e_i.$$

The first parts of the equalities of (9) and (8) are direct consequences of (6) and (10). In order to show the second part of (8), we use the second equality of (6) and (4):

$$\begin{aligned} e_j e_i g_j &= (e_j g_i g_j) g_j = e_j g_i \big(m l^{-1} e_j - m g_j + 1\big) \\ &= m e_j - m e_j g_i g_j + e_j g_i = m(e_j - e_j e_i) + e_j g_i. \end{aligned}$$

The second part of (9) follows from this by the anti-involution of Remark 2.2(i).

For the first part of (7), as the g_i and g_j are invertible this is $g_i g_j e_i g_j g_i = e_j$. By (6) the left side is $e_j e_i e_j$ which is e_j by (10).

Finally we derive the second part of (7).

$$\begin{aligned} g_j e_i g_j &= g_j e_i e_j e_i g_j = \big(m(e_j - e_i e_j) + g_i e_j\big) e_i g_j \\ &= m e_j e_i g_j - m e_i e_j e_i g_j + g_i e_j e_i g_j \\ &= m\big(m(e_j - e_j e_i) + e_j g_i\big) - m e_i g_j + g_i\big(m(e_j - e_j e_i) + e_j g_i\big) \\ &= m^2 e_j - m^2 e_j e_i + m(e_j g_i - e_i g_j + g_i e_j) - m g_i e_j e_i + g_i e_j g_i \\ &= g_i e_j g_i + m^2 e_j - m^2 e_j e_i + m(e_j g_i - e_i g_j + g_i e_j) \\ &\quad - m\big(m(e_i - e_j e_i) + g_j e_i\big) \\ &= g_i e_j g_i + m^2 e_j - m^2 e_i + m(e_j g_i - e_i g_j + g_i e_j - g_j e_i). \qquad \square \end{aligned}$$

The above identities suffice for a full determination of the BMW algebra associated with the braid group on 3 braids.

Corollary 2.4. *The BMW algebra of type* A_2 *has dimension* 15 *and is spanned by the monomials*:

$$\begin{gathered} 1, \\ g_1, g_2, e_1, e_2, \\ g_1 g_2, g_1 e_2, g_2 g_1, g_2 e_1, e_1 g_2, e_1 e_2, e_2 g_1, e_2 e_1, \\ g_1 g_2 g_1, g_1 e_2 g_1. \end{gathered}$$

Proof. Let B be the BMW algebra of type A_2. Of the sixteen possible words of length 2 the eight consisting of two elements with the same index can be reduced to words of length 1. For, by (D1) g_i^2 can be written as a linear combination of g_i, e_i and 1 and by (5) e_i^2 is a scalar multiple of e_i. Finally, by relation (R1) the remaining four words reduce to e_i.

Now consider words of length 3. By the knowledge that $x^{-1}e_i$ is an idempotent and relation (10) it is clear that no words of length 3 can occur containing only e's. Words containing only g's can be reduced if two g's with the same index occur next to each other. This leaves two possible words $g_i g_j g_i$ either of which can be rewritten to the other one by (B1).

If a word contains e's and g's, no e and g may occur next to each other having the same index as this can be reduced by relation (R1). So the only sequences of indices allowed here are i, j, i and j, i, j. If a g occurs in the middle, we can reduce the word by relation (R2) or (6). This leaves the case with an e in the middle. By (8), (9), and (10) these words reduce unless both the other elements are g's. Finally by (7) the two words left, viz. $g_i e_j g_i$ and $g_j e_i g_j$, are equal up to some terms of shorter length, so at most one is in the basis.

All words of length 4 that can be made by multiplication with a generator from the two words left of length 3, can be reduced. First consider $g_i g_j g_i$. Multiplication by a g gives, immediately or after applying (B2), a reducible g^2 component. Similarly, multiplication by an e will result in a reducible $e_i g_i$ word part. This leaves us with multiples of $g_i e_j g_i$. As noted above, they can be expressed as a linear combination of $g_j e_i g_j$ and terms of shorter length. Again, multiplication by g leads to a g^2 component and the word can be reduced. Multiplication by e will always enable application of relation (R2) to the constructed word and can therefore be reduced, proving that no reduced words of length 4 occur in B.

Finally, by use of the 15 elements as a basis, one can construct an algebra satisfying all relations of the BMW algebra, so the dimension of B is indeed 15. This is done in [17] and later in this paper. □

Proposition 2.5. *The following identities hold for $i \not\sim j$:*

$$e_i g_j = g_j e_i, \tag{13}$$

$$e_i e_j = e_j e_i. \tag{14}$$

Proof. By (D1), the e_i are defined as polynomials in g_i and belong to the subalgebra of B generated by g_i. By (B1) this subalgebra commutes with g_j. □

Proposition 2.6. *There is a unique semilinear automorphism of B of order 2 determined by*

$$g_i \mapsto -g_i^{-1}, \qquad e_i \mapsto e_i, \qquad l \mapsto -l^{-1}, \qquad m \mapsto m.$$

It commutes with the opposition involution of Remark 2.2(i).

Proof. Using the identities proved above, it is readily verified that the defining relations of B are preserved. □

We recall the definition of the BMW algebra as given in [17]; however, we take the parameters q, r to be indeterminates over the field.

Definition 2. Let q, r be indeterminates. The Birman–Murakami–Wenzl algebra BMW_k is the algebra over $\mathbb{C}(r, q)$ generated by $1, g_1, g_2, \ldots, g_{k-1}$, which are assumed to be invertible, subject to the relations:

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},$$
$$g_i g_j = g_j g_i \quad \text{if } |i - j| \geqslant 2,$$
$$e_i g_i = r^{-1} e_i,$$
$$e_i g_{i-1}^{\pm 1} e_i = r^{\pm 1} e_i,$$

where e_i is defined by the equation $(q - q^{-1})(1 - e_i) = g_i - g_i^{-1}$.

We now show that our definition of the BMW algebra of type A_n coincides with this one.

Theorem 2.7. *Let $n \geqslant 2$. The BMW algebra B of type A_{n-1} is the Birman–Murakami–Wenzl algebra BMW_n where $l = r$ and $m = q^{-1} - q$.*

Proof. To show both definitions are of the same algebra, we take our parameters $l = r$ and $m = q^{-1} - q$. The first two relations for both algebras are the same. It is evident from the definition of e_i in both BMW_n and B that g_i and e_i commute, so the third relation for BMW_n is equivalent to (2) and (R1) for B. Also the relation $e_i g_{i-1} e_i = l e_i$ for BMW_n is equivalent to (R2) for B. To see that g_i and e_i in B satisfy $e_i g_{i-1}^{-1} e_i = l^{-1} e_i$, the final defining relation for BMW_n, observe that, for $i \sim j$, by (3), (R2), (5), (10), and (1),

$$e_i g_j^{-1} e_i = e_i (g_j + m - m e_j) e_i = (l + mx - m) e_i = l^{-1} e_i.$$

The definition of e_i follows from Remark 2.2(ii). This shows that B is a homomorphic image of BMW_n. To go the other way it is shown in [17, (4)] that $e_i g_{i+1}^{\pm 1} e_i = r^{\pm 1} e_i$ and so all the relations of B are verified for BMW_n except (D1). This follows from (10) in [17] which when corrected reads $g_i^2 = (q - q^{-1})(g_i - r^{-1} e_i) + 1$. The invertibility of the g_i follows from (3). This shows the algebras are isomorphic. $\square$

Although it is not needed for our computations, there is a cubic relation which is sometimes instructive.

Proposition 2.8. *The elements g_i of B satisfy the cubic relation*

$$\left(g_i^2 + m g_i - 1\right)\left(g_i - l^{-1}\right) = 0.$$

Proof. By (D1) and (2), we have

$$\left(g_i^2 + m g_i - 1\right)\left(g_i - l^{-1}\right) = e_i\left(g_i - l^{-1}\right) = 0. \qquad \square$$

In [17, Proposition 3.2], it is shown that the algebras of type A_{n-1}, the so-called BMW algebras, are finite dimensional. This uses in a crucial way that the symmetric group $S_n \cong W(A_{n-1})$ is doubly transitive on the cosets of S_{n-1}. This is not true for the other algebras. However, we provide a proof of finite dimensionality which applies to the algebras of type A_n as well.

Let (W, R) be the Coxeter system of type M and let $\{r_1, \ldots, r_n\} = R$. Assume furthermore that M is spherical. Then the number of positive roots, $|\Phi^+|$, is the length of the longest word in the generators r_i of W. This means that any product in B of g_i and e_i of longer length can be rewritten by using the relations (B1) and (B2) until one of g_i^2, $g_i e_i$, $e_i g_i$, e_i^2 occurs as a subproduct for some i. In the Coxeter group, r_i has order 2 so we can remove the square and obtain a word of shorter length. In our algebra, we can rewrite the four words to obtain a linear combination of words of shorter length. This leads to the following result.

Proposition 2.9. *If the diagram M is spherical, then any word in the generators of B of length greater than $|\Phi^+|$ in g_i, g_i^{-1}, e_i can be expressed as a sum of words of smaller length by using the defining relations of B. In particular, B is finite dimensional.*

Proof. We can express g_i^{-1} by e_i and g_i to get sums of words in g_i and e_i. Suppose w is a word in g_i and e_i of length greater than $|\Phi^+|$. Consider the word in the Coxeter group w' in r_i where each g_i, e_i in w is replaced by r_i. Notice that if $i \not\sim j$ that both r_i and r_j commute and that both e_i and g_i commute with both e_j and g_j. In particular, the same changes can be made without changing w or w'. Suppose the relation (B2) is used in w', $r_j r_i r_j = r_i r_j r_i$. Consider the same term in w where r_i are replaced by g_i, or e_i and the same for r_j. We showed in the previous sections that all possible ways of replacing the r_i and r_j by e and g elements reduces the word except for $g_i g_j g_i = g_j g_i g_j$ and $g_i e_j g_i = g_j e_i g_j + \omega$, where ω is a linear combination of monomials of degree less than 3. In fact they give words of length 2 or, in the case $e_j g_i^{\pm 1} e_j$, length 1. If we arrive at $e_i g_i = g_i e_i$ we can replace it by (R1) with $l^{-1} e_i$ of shorter length. If we arrive at g_i^2 we use (4) to express it as a sum of words with g_i^2 replaced with e_i, g_i, and the identity. The same holds for g_i^{-2} using the definition. If we arrive at e_i^2 we can replace it with a multiple of itself. In all cases we can reduce the length.

It is now clear that any word in g_i, e_i can be written as a sum of the words of length at most $|\Phi^+|$ in g_i, g_i^{-1}, and e_i. □

Proof of Theorem 1.1. This is a direct consequence of the above proposition. □

3. Artin group properties

In this section, M is a connected, simply laced, spherical Coxeter diagram. This means $M = A_n$ $(n \geqslant 1)$, D_n $(n \geqslant 4)$, or E_n $(n \in \{6, 7, 8\})$. We shall often abbreviate this condition by writing $M \in \mathrm{ADE}$.

We let (A, S) be an Artin system of type M, that is, a pair consisting of an Artin group A of type M with distinguished generating set $\{s_1, \ldots, s_n\}$ corresponding to the nodes of M.

Similarly, we let (W, R) be the Coxeter system of type M, where R is the set of fundamental reflections $r_1, \ldots, r_n$. We shall write Φ for the root system associated with (W, R) and Φ^+ for the set of positive roots with respect to simple roots $\alpha_1, \ldots, \alpha_n$ whose corresponding reflections are $r_1, \ldots, r_n$. There is a map $\psi : W \to A$ sending x to the element $\psi(x) = s_{i_1} \cdots s_{i_t}$ whenever $x = r_{i_1} \cdots r_{i_t}$ is an expression for x as a product of elements of R of minimal length. For $\beta \in \Phi$, we shall denote by r_β the reflection with root β and by s_β its image $\psi(r_\beta)$ in A. For a subset X of W we write $\psi(X)$ to denote $\{\psi(w) \mid w \in X\}$. The map ψ is a section of the morphism of groups $\pi : A \to W$ determined by $s_i \mapsto r_i$, that is, $\pi \circ \psi$ is the identity on W.

Let B be the BMW algebra of type M over $\mathbb{Q}(l, x)$. By means of the composition of ψ and the morphism of groups $A \to B^\times$, we find a map $W \to B$. We shall write $\widehat{w}$ or, if $r_{i_1} \cdots r_{i_t}$ is a reduced expression for w, also $\widehat{i_1 \cdots i_t}$ to denote the image in $B^\times$ of w under this map. In particular, $g_i = \widehat{r_i} = \hat{i}$.

Let $g \in A$. By $g^{-\mathrm{op}}$ we denote the anti-involution op of B introduced in Remark 2.2(i) applied to the inverse of the image of g in B, which is the same as the inverse of the anti-involution applied to g, viewed as an element of B.

Lemma 3.1. *Let i, j be nodes of M. There is a unique element of minimal length in W, denoted by w_{ji}, such that $w_{ji}\alpha_j = \alpha_i$. It has the following properties.*

- (i) *If $i = i_1 \sim i_2 \sim \cdots \sim i_q = j$ is the geodesic in M from i to j, then $\widehat{w_{ji}} = \widehat{i_2}\widehat{i_1}\widehat{i_3}\widehat{i_2} \cdots \widehat{i_{q-1}}\widehat{i_{q-2}}\widehat{i_q}\widehat{i_{q-1}}$.*
- (ii) $w_{ij}^{-1} = w_{ji}$.
- (iii) $\widehat{w_{ij}}^{\mathrm{op}} = \widehat{w_{ji}}$.
- (iv) $\widehat{w_{ij}} e_i = e_j e_{i_{q-1}} \cdots e_{i_2} e_i = e_j \widehat{w_{ij}}$.
- (v) $\widehat{w_{ij}} e_i = \widehat{w_{ij}}^{-\mathrm{op}} e_i = \widehat{w_{ji}}^{-1} e_i$.

Proof. Consider the graph Γ whose nodes are the elements of Φ^+ and in which two nodes α, β are adjacent whenever there is a node k of M such that $r_k\alpha = \beta$. An expression $w = r_{i_1} \cdots r_{i_t}$ of an element w of W satisfying $w\alpha_j = \alpha_i$ represents a path $\alpha_j, r_{i_t}\alpha_j, \ldots, r_{i_2} \cdots r_{i_t}\alpha_j, w\alpha_j = \alpha_i$ from α_j to α_i in Γ. Clearly, if w is of minimal length then this path is a geodesic. This geometric setting readily leads to a proof of (i).

A geodesic in Γ from α to β is given by a backwards traversal of the geodesic from β to α. The corresponding element of W is w^{-1}, whence (ii) and (iii).

Finally, (iv) and (v) follow by induction from (i) and, respectively, (6) and (9). □

For a positive root β, we write $\mathrm{ht}(\beta)$ to denote its height, that is, the sum of its coefficients with respect to the α_i. Furthermore, the support of β, notation $\mathrm{Supp}(\beta)$, is the set of $k \in \{1, \ldots, n\}$ such that the coefficient of α_k in β is non-zero.

Proposition 3.2. *For each node i of M and each positive root β there is a unique element $w \in W$ of minimal length such that $w\alpha_i = \beta$. This element satisfies the following properties.*

(i) *If* $\beta = \alpha_j$ *for some* j, *then* $w = w_{ij}$.
(ii) *If* j *is the unique node of* M *in* $\mathrm{Supp}(\beta)$ *nearest to* i, *then* $l(w) = \mathrm{ht}(\beta) + l(w_{ij}) - 1$.

Proof. Suppose first that i lies in the support of β. Then β can be obtained from α_i by building up with addition of one fundamental root at a time, which corresponds to finding an element w of W by multiplication to the right of the fundamental reflection corresponding to the newly added fundamental root. This shows that there exists $w \in W$ of length at most $\mathrm{ht}(\beta) - 1$ such that $w\alpha_i = \beta$. But the height of β is clearly at most $l(w) + 1$, so the minimal length of any element w of W so that $w\alpha_i = \beta$ must be $\mathrm{ht}(\beta) - 1$.

Next suppose that i does not lie in the support of β and let j be the nearest node to i in $\mathrm{Supp}(\beta)$. Then, with $y \in W$ as in the first paragraph with respect to β and j so that $y\alpha_j = \beta$ and $l(y) = \mathrm{ht}(\beta) - 1$, we have that $yw_{ji}\alpha_i = \beta$ and that $l(yw_{ij}) \leqslant l(w) + l(w_{ij}) = \mathrm{ht}(\beta) + l(w_{ij}) - 1$. On the other hand, in order to transform α_i into β by a chain of roots differing by a fundamental root, we need to apply each root but i and j on the geodesic in M from i to j at least twice (once for creation of the presence of the node in the support, and one for making it vanish). We also need both i and j at least once. Hence, in order to make a fundamental root of $\mathrm{Supp}(\beta)$ occur in the image $u\alpha_i$ of α_i of some $u \in W$, we need $l(u) \geqslant l(w_{ij})$, with equality only if $u = w_{ij}$ and $u\alpha_i = \alpha_j$. Notice that the fundamental reflections in w_{ij} except for α_j do not contribute at all to the creation of the fundamental nodes in $\mathrm{Supp}(\beta)$, so that the estimate for the fundamental roots needed to build up β stays as before. Taking $w = yu$ we find $l(w) = l(yu) = l(y) + l(u) = l(y) + l(w_{ij}) = \mathrm{ht}(\beta) + l(w_{ij}) - 1$.

Next we prove uniqueness of w as stated. Suppose $v \in W$ also satisfies $l(v) = \mathrm{ht}(\beta) + l(w_{ij}) - 1$. As argued above, we must have $v = v'w_{ji}$ and $l(v) = l(v') + l(w_{ji})$ so, without loss of generality, we may assume $i = j$ lies in the support of β. If $l(w) = 0$ then there is nothing to show. Suppose therefore $l(w) > 0$ and apply induction on $l(w)$. Take nodes k, h of M such that $l(r_k w) < l(w)$ and $l(r_h v) < l(v)$ while $r_k\beta = \beta - \alpha_k$ and $r_h\beta = \beta - \alpha_h$. Such k and h exist by the way β is built up of fundamental roots via w and v, respectively. Notice that $(\beta, \alpha_k) = (\beta, \alpha_h) = 1$. Now consider $(\beta - \alpha_k, \alpha_h)$. The value equals -1 if $k = h$; 1 if $h \neq k \not\sim h$; and 2 if $k \sim h$. In the first case, we apply induction to $(r_h w)\alpha_i = \beta - \alpha_h = (r_h v)\alpha_i$, and find $r_h w = r_h v$, whence $w = v$.

In the non-adjacent case, $\beta - \alpha_h - \alpha_k$ is also a root, so there is a unique minimal $u \in W$ such that $u\alpha_i = \beta - \alpha_h - \alpha_k$. Now $r_h r_k u\alpha_i = \beta = w\alpha_i = v\alpha_i$, so $r_h w\alpha_i = r_k u\alpha_i$ and $r_k v\alpha_i = r_h u\alpha_i$, whence, by induction, both $r_h w = r_k u$ and $r_h u = r_k v$. But then $w = r_h r_k u = r_k r_h u = v$.

Finally, if $k \sim h$, we find $(\beta - \alpha_k, \alpha_h) = 2$, whence $\beta = \alpha_h + \alpha_k$. But then i must be either h or k. Assuming (without loss of generality) $i = h$, we find $w = r_k$ and $v = r_h = r_i$, a contradiction with $v\alpha_i = \alpha_i + \alpha_h$.

This establishes that w is unique, and finishes the proof of the lemma. $\square$

Definition 3.3. For a node i of M and a positive root β we denote by $w_{\beta,i}$ the unique element (by the above proposition) of minimal length in W for which $w_{\beta,i}\alpha_i = \beta$. We denote by D_i the set $\{w_{\beta,i} \mid \beta \in \Phi^+\}$.

If $w \in D_i$ then wr_iw^{-1} is a shortest expression of the reflection corresponding to $w\alpha_i$ as a conjugate of r_i.

Corollary 3.4. *For each node i of M, the set D_i satisfies the following properties, where j is a node of M.*

(i) *If $r_jv \in D_i$ and $v \in W$ with $l(r_jv) = l(v) + 1$, then $v \in D_i$.*
(ii) $w_{ij} \in D_i$.

Lemma 3.5. *If i and j are nodes of M, then $\widehat{w_{\alpha_j,i}}e_i = \widehat{w_{ij}}e_i$.*

Proof. Building up $w_{\alpha_j,i}$ from the right, and letting the intermediate results act on α_i, we find a shortest path $i = i_1 \sim i_2 \sim \cdots \sim i_t = j$ in M from i to j. The element $\widehat{w_{ij}}$ represents the corresponding element $\widehat{i_{t-1}i_t} \cdots \widehat{i_2i_3}\widehat{ii_2}$ of B. □

Lemma 3.6. *For all nodes i, j, k of M we have $\widehat{w_{ki}}\widehat{w_{jk}}e_j = \widehat{w_{ji}}e_j$.*

Proof. Denote by $i = i_1 \sim i_2 \sim \cdots \sim i_q = k$ the geodesic from i to k and by $k = k_1 \sim k_2 \sim \cdots \sim k_p = j$ the geodesic from k to j. Then there is an $m \in \{1, \ldots, q\}$ such that $k = k_1 = i_q \sim k_2 = i_{q-1} \sim \cdots \sim k_m = i_{q-m+1}$ and $k_{m+1} \neq i_{q-m}$. Then the geodesic from i to j is $i = i_1 \sim i_2 \sim \cdots \sim i_{q-m} \sim k_m \sim k_{m+1} \sim \cdots \sim k_{p-1} \sim k_p$ and so

$$\begin{aligned}
\widehat{w_{ki}}\widehat{w_{jk}}e_j &= \widehat{w_{ki}}e_{k_1}\cdots e_{k_p} \\
&= e_{i_1}\cdots e_{i_q}e_{k_1}\cdots e_{k_p} \\
&= e_{i_1}\cdots e_{i_{q-m}}e_{k_m}\cdots e_{k-1}e_ke_{k-1}\cdots e_{k_p} \\
&= e_{i_1}\cdots e_{i_{q-m}}e_{k_m}\cdots e_{k-1}e_ke_{k-1}\cdots e_{k_m}e_{k_{m+1}}\cdots e_{k_p} \\
&= e_{i_1}\cdots e_{i_{q-m}}e_{k_m}e_{k_{m+1}}\cdots e_{k_p} \\
&= \widehat{w_{ji}}e_j. \qquad \square
\end{aligned}$$

For $\alpha, \beta \in \Phi^+$ with $\alpha \leqslant \beta$ (that is, for each i, the difference of the coefficient of α_i in β and the coefficient of α_i in α is non-negative), let $w_{\beta,\alpha}$ be the (unique) shortest element of W mapping α to β. Clearly, $l(w_{\beta,\alpha}) = \mathrm{ht}(\beta) - \mathrm{ht}(\alpha)$. Thus, $w_{\beta,i} = w_{\beta,\alpha_i}$ if $i \in \mathrm{Supp}(\beta)$. For a positive root β, set $d_\beta = \psi(w_{\alpha_0,\beta}^{-1}) \in A$. This implies that $s_{\alpha_0} = d_\beta^{\mathrm{op}} s_\beta d_\beta$. For a node i such that α_i is orthogonal to β, we shall need the following Artin group element.

$$h_{\beta,i} = d_\beta^{-1} s_i d_\beta. \tag{15}$$

Lemma 3.7. *The following relations hold for elements $h_{\gamma,k}$ of the Artin group A, where we are always assuming that γ is a positive root and $(\alpha_k, \gamma) = 0$:*

$$h_{\beta,i}h_{\beta,j} = h_{\beta,j}h_{\beta,i} \qquad \text{if } i \not\sim j, \tag{16}$$

$$h_{\beta,i}h_{\beta,j}h_{\beta,i} = h_{\beta,j}h_{\beta,i}h_{\beta,j} \quad \textit{if } i \sim j, \tag{17}$$

$$h_{\beta+\alpha_j,i} = h_{\beta,i} \quad \textit{if } i \not\sim j, \tag{18}$$

$$h_{\beta+\alpha_j,i} = h_{\beta-\alpha_i,j} \quad \textit{if } i \sim j, \tag{19}$$

$$h_{\beta-\alpha_i-\alpha_j,i} = h_{\beta,j} \quad \textit{if } i \sim j, \tag{20}$$

$$h_{\beta+\alpha_i+\alpha_j,j} = h_{\beta,i} \quad \textit{if } i \sim j, \tag{21}$$

$$h_{\alpha_i,j} = h_{\alpha_j,i} \quad \textit{if } i \textit{ and } j \textit{ are at distance } 2 \textit{ in } M, \tag{22}$$

$$h_{\alpha_j,k} = h_{\alpha_i,k} \quad \textit{if } i \sim j. \tag{23}$$

Proof. The rules are all straightforward applications of corresponding rules for d_β. We prove (19) and (23) and leave the rest to the reader.

For rule (19), we have $d_{\beta-\alpha_i} = s_i s_j d_{\beta+\alpha_j}$ in the Artin group whereas $i \sim j$, $(\alpha_i, \beta) = -1$, and $(\alpha_j, \beta) = 1$, so $h_{\beta-\alpha_i,j}$ is the Hecke algebra element corresponding to the Artin group element $d_{\beta+\alpha_i}^{-1} s_j d_{\beta+\alpha_i} = d_{\beta-\alpha_j}^{-1} s_j^{-1} s_i^{-1} s_j s_i s_j d_{\beta-\alpha_j} = d_{\beta-\alpha_j}^{-1} s_i d_{\beta-\alpha_j}$, and so $h_{\beta-\alpha_i,j}$ coincides with $h_{\beta-\alpha_j,i}$.

We finish with (23). It is a direct consequence of $s_i^{-1} d_{\alpha_j} = d_{\alpha_i+\alpha_j} = s_j^{-1} d_{\alpha_i}$ and the fact that k is adjacent to neither i nor j:

$$h_{\alpha_j,k} = d_{\alpha_j}^{-1} k d_{\alpha_j} = d_{\alpha_i}^{-1} s_j s_i^{-1} s_k s_i s_j^{-1} d_{\alpha_j} = d_{\alpha_i}^{-1} s_k d_{\alpha_i} = h_{\alpha_i,k}. \qquad \square$$

As before, let C be the set of nodes i of M for which α_i is orthogonal to the highest root α_0 of Φ^+.

Lemma 3.8. *The following properties hold for C.*

(i) *If i is a node of M and $\beta \in \Phi^+$ satisfies $(\alpha_i, \beta) = 0$, then there is a node j of C such that $h_{\beta,i} = s_j$.*
(ii) *For each j in C there exist non-adjacent nodes i, k with $h_{\alpha_i,k} = s_j$.*

Proof. (i) If $\beta = \alpha_0$, then i is a node orthogonal to α_0 and so $h_{\beta,i} = s_i$ and i belongs to C by definition of C. We continue by induction with respect to the height of β. Assume $\mathrm{ht}(\beta) < \mathrm{ht}(\alpha_0)$. Then there is a node j such that $(\alpha_j, \beta) = -1$, so $\gamma = \beta + \alpha_j$ is a root, whence $d_\beta = s_j d_\gamma$. If $i \not\sim j$, then, by (18), $h_{\beta,i} = h_{\gamma,i}$. Otherwise, by (21) $h_{\beta,i} = h_{\gamma+\alpha_i,j}$. In both cases the expression found for $h_{\beta,i}$ is as required by the induction hypothesis.

(ii) Let j be a node in C. Then $h_{\alpha_0,j} = \hat{j}$. Let β be a minimal positive root for which there exists a node k with $(\alpha_k, \beta) = 0$ and $h_{\beta,k} = \hat{j}$. If $\mathrm{ht}(\beta) > 1$, take a node i such that $(\alpha_i, \beta) = 1$. By Lemma 3.7, either $i \sim k$ and $h_{\beta-\alpha_i-\alpha_k,i} = \hat{j}$, or $(\alpha_i, \alpha_k) = 0$ and $h_{\beta-\alpha_i,k} = \hat{j}$. Therefore, we may assume $\mathrm{ht}(\beta) = 1$, and so $\beta = \alpha_i$ for some i with $(\alpha_i, \alpha_k) = 0$. $\square$

Lemma 3.9. *If i is a node of M and β a positive root such that $(\alpha_i, \beta) = 0$, then*

$$s_i s_\beta = s_\beta s_i.$$

Proof. We proceed by induction on $\mathrm{ht}(\beta)$. If $\mathrm{ht}(\beta) = 1$, then $\beta = \alpha_j$. As $(\alpha_i, \beta) = 0$, we have $i \not\sim j$ and so $s_i s_\beta = s_i s_j = s_j s_i = s_\beta s_i$ by the braid relations.

Assume now that $\mathrm{ht}(\beta) > 1$. Let j be a node of M such that $(\alpha_j, \beta) = 1$, so $\beta - \alpha_j$ is a positive root. Then $s_\beta = s_j s_{\beta-\alpha_j} s_j$. If $j \not\sim i$, then $(\alpha_i, \beta - \alpha_j) = 0$, so, by the induction hypothesis, $s_i s_{\beta-\alpha_j} = s_{\beta-\alpha_j} s_i$, whence $s_i s_\beta = s_i s_j s_{\beta-\alpha_j} s_j = s_j s_i s_{\beta-\alpha_j} s_j = s_j s_{\beta-\alpha_j} s_i s_j = s_j s_{\beta-\alpha_j} s_j s_i = s_\beta s_i$. Otherwise, $j \sim i$, and $\gamma = \beta - \alpha_i - \alpha_j$ is a positive root with $(\alpha_j, \gamma) = 0$ and $s_\beta = s_j s_i s_\gamma s_i s_j$. By the induction hypothesis, $s_j s_\gamma = s_\gamma s_j$, whence $s_i s_\beta = s_i s_j s_i s_\gamma s_i s_j = s_j s_i s_j s_\gamma s_i s_j = s_j s_i s_\gamma s_j s_i s_j = s_j s_i s_\gamma s_i s_j s_i = s_\beta s_i$. □

4. Some ideals of the BMW algebra

In this section, let M be a simply laced Coxeter diagram (not necessarily spherical). In the BMW algebra B of type M, the e_i generate an ideal (by which we mean a 2-sided ideal). Taking products of e_i's for non-adjacent nodes i of M, we obtain further ideals.

Definition 4.1. Let Y be a coclique of M, that is, a subset of the nodes of M in which no two nodes are adjacent. The *ideal of type Y* is the (2-sided) ideal of B generated by e_Y, where

$$e_Y = \prod_{y \in Y} e_y.$$

The element e_Y is well defined as the product does not depend on the order of the e_y in view of (14). The ideal $Be_Y B$ is denoted by I_Y. By I_j, for $j = 1, \ldots, n$, we denote the ideal generated by all I_Y for Y a coclique of size j.

Since the e_i are scalar multiples of idempotents, so are their products e_Y for Y a coclique of M.

Proposition 4.2. *Let X, Y be cocliques of M.*

- (i) *If $X \subseteq Y$ then $I_Y \subseteq I_X$.*
- (ii) *If $\{r_j \mid j \in X\}$ is in the same W-orbit as $\{r_j \mid j \in Y\}$ then $I_X = I_Y$.*
- (iii) *The quotient algebra B/I_1 is the Hecke algebra of type M over $\mathbb{Q}(l, x)$, with parameter m.*

Proof. (i) is immediate from the definition of I_Y and the commutation of the e_i for $i \in Y$.

(ii) For $|X| = |Y| = 1$, say $X = \{i\}$ and $Y = \{j\}$, this follows from the existence of the invertible element $\widehat{w_{ij}}$ as in Lemma 3.1(iv). More generally, by [11], there exists $w \in W$ such that $\widehat{w}\widehat{X}\widehat{w}^{-1} = \widehat{Y}$. This implies $\widehat{w} e_X \widehat{w}^{-1} = e_Y$, whence $I_X = I_Y$.

(iii) By (6), invertibility of the g_i and connectedness of M, the ideal I_1 coincides with $I_{\{j\}}$ for any node j of M. Consequently, the quotient ring B/I_1 is obtained by setting $e_i = 0$ for all i. This means that the braid relations (B1) and (B2) and (D1) are the defining relations for B/I_1 in terms of g_i. Now (D1) reads $g_i^2 + mg_i - 1 = 0$, so we obtain the defining relations of the Hecke algebra. $\square$

By (i), we have the chain of ideals

$$I_1 \supset I_2 \supset \cdots \supset I_k,$$

where k is the maximal coclique size of M. By analogy with the BMW algebra of type A_n and computer results for D_4 we expect this is a strictly decreasing series of ideals. We already know from (iii) of the above proposition that I_1 is properly contained in B. Straightforward calculations for the Lawrence–Krammer representation, described in [7] and in [16] for the non-spherical types, show that (D1), (R1), (R2) are also satisfied, so it is a representation of B. Furthermore it can be seen that e_i is not represented as 0 but e_ie_j is for any two distinct non-adjacent nodes i, j of M. These calculations will be presented in a more general setting later, in Section 6. As a consequence I_2 is properly contained in I_1. This follows also of course from Theorem 1.2.

It is also clear from the definition that $I_j = \{0\}$ when j is bigger than the maximal coclique size of M. These sizes are $\lfloor (n+1)/2 \rfloor$ for A_n; $\lfloor n/2 \rfloor + 1$ for D_n; 3 for E_6; and 4 for both E_7 and E_8.

5. Structure of I_1/I_2

Throughout this section, M is a connected simply laced spherical diagram. This means $M \in \mathrm{ADE}$. By B we denote the corresponding BMW algebra over $\mathbb{Q}(l, x)$, by (A, S) the corresponding Artin system, and by (W, R) the corresponding Coxeter system. Furthermore, Φ^+ is the set of positive roots associated with (W, R) and C the set of nodes i of M with α_i orthogonal to the highest root of Φ^+.

We now prepare for considerations of B modulo I_2. This is indicated in the statements. The aim is to find a linear spanning set for I_1/I_2 of size $|\Phi^+|^2|W_C|$. In particular, we obtain an upper bound for $\dim(I_1/I_2)$, which by Theorem 1.2 will be an equality.

Let i be a node of M and let Z_i be the subalgebra (not necessarily containing the identity) of B generated by all elements of the form $\widehat{w_{ji}}\hat{k}\widehat{w_{ij}}e_i$ for j and k non-adjacent nodes of M. We allow for j and k to be equal, so that, in case $M = \mathrm{A}_2$, the subalgebras Z_i are one-dimensional (scalar multiples of e_i). By Lemma 3.1(iv), (v), the generators can be written in various ways:

$$e_i\widehat{w_{ji}}\hat{k}\widehat{w_{ji}}^{-1} = \widehat{w_{ji}}\hat{k}\widehat{w_{ji}}^{-1}e_i = \widehat{w_{ji}}\hat{k}\widehat{w_{ij}}e_i.$$

We will need an integral version of Z_i and B. We shall work with the coefficient ring $E = \mathbb{Q}(x)[l^{\pm}]$ inside our field $\mathbb{Q}(l, x)$. Observe $m \in E$ by (1). Let $B^{(0)}$ be the subalgebra of B over E generated by all g_i and e_i, and let $Z_i^{(0)}$ be the subalgebra of Z_i over E generated by the same elements as taken above for generating Z_i. Then $Z_i^{(0)}$ is a subalgebra of $B^{(0)}$.

Proposition 5.1. *The subalgebra $Z_i^{(0)}$ of $B^{(0)}$ satisfies the following properties.*

(i) *It centralizes e_i and has identity element $x^{-1}e_i$.*
(ii) $Z_i^{(0)} = \widehat{w_{ji}} Z_j^{(0)} \widehat{w_{ji}}^{-1}$ *for all nodes j of M.*
(iii) *The scaled versions $x^{-1}e_i\widehat{w_{ji}}\hat{k}\widehat{w_{ji}}^{-1}$ of the generators of $Z_i^{(0)}$ satisfy the quadratic relation $X^2 + mX - 1_i = 0 \bmod I_2$, where 1_i stands for the identity element $x^{-1}e_i$ of $Z_i^{(0)}$.*

Proof. (i) Since $x^{-1}e_i$ is an idempotent (cf. (5)), it suffices to verify that the generators of Z_i centralize e_i. This follows from the following computation, in which Lemmas 3.1 and 3.6 are used.

$$\widehat{w_{ji}}\hat{k}\widehat{w_{ji}}^{-1}e_i = \widehat{w_{ji}}\hat{k}e_j\widehat{w_{ij}} = \widehat{w_{ji}}\hat{k}e_j\widehat{w_{ji}}^{-1} = \widehat{w_{ji}}e_j\hat{k}\widehat{w_{ji}}^{-1} = e_i\widehat{w_{ji}}\hat{k}\widehat{w_{ji}}^{-1}.$$

(ii) For the generator $e_h\widehat{w_{jh}}\hat{k}\widehat{w_{jh}}^{-1}$ of $Z_h^{(0)}$, where $j \perp k$, we have

$$\begin{aligned}\widehat{w_{hi}}e_h\widehat{w_{jh}}\hat{k}\widehat{w_{jh}}^{-1}\widehat{w_{hi}}^{-1} &= \widehat{w_{hi}}\widehat{w_{jh}}e_j\hat{k}\widehat{w_{jh}}^{-1}\widehat{w_{hi}}^{-1} = \widehat{w_{ji}}e_j\hat{k}\widehat{w_{jh}}^{-1}\widehat{w_{hi}}^{-1}\\ &= \widehat{w_{ji}}\hat{k}e_j\widehat{w_{jh}}^{-1}\widehat{w_{hi}}^{-1} = \widehat{w_{ji}}\hat{k}e_j\widehat{w_{hj}}\widehat{w_{hi}}^{-1}\\ &= \widehat{w_{ji}}\hat{k}\widehat{w_{hj}}e_h\widehat{w_{hi}}^{-1} = \widehat{w_{ji}}\hat{k}\widehat{w_{hj}}e_h\widehat{w_{ih}}\\ &= \widehat{w_{ji}}\hat{k}e_j\widehat{w_{hj}}\widehat{w_{ih}} = \widehat{w_{ji}}\hat{k}e_j\widehat{w_{ij}}\\ &= \widehat{w_{ji}}\hat{k}e_j\widehat{w_{ji}}^{-1} = \widehat{w_{ji}}e_j\hat{k}\widehat{w_{ji}}^{-1} = e_i\widehat{w_{ji}}\hat{k}\widehat{w_{ji}}^{-1},\end{aligned}$$

whence $\widehat{w_{hi}} Z_h^{(0)} \widehat{w_{hi}}^{-1} \subseteq Z_i^{(0)}$. The rest follows easily.

(iii) Substituting $x^{-1}e_i\widehat{w_{ji}}\hat{k}\widehat{w_{ji}}^{-1}$ for X, we find

$$\begin{aligned}&\bigl(x^{-1}e_i\widehat{w_{ji}}\hat{k}\widehat{w_{ji}}^{-1}\bigr)^2 + m\bigl(x^{-1}e_i\widehat{w_{ji}}\hat{k}\widehat{w_{ji}}^{-1}\bigr) - x^{-1}e_i\\ &\quad = x^{-1}e_i\widehat{w_{ji}}\bigl(\hat{k}^2 + m\hat{k} - 1\bigr)\widehat{w_{ji}}^{-1} = x^{-1}e_i\widehat{w_{ji}}e_k\widehat{w_{ji}}^{-1} \in Be_je_kB \subseteq I_2. \qquad \square\end{aligned}$$

We recall that $w_{\beta,i} \in W$ is the element of minimal length with the property that $w_{\beta,i}\alpha_i = \beta$ with $\alpha_i, \beta \in \Phi^+$.

Lemma 5.2. *Suppose i, j, and k are distinct nodes of M. Then*

$$e_i\hat{j}e_k = \begin{cases} e_ie_k\hat{j} & \text{if } j \not\sim k \text{ and } i \not\sim k,\\ \widehat{w_{\alpha_i,k}}e_k\hat{j} & \text{if } j \not\sim k \text{ and } i \sim k,\\ \widehat{w_{\alpha_i,k}}e_k(\hat{i}+m) - me_ie_k & \text{if } j \sim k \text{ and } i \sim j,\\ \widehat{w_{\alpha_i,k}}e_k\widehat{jkikj} & \text{if } j \sim k \text{ and } i \sim k,\\ e_ie_k\widehat{w_{ik}}\hat{j}\widehat{w_{ki}} & \text{if } j \sim k,\ i \not\sim j,\ \text{and } i \not\sim k.\end{cases}$$

In each case the result is in $\widehat{w_{\alpha_i,k}} Z_k^{(0)} + I_2$.

Proof. In the first two cases as $j \not\sim k$ we have $e_i\hat{j}e_k = e_ie_k\hat{j}$. If $i \not\sim k$, e_ie_k is in I_2. If $i \sim k$, $e_ie_k = w_{\alpha_i,k}e_k$. These are the only possibilities when $j \not\sim k$.

Suppose next that $j \sim k$. In the last case e_i commutes with $\hat{j}$ and e_ie_k is in I_2. Suppose then $i \sim j$. Of course then $i \not\sim k$ since the type is spherical. Now by (R2)

$$\begin{aligned} e_i\hat{j}e_k &= (e_ie_je_i)\hat{j}e_k = \big(e_ie_j\widehat{ij}\big)\hat{j}e_k \\ &= e_ie_j\hat{i}\big(1 - m\hat{j} + ml^{-1}e_j\big)e_k = e_ie_j\hat{i}e_k - me_ie_je_ie_k + me_ie_je_k \\ &= e_ie_je_k\big(\hat{i} + m\big) - me_ie_k = \widehat{w_{\alpha_i,k}}e_k\big(\hat{i} + m\big) - me_ie_k. \end{aligned}$$

As $e_ie_k \in I_2$ the result follows.

Finally, if $i \sim k$ then necessarily $i \not\sim j$, and

$$e_i\hat{j}e_k = e_ie_ke_i\hat{j}e_k = e_ie_k\hat{j}e_ie_k = e_ie_ke_i\hat{j}\widehat{ki} = e_ie_k\widehat{ikjki} = e_ie_k\widehat{jkikj}.$$

In each of the cases the elements are in $\widehat{w_{\alpha_i,k}}Z_k^{(0)} + I_2$ from the definition. □

If some of i, j, k are equal, similar results follow from the defining relations and Propositions 2.3 and 2.5.

Lemma 5.3. *Let $i, j, k \in \{1, \ldots, n\}$ and let γ be the shortest path from j to k. Then*

$$\hat{i}\widehat{w_{\alpha_j,k}}e_k = \begin{cases} \widehat{w_{\alpha_j,k}}e_k\hat{i} & \text{if } i \not\sim \text{any point of } \gamma, \\ l^{-1}\widehat{w_{\alpha_j,k}}e_k & \text{if } i = j, \\ \widehat{w_{\alpha_j,k}}e_k\widehat{h'} \bmod I_2 & \text{if } i \in \gamma,\ i \neq j, \\ & \quad h' \text{ on the path from } i \text{ to } j, \\ & \quad h' \text{ at distance } 2 \text{ to } i \text{ in } M, \\ \widehat{w_{\alpha_j,k}}e_k\widehat{w_{h'k}}\hat{i}\widehat{w_{kh'}} & \text{if } i \notin \gamma,\ i \sim h,\ h \in \gamma, \\ & \quad h \neq j,\ h' \sim h, \text{ and} \\ & \quad h' \text{ on the path from } h \text{ to } j, \\ \widehat{w_{\alpha_i+\alpha_j,k}}e_k + m\widehat{w_{\alpha_i,k}}e_k - m\widehat{w_{\alpha_j,k}}e_k & \text{if } i \in \gamma \text{ and } i \sim j, \\ \widehat{w_{\alpha_i+\alpha_j,k}}e_k & \text{if } i \notin \gamma \text{ and } i \sim j. \end{cases}$$

Also

$$e_i\widehat{w_{\alpha_j,k}}e_k = \begin{cases} x\widehat{w_{\alpha_j,k}}e_k & \text{if } i = j, \\ 0 \bmod I_2 & \text{if } i \not\sim j, \\ \widehat{w_{\alpha_i,k}}e_k & \text{if } i \sim j. \end{cases}$$

In each case, the result is in $\widehat{w_{r_i\alpha_j,k}}Z_k^{(0)} + m\widehat{w_{\alpha_j,k}}Z_k^{(0)} + m\widehat{w_{\alpha_i,k}}Z_k^{(0)} + I_2$.

Proof. Consider the shortest path $\gamma = k, \ldots, j$ from k to j in M. If i is non-adjacent to each element of this path, then the statement holds. Also if $i = j$ the statement follows immediately. This leaves two possibilities, i is in γ, or i is not in γ but is adjacent to some h in γ.

Assume that i occurs in γ. If $i \sim j$, then by (9)

$$\begin{aligned}\hat{i}\widehat{w_{\alpha_j,k}}e_k &= \hat{i}e_j e_i \cdots e_k \\ &= \hat{j}^{-1}\widehat{w_{ki}}e_k = \widehat{w_{\alpha_i+\alpha_j,k}}e_k + m\widehat{w_{ki}}e_k - m\widehat{w_{kj}}e_k.\end{aligned}$$

Suppose, therefore, that $i \not\sim j$. Then $\hat{i}\widehat{w_{kj}}e_k = e_j \cdots e_{h'}\hat{i}e_h e_i e_{i'} \cdots e_k$ with $h' \sim h \sim i \sim i'$. Substitution of $\hat{i}e_h e_i = \hat{h}e_i - me_h e_i + me_i$ and use of Lemma 5.3 gives

$$\begin{aligned}\hat{i}\widehat{w_{kj}}e_k &= e_j \cdots e_{h'}\hat{i}e_h e_i \cdots e_k = e_j \cdots e_{h'}\big(\hat{h}e_i - me_h e_i + me_i\big)e_{i'} \cdots e_k \\ &= e_j \cdots e_{h'}\hat{h}e_i e_{i'} \cdots e_k - m\widehat{w_{kj}}e_k + me_j \cdots e_{h'}e_i \cdots e_k \\ &\in e_j \cdots e_{h'}e_h e_i\big(\widehat{h'} + m\big)e_{i'} \cdots e_k - m\widehat{w_{kj}}e_k + I_2 \\ &= e_j \cdots e_{h'}e_h e_i\widehat{h'}e_{i'} \cdots e_k + I_2 = e_j \cdots e_{h'}e_h e_i \cdots e_k\widehat{h'} + I_2 \\ &= \widehat{w_{kj}}e_k h' + I_2.\end{aligned}$$

Next assume i is not in γ but is adjacent to some h in γ. Suppose there exists $h' \sim h$ in γ, so

$$\hat{i}\widehat{w_{kj}}e_k = e_j \cdots e_{h'}\hat{i}e_h \cdots e_k.$$

With the use of $e_{h'} = e_{h'} \cdots e_k \cdots e_{h'} = \widehat{w_{kh'}}e_k\widehat{w_{h'k}}$ this becomes

$$\begin{aligned}\hat{i}\widehat{w_{kj}}e_k &= \widehat{w_{h'j}}e_{h'}\hat{i}e_h \cdots e_k = \widehat{w_{h'j}}\hat{i}e_{h'}\widehat{w_{kh'}} = \widehat{w_{h'j}}e_{h'}\hat{i}\widehat{w_{kh'}} \\ &= \widehat{w_{h'j}}\widehat{w_{kh'}}e_k\widehat{w_{h'k}}\hat{i}\widehat{w_{kh'}} = \widehat{w_{kj}}e_k\widehat{w_{h'k}}\hat{i}\widehat{w_{kh'}}.\end{aligned}$$

It is easy to verify that $\widehat{w_{h'k}}\hat{i}\widehat{w_{kh'}}$ commutes with e_k.

We are left with the case where i is not in γ but is adjacent to j, an end node of γ. Then $\hat{i}\widehat{w_{kj}}e_k = \hat{i}\widehat{w_{\alpha_j,k}}e_k = \widehat{w_{\alpha_i+\alpha_j,k}}e_k$. This ends the proof of the equalities involving $\hat{i}\widehat{w_{kj}}e_k$.

We now consider $e_i\widehat{w_{kj}}e_k$. If $i = j$, we have trivially $e_i\widehat{w_{kj}}e_k \in \widehat{w_{kj}}Z_k^{(0)}$. So let $i \neq j$. If $i \not\sim j$ we find $e_i\widehat{w_{kj}}e_k = e_i e_j \cdots \widehat{w_{kj}}e_k \in I_2$. So assume $i \sim j$.

If i occurs in γ, the path γ begins with $j \sim i$ and so

$$e_i\widehat{w_{kj}}e_k = e_i e_j e_i \cdots e_k = e_i \cdots e_k = \widehat{w_{ki}}e_k$$

and if i does not occur in γ, we have $e_i\widehat{w_{kj}}e_k = e_i e_j \cdots e_k = \widehat{w_{ki}}e_k$. □

Let i be a node of M and $\beta \in \Phi^+$. We shall use the following notation.

- $\mathrm{Geod}(i,\beta)$ is the set of nodes of the shortest path from i to a node in the support of β that are not in the support themselves. So $\mathrm{Geod}(i,\beta)=\emptyset$ if $i\in\mathrm{Supp}(\beta)$.
- $\mathrm{Proj}(i,\beta)$ is the node in the support of β nearest i. So $\mathrm{Proj}(i,\beta)=i$ if $i\in\mathrm{Supp}(\beta)$.
- $C_{\beta,i}$ is the coefficient of α_i in the expression of β as a linear combination of the fundamental roots. So $\beta=\sum_i C_{\beta,i}\alpha_i$.
- $J_{\beta,k}$ is the subset of M of all nodes j such that $(\alpha_j,\beta)=1$ and $\hat{j}\,\widehat{w_{\beta-\alpha_j,h}}=\widehat{w_{\beta,h}}$, where $h=\mathrm{Proj}(\beta,k)$. This set is empty only if β is a fundamental root.

For i a node of M, denote by $i^\perp$ the set of all nodes distinct and non-adjacent to i.

Lemma 5.4. *Let β be a root and let k be a node of M such that $i=\mathrm{Proj}(\beta,k)$ satisfies $(\alpha_i,\beta)=0$ and $C_{\beta,i}=1$. If $J_{\beta,k}\cap i^\perp=\emptyset$ then*

$$\hat{i}\,\widehat{w_{\beta,k}}e_k=\widehat{w_{\beta,k}}^{-\mathrm{op}}e_k\widehat{w_{\beta,k}}^{\mathrm{op}}\hat{i}\,\widehat{w_{\beta,k}}\in\widehat{w_{\beta,k}}^{-\mathrm{op}}Z_k^{(0)}.$$

Proof. We only have to prove that $e_k\widehat{w_{\beta,k}}^{\mathrm{op}}\hat{i}\,\widehat{w_{\beta,k}}$ belongs to $Z_k^{(0)}$. Moreover,

$$e_k\widehat{w_{\beta,k}}^{\mathrm{op}}\hat{i}\,\widehat{w_{\beta,k}}=e_k\widehat{w_{ik}}\,\widehat{w_{\beta,i}}^{\mathrm{op}}\hat{i}\,\widehat{w_{\beta,i}}\,\widehat{w_{ki}}$$

and $J_{\beta,k}=J_{\beta,i}$, so, by Proposition 5.1(ii), it suffices to consider the case where $k=i$.

We prove this by induction on the height of β. The smallest possible root that satisfies the conditions of the lemma is a root of the form $\alpha_j+\alpha_i+\alpha_h$ with $j\sim i\sim h$. In this case $\widehat{w_{\beta,i}}=\widehat{hj}$. Straightforward computations give

$$e_i\widehat{w_{\beta,i}}^{\mathrm{op}}\hat{i}\,\widehat{w_{\beta,i}}=e_i\widehat{jh}\hat{i}\widehat{hj}=e_i\widehat{jihij}=e_i\widehat{w_{ji}}\hat{h}\widehat{w_{ij}}=e_i\widehat{w_{ij}}^{\mathrm{op}}\hat{h}\widehat{w_{ij}},$$

which belongs to $Z_k^{(0)}$ by definition.

Let β be a positive root of height at least 4 and assume that the lemma holds for all positive roots of height less than $\mathrm{ht}(\beta)$. Now $\widehat{w_{\beta,k}}e_k=\widehat{w_{\beta,i}}e_i\cdots e_k$ with no i in $w_{\beta,i}$. Let $j\in J_{\beta,k}$. Then, by the hypothesis $J_{\beta,k}\cap i^\perp=\emptyset$, we have $i\sim j$. Clearly $w_{\beta,i}=jw_{\beta-\alpha_j,i}$. As $(\alpha_i,\beta)=0$ and $C_{\beta,i}=1$, the sum of $C_{\beta,j}$ for j running over the neighbors of i in M, must be 2. Hence there are either two nodes j, h say, in M with $C_{\beta,j}=C_{\beta,h}=1$ or there is a single node j of M adjacent to i with $C_{\beta,j}=2$. In the former case, as $\mathrm{ht}(\beta)\geqslant 4$, there is an end node p of β distinct from j, i, h and non-adjacent to i with $C_{\beta,p}=1$, which implies $(\alpha_p,\beta)=1$, whence $p\in J_{\beta,i}\cap i^\perp$, a contradiction. Hence i is an end node of β and has a neighbor j with $C_{\beta,j}=2$ and $(\alpha_j,\beta)=1$. This implies that $\widehat{w_{\beta-\alpha_j,i}}=\widehat{w_{\gamma,j}}\hat{j}$, where $\gamma=\beta-\alpha_i-\alpha_j$. As $(\alpha_j,\gamma)=0$ and $J_{\gamma,j}\cap j^\perp\subseteq J_{\beta,i}\cap i^\perp=\emptyset$, we can apply induction to find $e_j\widehat{w_{\gamma,j}}^{\mathrm{op}}\hat{j}\widehat{w_{\gamma,j}}$ belongs to $Z_j^{(0)}$. Consequently,

$$\begin{aligned}e_i\widehat{w_{\beta,i}}^{\mathrm{op}}\hat{i}\,\widehat{w_{\beta,i}}&=e_i\hat{j}\widehat{w_{\gamma,j}}^{\mathrm{op}}\widehat{jij}\widehat{w_{\gamma,j}}\hat{j}=e_i\hat{j}\widehat{w_{\gamma,j}}^{\mathrm{op}}\widehat{iji}\widehat{w_{\gamma,j}}\hat{j}=e_i\widehat{ji}\widehat{w_{\gamma,j}}^{\mathrm{op}}\hat{j}\widehat{w_{\gamma,j}}\widehat{ij}\\&\in\widehat{w_{ji}}Z_j^{(0)}\widehat{w_{ij}}=Z_i^{(0)}.\qquad\square\end{aligned}$$

Lemma 5.5. *Let β be a root and let i be a node with $(\alpha_i,\beta)=0$. Then the following hold.*

(i) *If j is a node in $J_{\beta,k} \cap i^\perp$ then* $\hat{i}\widehat{w_{\beta,k}}e_k = \hat{j}\hat{i}\widehat{w_{\beta-\alpha_j,k}}e_k$.

(ii) *If* $i = \mathrm{Proj}(\beta,k)$ *and* $C_{\beta,i} = 1$ *and* $J_{\beta,k} \cap i^\perp = \emptyset$, *then* $\widehat{w^{\mathrm{op}}_{\beta,k}}\hat{i}\widehat{w_{\beta,k}} \in Z_k^{(0)}$ *and*

$$\hat{i}\widehat{w_{\beta,k}}e_k = \widehat{w_{\beta,k}}^{-\mathrm{op}}e_k\left(\widehat{w^{\mathrm{op}}_{\beta,k}}\hat{i}\widehat{w_{\beta,k}}\right).$$

(iii) *If* $i \neq \mathrm{Proj}(\beta,k)$ *or* $C_{\beta,i} > 1$, *then, for* $j \in J_{\beta,k} \setminus i^\perp$,

$$\hat{i}\widehat{w_{\beta,k}}e_k = \widehat{ji}\hat{j}\widehat{w_{\beta-\alpha_j-\alpha_i,k}}e_k.$$

In each case, $\hat{i}\widehat{w_{\beta,k}}e_k \in \widehat{w_{\beta,k}}Z_k^{(0)}$.

Proof. (i) Straightforward from $\widehat{ij} = \widehat{ji}$.

For the remainder of the proof, we can and will assume there is a node j with $(\alpha_j,\beta) = 1$, $w_{\beta,k} = r_j w_{\beta-\alpha_j,k}$ and $i \sim j$. Then $(\alpha_i, \beta - \alpha_j) = 1$.

(ii) This follows from Lemma 5.4.

(iii) Here $\widehat{w_{\beta,k}} = \widehat{ji}\widehat{w_{\beta-\alpha_j-\alpha_i,k}}$ and the statement follows from the braid relation $\widehat{iji} = \widehat{jij}$. □

Theorem 5.6. *Let B be a BMW-algebra of type $M \in$ ADE, let $\beta \in \Phi^+$, and let i, k be nodes of M.*

If $(\alpha_i,\beta) = -1$, *then*

$$\hat{i}\widehat{w_{\beta,k}}e_k = \begin{cases} \widehat{w_{\beta+\alpha_i,k}}e_k & \text{if } i \notin \mathrm{Geod}(k,\beta), \\ \widehat{w_{\beta+\alpha_i,k}}e_k - m\widehat{w_{\beta,k}}e_k + m\widehat{w_{\alpha_i,k}}e_k\widehat{w_{\beta,h}} & \text{if } i \in \mathrm{Geod}(k,\beta) \text{ and} \\ & \quad h = \mathrm{Proj}(k,\beta). \end{cases}$$

If $(\alpha_i,\beta) = 1$, *then*

$$\hat{i}\widehat{w_{\beta,k}}e_k = \begin{cases} \widehat{w_{\beta-\alpha_i,k}}e_k - m\widehat{w_{\beta,k}}e_k + ml^{-1}e_i\widehat{w_{\beta-\alpha_i,k}}e_k & \text{if } i \in J_{\beta,k}, \\ \widehat{w_{\beta-\alpha_i,k}}e_k & \text{if } i \notin J_{\beta,k}. \end{cases}$$

If $(\alpha_i,\beta) = 0$, *then*

$$\hat{i}\widehat{w_{\beta,k}}e_k = \begin{cases} \widehat{w_{\beta,k}}e_k(\widehat{w_{\beta,k}}^{-1}\hat{i}\widehat{w_{\beta,k}}) & \text{if } i \notin \mathrm{Supp}(\beta), \\ \widehat{ji}\widehat{w_{\beta-\alpha_j,k}}e_k & \text{if } j \in J_{\beta,k} \cap i^\perp, \\ \widehat{w_{\beta,k}}^{-\mathrm{op}}e_k(\widehat{w^{\mathrm{op}}_{\beta,k}}\hat{i}\widehat{w_{\beta,k}}) & \text{if } C_{\beta,i} = 1,\ i = \mathrm{Proj}(\beta,k), \\ & \quad \text{and } J_{\beta,k} \cap i^\perp = \emptyset, \\ \widehat{jij}\widehat{w_{\beta-\alpha_j-\alpha_i,k}}e_k & \text{if } j \in J_{\beta,k} \setminus i^\perp \text{ and } i \in J_{\beta-\alpha_j,k}. \end{cases}$$

If $(\alpha_i,\beta) = 2$, *then* $\beta = \alpha_i$ *and* $\hat{i}\widehat{w_{\beta,k}}e_k = l^{-1}\widehat{w_{\beta,k}}e_k$.

In each case, the result is in $\widehat{w_{\gamma,k}}Z_k^{(0)} + m\widehat{w_{\beta,k}}Z_k^{(0)} + m\widehat{w_{\alpha_i,k}}Z_k^{(0)} + I_2$, *where* $\gamma = \beta$ *if* $\beta = \alpha_i$ *and* $\gamma = r_i\beta$ *otherwise.*

Proof. By Lemma 5.3 the theorem holds for all fundamental roots β in Φ^+. Suppose β is a non-fundamental root in Φ^+, and consider $\hat{i}\widehat{w_{\beta,k}}e_k$. Now $(\alpha_i,\beta)<2$, for otherwise $\beta=\alpha_i$. First let $(\alpha_i,\beta)=1$. If $i\in J_{\beta,k}$, then

$$\hat{i}\widehat{w_{\beta,k}}e_k=\hat{i}^2\widehat{w_{\beta-\alpha_i,k}}e_k=\widehat{w_{\beta-\alpha_i,k}}e_k-m\widehat{w_{\beta,k}}e_k+ml^{-1}e_i\widehat{w_{\beta-\alpha_i,k}}e_k.$$

Assume $i\notin J_{\beta,k}$ then $i=\mathrm{Proj}(k,\beta)$ and $C_{\beta,i}=1$. There must be a single node $j\in\mathrm{Supp}(\beta)\setminus i^{\perp}$ with $C_{\beta,j}=1$, and the remaining nodes in the support of β are on the side of j in M other than i. This means $\widehat{w_{\beta,k}}=\hat{u}\hat{j}\widehat{w_{\alpha_i,k}}$ where the elements in u are on the side of j other than i and so i commutes with u. Now $\hat{i}\hat{u}\hat{j}\widehat{w_{\alpha_i,k}}=\hat{u}\hat{i}\hat{j}\widehat{w_{\alpha_i,k}}=\hat{u}\widehat{w_{\alpha_j,k}}$ so $\hat{i}\widehat{w_{\beta,k}}e_k=\widehat{w_{\beta-\alpha_i,k}}e_k$ as required.

Next let $(\alpha_i,\beta)=0$ and assume i is not in the support of β. Put $h=\mathrm{Proj}(k,\beta)$ and $\rho=\mathrm{Geod}(k,\beta)$. If i is not in ρ and not adjacent to an element of ρ, then $\hat{i}$ commutes with $\widehat{w_{\beta,k}}$ so $\widehat{w_{\beta,k}}^{-1}\hat{i}\widehat{w_{\beta,k}}=\hat{i}$ and $\hat{i}\widehat{w_{\beta,k}}e_k=\widehat{w_{\beta,k}}e_k\hat{i}$.

If i is in ρ or adjacent to an element of ρ, then $\hat{i}$ commutes with $\widehat{w_{\beta,h}}$ where $\widehat{w_{\beta,k}}=\widehat{w_{\beta,h}}\widehat{w_{\alpha_h,k}}$. Now

$$\widehat{w_{\beta,k}}^{-1}\hat{i}\widehat{w_{\beta,k}}=\widehat{w_{\alpha_h,k}}^{-1}\hat{i}\widehat{w_{\alpha_h,k}}.$$

We know that $i\not\sim h$ so $\widehat{w_{\alpha_h,k}}^{-1}\hat{i}\widehat{w_{\alpha_h,k}}e_k\in Z_k^{(0)}$ by Lemma 5.3. We conclude

$$\begin{aligned}\hat{i}\widehat{w_{\beta,k}}e_k&=\widehat{w_{\beta,k}}\widehat{w_{\beta,k}}^{-1}\hat{i}\widehat{w_{\beta,k}}e_k=\widehat{w_{\beta,k}}\widehat{w_{\alpha_h,k}}^{-1}\hat{i}\widehat{w_{\alpha_h,k}}e_k\\&=\widehat{w_{\beta,k}}e_k\big(\widehat{w_{\alpha_h,k}}^{-1}\hat{i}\widehat{w_{\alpha_h,k}}\big)\in\widehat{w_{\beta,k}}Z_k^{(0)}.\end{aligned}$$

If $(\alpha_i,\beta)=0$ with $i\in\mathrm{Supp}(\beta)$, then the assertion follows from Lemma 5.5.

Finally let $(\alpha_i,\beta)=-1$. Here $\hat{i}\widehat{w_{\beta,k}}e_k=\widehat{w_{\beta+\alpha_i,k}}e_k$ by definition if i is not in $\mathrm{Geod}(k,\beta)$. So suppose $i\in\mathrm{Geod}(k,\beta)$. Write $h=\mathrm{Proj}(k,\beta)$. Since $(\alpha_i,\beta)=-1$, we must have $i\sim h$. Therefore $\widehat{w_{\beta,k}}=\widehat{w_{\beta,h}}\widehat{w_{kh}}$ and $\widehat{w_{\beta,k}}e_k=\widehat{w_{\beta,h}}e_he_i\cdots e_k$. The set $\mathrm{Supp}(\beta)\setminus\{h\}$ is a connected component of the Dynkin diagram connected to h and disconnected from $\mathrm{Geod}(k,\beta)$. Hence $\hat{h}$ does not appear in $\widehat{w_{\beta,h}}$. This means $\hat{i}$ commutes with $\widehat{w_{\beta,h}}$. Moreover, by definition of $w_{\beta,h}$, we have $\widehat{w_{\beta,h}}\hat{h}=\widehat{w_{\beta+\alpha_i,i}}$ and so $\widehat{w_{\beta,h}}\hat{h}\widehat{w_{ki}}=\widehat{w_{\beta+\alpha_i,k}}$. Consequently, by (9),

$$\begin{aligned}\hat{i}\widehat{w_{\beta,h}}e_he_i\cdots e_k&=\widehat{w_{\beta,h}}\hat{i}e_he_i\cdots e_k=\widehat{w_{\beta,h}}\big(\hat{h}+m(1-e_h)\big)e_i\cdots e_k\\&=\widehat{w_{\beta,h}}\hat{h}\widehat{w_{ki}}e_k+m\widehat{w_{ki}}e_k\widehat{w_{\beta,h}}-m\widehat{w_{\beta,h}}e_h\widehat{kh}\\&=\widehat{w_{\beta+\alpha_i,k}}e_k+m\widehat{w_{ki}}e_k\widehat{w_{\beta,h}}-m\widehat{w_{\beta,k}}e_k.\qquad\Box\end{aligned}$$

Corollary 5.7. *Let B be a BMW-algebra of type $M\in$ ADE, let $\beta\in\Phi^+$, and let i,k be nodes of M.*

(i) $\widehat{w_{\beta,k}}^{-\mathrm{op}}e_k\in\widehat{w_{\beta,k}}e_k+m\sum_{\mathrm{ht}(\gamma)<\mathrm{ht}(\beta)}\widehat{w_{\gamma,k}}Z_k^{(0)}+I_2$,

(ii) $e_i\widehat{w_{\beta,k}}e_k\in\widehat{w_{\alpha_i,k}}Z_k^{(0)}+I_2$,

(iii) $\hat{i}\,\widehat{w_{\beta,k}}e_k \in \sum_{\gamma\in H_{\beta,i}} \widehat{w_{\gamma,k}}Z_k^{(0)} + I_2$,

where $H_{\beta,i} = \{\delta \in \Phi^+ \mid \mathrm{ht}(\delta) < \mathrm{ht}(\beta)\} \cup \{\beta, \beta+\alpha_i\} \cap \Phi^+$.

Proof. We prove the statements simultaneously by induction on the height of β. If β is a fundamental root then statement (i) holds by Lemma 3.1 and the statements (ii) and (iii) by Lemma 5.3.

Let $\beta \in \Phi^+$ with $\mathrm{ht}(\beta) \geqslant 2$ and assume the lemma holds for all $\gamma \in \Phi^+$ with $\mathrm{ht}(\gamma) < \mathrm{ht}(\beta)$. Let i, k be nodes and consider $\widehat{w_{\beta,k}}^{-\mathrm{op}}e_k$, $e_i\widehat{w_{\beta,k}}e_k$ and $\hat{i}\,\widehat{w_{\beta,k}}e_k$. There is (at least one) j such that $\widehat{w_{\beta,k}} = \hat{j}\,\widehat{w_{\beta-\alpha_j,k}}$; then $\mathrm{ht}(\beta-\alpha_j) = \mathrm{ht}(\beta) - 1$. Now

$$\begin{aligned}
\widehat{w_{\beta,k}}^{-\mathrm{op}}e_k &= \hat{j}^{-1}\widehat{w_{\beta-\alpha_j,k}}^{-\mathrm{op}}e_k \\
&\in \bigl(\hat{j} + m - me_j\bigr)\Biggl(\widehat{w_{\beta-\alpha_j,k}}e_k + m\sum_{\mathrm{ht}(\gamma)<\mathrm{ht}(\beta-\alpha_j)} \widehat{w_{\gamma,k}}Z_k^{(0)} + I_2\Biggr) \\
&= \widehat{w_{\beta,k}}e_k + m\widehat{w_{\beta-\alpha_j,k}}e_k - me_j\widehat{w_{\beta-\alpha_j,k}}e_k + m\sum_{\mathrm{ht}(\gamma)<\mathrm{ht}(\beta-\alpha_j)} \hat{j}\,\widehat{w_{\gamma,k}}Z_k^{(0)} \\
&\quad + m^2\sum_{\mathrm{ht}(\gamma)<\mathrm{ht}(\beta-\alpha_j)} \widehat{w_{\gamma,k}}Z_k^{(0)} - m^2\sum_{\mathrm{ht}(\gamma)<\mathrm{ht}(\beta-\alpha_j)} e_j\widehat{w_{\gamma,k}}Z_k^{(0)} + I_2 \\
&\subseteq \widehat{w_{\beta,k}}e_k + m\sum_{\mathrm{ht}(\gamma)<\mathrm{ht}(\beta)} \widehat{w_{\gamma,k}}Z_k^{(0)} + m\sum_{\mathrm{ht}(\gamma)<\mathrm{ht}(\beta)} e_j\widehat{w_{\gamma,k}}Z_k^{(0)} + I_2 \\
&\subseteq \widehat{w_{\beta,k}}e_k + m\sum_{\mathrm{ht}(\gamma)<\mathrm{ht}(\beta)} \widehat{w_{\gamma,k}}Z_k^{(0)} + I_2.
\end{aligned}$$

To see that $\sum_{\mathrm{ht}(\gamma)<\mathrm{ht}(\beta-\alpha_j)} \hat{j}\,\widehat{w_{\gamma,k}}Z_k^{(0)}$ is contained in $\sum_{\mathrm{ht}(\gamma)<\mathrm{ht}(\beta)} \widehat{w_{\gamma,k}}Z_k^{(0)}$, observe that by the induction hypothesis on (iii) we have

$$\hat{j}\,\widehat{w_{\gamma,k}}e_k \in \sum_{\delta\in H_{\gamma,i}} \widehat{w_{\delta,k}}Z_k^{(0)} + I_2.$$

Here $\mathrm{ht}(\delta) \leqslant \mathrm{ht}(\gamma)+1 < \mathrm{ht}(\beta)$ while $\mathrm{ht}(\gamma) < \mathrm{ht}(\beta-\alpha_j)$. The sum $\sum_{\mathrm{ht}(\gamma)<\mathrm{ht}(\beta)} e_j\widehat{w_{\gamma,k}}Z_k^{(0)}$ is in $\widehat{w_{\alpha_j,k}}Z_k^{(0)}$ by our induction hypothesis on (ii) and this gives (i) for β.

Now focus on $e_i\widehat{w_{\beta,k}}e_k = e_i\hat{j}\,\widehat{w_{\beta-\alpha_j,k}}e_k$. If $i = j$ then, by the induction hypothesis,

$$e_i\widehat{w_{\beta,k}}e_k = l^{-1}e_i\widehat{w_{\beta-\alpha_j,k}}e_k \in \widehat{w_{\alpha_i,k}}Z_k^{(0)} + I_2.$$

If $i \not\sim j$ then

$$e_i\hat{j}\,\widehat{w_{\beta-\alpha_j,k}}e_k = \hat{j}e_i\widehat{w_{\beta-\alpha_j,k}}e_k \in \hat{j}\,\widehat{w_{\alpha_i,k}}Z_k^{(0)} + I_2$$

and by Lemma 5.3 this is contained in $\widehat{w_{\alpha_i,k}}Z_k^{(0)} + I_2$.

So, for the remainder of the proof, we may (and shall) assume $i \sim j$. By (9), we have $e_i \hat{j} = e_i e_j \hat{i} + m e_i e_j - m e_i$, so

$$e_i \widehat{w_{\beta,k}} e_k = e_i \hat{j} \widehat{w_{\beta-\alpha_j,k}} e_k = e_i e_j \hat{i} \widehat{w_{\beta-\alpha_j,k}} e_k + m e_i e_j \widehat{w_{\beta-\alpha_j,k}} e_k - m e_i \widehat{w_{\beta-\alpha_j,k}} e_k.$$

By our induction hypothesis the last two terms are in $\widehat{w_{\alpha_i,k}} Z_k^{(0)} + I_2$. This leaves the first term, $e_i e_j \hat{i} \widehat{w_{\beta-\alpha_j,k}} e_k$. Because $(\beta - \alpha_j, \alpha_i) = (\beta, \alpha_i) + 1$ the inner product of α_i with $\beta - \alpha_j$ can only take values 0, 1, and 2 and thus $H_{\beta-\alpha_j,i}$ consists of roots with height at most $\mathrm{ht}(\beta - \alpha_j)$.

The induction hypothesis on (iii) now gives

$$\hat{i} \widehat{w_{\beta-\alpha_j,k}} e_k \in \sum_{\gamma \in H_{\beta-\alpha_j,i}} \widehat{w_{\gamma,k}} Z_k^{(0)}$$

where $\mathrm{ht}(\gamma) < \mathrm{ht}(\beta)$ for all γ. By applying the induction hypothesis twice we obtain

$$e_i e_j \hat{i} \widehat{w_{\beta-\alpha_j,k}} e_k \in \sum_{\mathrm{ht}(\gamma) \in H_{\beta-\alpha_j,i}} e_i e_j \widehat{w_{\gamma,k}} Z_k^{(0)} + I_2 \subseteq e_i \widehat{w_{\alpha_j,k}} Z_k^{(0)} \subseteq \widehat{w_{\alpha_i,k}} Z_k^{(0)} + I_2.$$

This establishes (ii). Finally consider $\hat{i} \widehat{w_{\beta,k}} e_k$. If $(\alpha_i, \beta) = -1$ then $\beta + \alpha_i \in \Phi^+$ and the statement holds by Theorem 5.6. Also, if $(\alpha_i, \beta) = 1$ then Theorem 5.6 applies. Here $e_i \widehat{w_{\beta-\alpha_i,k}} e_k \in \widehat{w_{\alpha_i,k}} Z_k^{(0)} + I_2$ by the induction hypothesis for (ii).

For the remainder of the proof we assume $(\alpha_i, \beta) = 0$. Again Theorem 5.6 gives an expression for $\hat{i} \widehat{w_{\beta,k}} e_k$ in each of the four cases discerned. In the first cases, where $i \notin \mathrm{Supp}(\beta)$, the statement is immediate from this expression. By our induction hypothesis for (iii) the second case gives an expression contained in $\sum_{\gamma \in H_{\beta-\alpha_j,i}} \hat{j} \widehat{w_{\gamma,k}} Z_k + I_2$ whence in $\sum_{\gamma \in H_{\beta,i}} \widehat{w_{\gamma,k}} Z_k + I_2$. Now the fourth case goes by the same argument and only the third case remains to be verified. Above we have shown that

$$\widehat{w_{\beta,k}}^{-\mathrm{op}} e_k \in \widehat{w_{\beta,k}} e_k + m \sum_{\mathrm{ht}(\gamma) < \mathrm{ht}(\beta)} \widehat{w_{\gamma,k}} Z_k^{(0)} + I_2$$

and that completes the proof. □

We shall use the following lemma to derive an upper bound for $\dim(Z_i)$ from Theorem 5.6.

Lemma 5.8. *Suppose F is a field, E is a subring of F which is a principal ideal domain. If V is a vector space over F and $V^{(0)}$ is an E-submodule of V containing a spanning set of V, then $V^{(0)}$ is a free E-module on a basis of V. Moreover, if $a \in E$ generates a maximal ideal of E, then*

$$\dim_F(V) = \dim_{E/aE}\big(V^{(0)}/aV^{(0)}\big).$$

Proof. As E is a principal ideal domain, it is well known, see [10, Theorem 12.5], that each E-module of finite rank without torsion is free. Applying this observation to $V^{(0)}$, we let X be a basis of the E-module $V^{(0)}$. By the hypothesis that $V^{(0)}$ spans V, it is also a basis of V, so $\dim_F(V) = |X|$. On the other hand, X maps onto a basis of $V^{(0)}/aV^{(0)}$ over E/aE (for, it clearly maps onto a spanning set and if $\sum_{x\in X} \lambda_x x = 0 \bmod aV^{(0)}$ for $\lambda_x \in E$, then, as $V^{(0)} = EX$, with X a basis, we have $\lambda_x = 0 \bmod a$ for each $x \in X$, so the linear relation in $V^{(0)}/aV^{(0)}$ is the trivial one). This proves $\dim_F(V) = |X| = \dim_{E/aE}(V^{(0)}/aV^{(0)})$. □

Corollary 5.9. *Let $M \in$ ADE and let i be a node of M. Then $\widehat{D_i} Z_i \widehat{D_i}^{\mathrm{op}}$ is a linear spanning set for I_1/I_2. Moreover, the dimension of Z_i is at most $|W_C|$.*

Proof. By Lemma 3.6 I_1 is spanned by a set of multiples of e_i by generators g_j, so $I_1 = Be_iB$. According to Theorem 5.6 and Corollary 5.7, $Be_i = \widehat{D_i} Z_i + I_2$. Applying Remark 2.2, we derive from this that $e_iB = Z_i \widehat{D_i}^{\mathrm{op}} + I_2$ (observe that Z_i and I_2 are invariant under the anti-involution). Therefore, $I_1 = Be_iB = \widehat{D_i} Z_i \widehat{D_i}^{\mathrm{op}} + I_2$.

It remains to establish that the dimension of Z_i mod I_2 is at most $|W_C|$. To this end we consider the integral versions $Z_i^{(0)}$ and $B^{(0)}$ of Z_i and B over $E = \mathbb{Q}(x)[l^{\pm}]$ defined at the beginning of Section 5, and look at the quotients modulo $(l-1)$. Observe that, by (1), m belongs to the ideal $(l-1)E$.

A careful inspection of the identities in Theorem 5.6 and Corollary 5.7, shows

$$B^{(0)}e_i = \widehat{D_i} Z_i^{(0)} + I_2, \quad \text{and} \quad e_iB^{(0)}e_i = Z_i^{(0)} + I_2.$$

Since Be_i is linearly spanned by the set $\widehat{D_i} Z_i^{(0)}$ mod I_2, it is linearly spanned by $B_i^{(0)}e_i$ mod I_2. Consequently, $Z_i = e_iBe_i$ is linearly spanned by $e_iB_i^{(0)}e_i + I_2$, whence by $Z_i^{(0)}$ mod I_2.

For brevity of notation, we set $m_1 = l-1$. (The remainder of the proof would also work for $m_1 = l+1$.) Since $x^{-1}e_i$ is a central idempotent belonging to $Z_i^{(0)}$, we have

$$\begin{aligned} m_1B^{(0)} \cap \left(Z_i^{(0)} + I_2\right) &= m_1e_iB^{(0)}e_i \cap \left(Z_i^{(0)} + I_2\right) = m_1\left(Z_i^{(0)} + I_2\right) \cap \left(Z_i^{(0)} + I_2\right) \\ &= m_1\left(Z_i^{(0)} + I_2\right). \end{aligned}$$

Therefore, the quotient $Z_i^{(0)}/m_1Z_i^{(0)}$, viewed as a vector space over $\mathbb{Q}(x)$, is isomorphic to $(Z_i^{(0)} + m_1B_i^{(0)} + I_2)/(m_1B_i^{(0)} + I_2)$. But this algebra is readily seen to be a quotient of a subalgebra of the group algebra over $\mathbb{Q}(x)$ of the stabilizer in W of the simple root α_i, for the image of $\{\widehat{w} \mid w \in W\}$ modulo $m_1B^{(0)}$ is the group W and the image of the algebra $Z_i^{(0)}$ is generated by the products of the elements of the form $w_{ji}r_kw_{ji}$ for j and k distinct nonadjacent nodes of M, all of which are contained in the stabilizer in W of α_i. Consequently, the dimension of $Z_i^{(0)}/m_1Z_i^{(0)}$ over $\mathbb{Q}(x)$ is at most $|W_C|$, the order of the stabilizer in W of α_0 (a group conjugate to the stabilizer in W of α_i). By Lemma 5.8, applied with $F = \mathbb{Q}(x,l)$, $E = \mathbb{Q}(x)[l^{\pm}]$, $V = Z_i$, $V^{(0)} = Z_i^{(0)}$, and $a = m_1$, we see that Z_i has dimension at most $|W_C|$ over $\mathbb{Q}(l,x)$. □

6. Generalized Lawrence–Krammer representations

In this section we construct the analog of the Lawrence–Krammer representation of A with coefficients in Z_0, the Hecke algebra of type C, where C is the parabolic of the highest root centralizer. We show the representation factors through B/I_2. By taking an irreducible representation of Z_0, we find an irreducible representation of B/I_2. Finally, by counting dimensions of irreducible representations, we are able to conclude that all representations of B/I_2 that do not vanish on I_1 are of this generalized Lawrence–Krammer type, and we can finish the proof of Theorem 1.2.

Since the construction for disconnected M is a direct sum of the representations of B for the distinct connected components, we simply take M to be connected, so $M \in \mathrm{ADE}$. We let Φ be the root system in $\mathbb{R}^n$ of type M, and denote by $\alpha_1, \dots, \alpha_n$ the fundamental roots corresponding to the reflections $r_1, \dots, r_n$, respectively. As usual, by Φ^+ we denote the set of positive roots in Φ.

For a root β, the set of roots $\{\gamma \in \Phi \mid (\beta, \gamma) = 0\}$ is also a root system. Its type can be read off from M as follows: the extended Dynkin diagram $\widetilde{K}$ of the connected component K of M involving β (i.e., having nodes in the support of β) has a single node α_0 in addition to those of K; now take C to consist of all nodes of M that are not connected to α_0. Then the type of the roots orthogonal to β is $M|_C$. In fact, if $\beta = \alpha_0$, then $\{\alpha_i \mid i \in C\}$ is a set of fundamental roots of the root system $\{\gamma \in \Phi \mid (\beta, \gamma) = 0\}$. For A_n with $\beta = \alpha_0$ this is the diagram of type A_{n-2} on $\{2, \dots, n-1\}$, for D_n, it is the diagram of type $\mathrm{A}_1\mathrm{D}_{n-2}$ on $\{1\} \cup \{3, \dots, n\}$, for E_6 it is the diagram of type A_5 on $\{1, 3, 4, 5, 6\}$, for E_7 it is the diagram of type D_6 on $\{2, 3, 4, 5, 6, 7\}$, and for E_8 it is the diagram of type E_7 on $\{1, 2, 3, 4, 5, 6, 7\}$. Here we have used the labeling of [3].

Recall the coefficients of Z_0 are in $\mathbb{Q}(l, x)$. We take the coefficients of our representation in the Hecke algebra $Z_0^{(0)}$ of type $M|_C$ over the subdomain $\mathbb{Q}[l^{\pm 1}, m]$ of $\mathbb{Q}(l, x)$, where m is defined in (1). Observe that the fraction field of $\mathbb{Q}(l, m)$ coincides with $\mathbb{Q}(l, x)$. The generators z_i $(i \in C)$ of $Z_0^{(0)}$ satisfy the quadratic relations $z_i^2 + m z_i - 1 = 0$. For the proof of irreducibility at the end of this section, we need however a smaller version of this Hecke algebra, namely the subalgebra $Z_0^{(1)}$ with same generators z_i, but over $\mathbb{Q}[m]$. Thus, $Z_0^{(0)} = Z_0^{(1)}\mathbb{Q}[l^{\pm 1}]$.

By Lemma 3.8, the element $h_{\beta,i}$ of A defined in (15), where $\beta \in \Phi^+$ and i is a node with $(\alpha_i, \beta) = 0$, maps onto an element of $Z_0^{(1)}$ upon substitution of s_j by z_j and s_j^{-1} by $z_j + m$. We shall also write $h_{\beta,i}$ for the image of this element in $Z_0^{(1)}$.

We write $V^{(0)}$ for the free right $Z_0^{(0)}$ module with basis x_β indexed by $\beta \in \Phi^+$. The connection with [7] is given by $m = r - r^{-1}$, $l = 1/(tr^3)$. Recall that A^+ is the positive monoid of A.

Theorem 6.1. *Let $M \in \mathrm{ADE}$ and let A be the Artin group of type M. Then, for each $i \in \{1, \dots, n\}$ and each $\beta \in \Phi^+$, there are elements $T_{i,\beta}$ in $Z_0^{(1)}$ such that the following map on the generators of A determines a representation of A on $V^{(0)}$.*

$$s_i \mapsto \sigma_i = \tau_i + l^{-1} T_i,$$

where τ_i is determined by

$$\tau_i(x_\beta) = \begin{cases} 0 & \textit{if } (\alpha_i, \beta) = 2, \\ x_{\beta-\alpha_i} & \textit{if } (\alpha_i, \beta) = 1, \\ x_\beta h_{\beta,i} & \textit{if } (\alpha_i, \beta) = 0, \\ x_{\beta+\alpha_i} - m x_\beta & \textit{if } (\alpha_i, \beta) = -1, \end{cases}$$

and where T_i is the $Z_0^{(0)}$-linear map on $V^{(0)}$ determined by $T_i x_\beta = x_{\alpha_i} T_{i,\beta}$ on the generators of $V^{(0)}$ and by $T_{i,\alpha_i} = 1$.

When tensored with $\mathbb{Q}(x,l)$, the representation of A on $V^{(0)}$ becomes a representation on the vector space V which factors through the quotient B/I_2 of the BMW algebra B of type M over $\mathbb{Q}(x,l)$.

Throughout this section we use several properties of the elements $h_{\beta,i}$ listed in Lemma 3.7. In addition, we shall use the Hecke algebra relation for the image of $h_{\beta,i}$ in $Z_0^{(0)}$:

$$h_{\beta,i}^{-1} = h_{\beta,i} + m. \tag{24}$$

The proof of the theorem follows the lines of the proof in [7]. We shall first describe the part modulo l^{-1} of the representation of the Artin monoid A^+ on $V^{(0)}$.

Lemma 6.2. *There is a monoid homomorphism $A^+ \to \mathrm{End}(V^{(0)})$ determined by $s_i \mapsto \tau_i$ $(i = 1, \ldots, n)$.*

Proof. We must show that, if i and j are not adjacent, then $\tau_i \tau_j = \tau_j \tau_i$ and, if they are adjacent, then $\tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j$. We evaluate the expressions on each x_β and show they are equal. We begin with the case where $\beta = \alpha_i$. Suppose first that i and j are not adjacent. Then $\tau_i x_{\alpha_i} = 0$ and $\tau_j x_{\alpha_i} = x_{\alpha_i} h_{\beta,j}$. Now $\tau_j \tau_i x_{\alpha_i} = 0$ and $\tau_i \tau_j x_{\alpha_i} = \tau_i x_{\alpha_i} h_{\beta,j} = 0$, so the result holds. Suppose next that i and j are adjacent. Then $\tau_i x_{\alpha_i} = \tau_j x_{\alpha_j} = 0$ and $\tau_j x_{\alpha_i} = -m x_{\alpha_i} + x_{\alpha_i+\alpha_j}$. Now

$$\tau_i \tau_j \tau_i x_{\alpha_i} = \tau_i \tau_j (0) = 0, \quad \text{and}$$
$$\tau_j \tau_i \tau_j x_{\alpha_i} = \tau_j \tau_i (-m x_{\alpha_i} + x_{\alpha_i+\alpha_j}) = \tau_j (x_{\alpha_i+\alpha_j-\alpha_i}) = \tau_j x_{\alpha_j} = 0.$$

This ends the verification for the case where $\beta = \alpha_i$. We now divide the verifications into the various cases depending on the inner products (α_i, β) and (α_j, β). By the above, we may assume $(\alpha_i, \beta), (\alpha_j, \beta) \neq 2$.

First assume that $(\alpha_i, \alpha_j) = 0$. The computations verifying $\tau_i \tau_j = \tau_j \tau_i$ are summarized in the following table. The last column indicates the formulas that are used.

(α_i,β)	(α_j,β)	$\tau_i\tau_j x_\beta = \tau_j\tau_i x_\beta$	ref.
1	1	$x_{\beta-\alpha_i-\alpha_j}$	
1	−1	$x_{\beta+\alpha_j-\alpha_i} - m x_{\beta-\alpha_i}$	
1	0	$x_{\beta-\alpha_i} h_{\beta,j}$	(18)
0	0	$x_\beta h_{\beta,i} h_{\beta,j}$	(16)
0	−1	$x_{\beta+\alpha_j} h_{\beta,i} - m x_\beta h_{\beta,i}$	(18)
−1	−1	$m^2 x_\beta - m(x_{\beta+\alpha_i} + x_{\beta+\alpha_j}) + x_{\beta+\alpha_i+\alpha_j}$	

We demonstrate how to derive these expressions by checking the third line.

$$\tau_i\tau_j x_\beta = \tau_i(x_\beta h_{\beta,j}) = x_{\beta-\alpha_i} h_{\beta,j}.$$

In the other order,

$$\tau_j\tau_i x_\beta = \tau_j(x_{\beta-\alpha_i}) = x_{\beta-\alpha_i} h_{\beta-\alpha_i,j}.$$

Equality between $h_{\beta,j}$ and $h_{\beta-\alpha_i,j}$ follows from (18).

Suppose next that $i \sim j$. The same situation occurs except the computations are sometimes longer and one case does not occur. This is the case where $(\alpha_i,\beta) = (\alpha_j,\beta) = -1$. For then $\beta+\alpha_i$ is also a root, and $(\beta+\alpha_i,\alpha_j) = -1-1 = -2$. This means $\beta+\alpha_i = -\alpha_j$ and β is not a positive root. The table is as follows.

(α_i,β)	(α_j,β)	$\tau_i\tau_j\tau_i x_\beta = \tau_j\tau_i\tau_j x_\beta$	ref.
1	1	0	
1	−1	$x_\beta h_{\beta-\alpha_i,j} - m x_{\beta-\alpha_i} h_{\beta-\alpha_i,j}$	(19)
1	0	$x_{\beta-\alpha_i-\alpha_j} h_{\beta,j}$	(20)
0	0	$x_\beta h_{\beta,i} h_{\beta,j} h_{\beta,i}$	(17)
0	−1	$x_{\beta+\alpha_i+\alpha_j} h_{\beta,i} - m x_{\beta+\alpha_j} h_{\beta,i} - m x_\beta h^2_{\beta,i}$	(21), (24)
−1	−1	does not occur	

Lemma is proved. □

We next study the possibilities for the parameters $T_{k,\beta}$ occurring in Theorem 6.1. Recall that there we defined $\sigma_k = \tau_k + l^{-1}T_k$, where $T_k x_\beta = x_{\alpha_k} T_{k,\beta}$. We shall introduce $T_{k,\beta}$ as elements of the Hecke algebra $Z_0^{(0)}$ of type $M|_C$.

Proposition 6.3. *Set $T_{i,\alpha_i} = 1$ for all $i \in \{1,\ldots,n\}$. For $\sigma_i \mapsto \tau_i + l^{-1}T_i$ to define a linear representation of the group A on V, it is necessary and sufficient that the equations in Table* 1 *are satisfied for each $k, j = 1,\ldots,n$ and each $\beta \in \Phi^+$.*

Proof. The σ_k should satisfy the relations (B1), (B2). Substituting $\tau_k + l^{-1}T_k$ for σ_k, we find relations for the coefficients of l^{-i} with $i = 0,1,2,3$. The constant part involves only the τ_k. It follows from Lemma 6.2 that these equations are satisfied. We shall derive all of the equations of Table 1 below except for (39) from the l^{-1}-linear part and the remaining one from the l^{-1}-quadratic part of the relations.

The coefficients of l^{-1} lead to

$$T_i\tau_j = \tau_j T_i \quad \text{and} \quad T_j\tau_i = \tau_i T_j \quad \text{if } i \not\sim j, \tag{25}$$

$$\tau_j T_i \tau_j + T_j \tau_i \tau_j + \tau_j \tau_i T_j = \tau_i T_j \tau_i + T_i \tau_j \tau_i + \tau_i \tau_j T_i \quad \text{if } i \sim j. \tag{26}$$

We focus on the consequences of these equations for the $T_{i,\beta}$. First consider the case where $i \not\sim j$. Then $\tau_i x_{\alpha_j} = x_{\alpha_j} h_{\alpha_j,i}$ and so, for the various values of (α_i, β) we find the following equations

(α_i, β)	$T_j\tau_i x_\beta = \tau_i T_j x_\beta$	equation
0	$x_{\alpha_j} T_{j,\beta} h_{\beta,i} = x_{\alpha_j} h_{\alpha_j,i} T_{j,\beta}$	$T_{j,\beta} h_{\beta,i} = h_{\alpha_j,i} T_{j,\beta}$
1	$x_{\alpha_j} T_{j,\beta-\alpha_i} = x_{\alpha_j} h_{\alpha_j,i} T_{j,\beta}$	$T_{j,\beta-\alpha_i} = h_{\alpha_j,i} T_{j,\beta}$
−1	$x_{\alpha_j} T_{j,\beta+\alpha_i} - m x_{\alpha_j} T_{j,\beta} = x_{\alpha_j} h_{\alpha_j,i} T_{j,\beta}$	$T_{j,\beta+\alpha_i} = h^{-1}_{\alpha_j,i} T_{j,\beta}$
2	$0 = x_{\alpha_j} h_{\alpha_j,i} T_{j,\beta}$	$0 = T_{j,\beta}$

The first equation gives

$$T_{j,\beta} h_{\beta,i} = h_{\alpha_j,i} T_{j,\beta} \tag{27}$$

and the second

$$T_{j,\beta} = h^{-1}_{\alpha_j,i} T_{j,\beta-\alpha_i}. \tag{28}$$

The third case gives an equation that is equivalent to (28). The fourth equation is part of (39) in Table 1 (namely the part where $j \not\sim i$).

Next, we assume $i \sim j$. A practical rule is

$$\tau_i \tau_j x_{\alpha_i} = \tau_i(-m x_{\alpha_i} + x_{\alpha_i+\alpha_j}) = x_{\alpha_j}.$$

We distinguish cases according to the values of (α_i, β) and (α_j, β). Since each inner product, for distinct roots is one of 1, 0, −1, there are six cases to consider up to interchanges of i and j. However, as in the proof of Lemma 6.2 for $i \sim j$, the case $(\alpha_i, \beta) = (\alpha_j, \beta) = -1$ does not occur.

For the sake of brevity, let us denote the images of the left-hand side and the right-hand side of (26) on x_β by LHS and RHS, respectively.

Case $(\alpha_i, \beta) = (\alpha_j, \beta) = 1$. Then $(r_i\beta, \alpha_j) = (\beta - \alpha_i, \alpha_j) = 2$, so $\beta = \alpha_i + \alpha_j$. Now

$$\text{RHS} = x_{\alpha_j}(T_{i,\beta} - m T_{j,\alpha_j}) + x_\beta T_{j,\alpha_j}.$$

Comparison with the same expression but then j and i interchanged yields LHS. This leads to the equations $T_{i,\beta} = m T_{j,\alpha_j}$ and $T_{j,\alpha_j} = T_{i,\alpha_i}$. In view of the latter, and connectedness of the diagram there is an element z in $Z_0^{(0)}$ such that

$$T_{i,\alpha_i} = z \quad \text{for all } i. \tag{29}$$

Consequently, the former equation reads

$$T_{i,\beta} = mz. \tag{30}$$

By the requirement $T_{i,\alpha_i} = 1$ in the hypotheses, we must have $z = 1$.

Case $(\alpha_i, \beta) = (\alpha_j, \beta) = 0$. This gives

$$\mathrm{RHS} = x_{\alpha_j}(T_{i,\beta} - mT_{j,\beta}h_{\beta,i}) + x_{\alpha_j+\alpha_i}T_{j,\beta}h_{\beta,i} + x_{\alpha_i}T_{i,\beta}h_{\beta,j}h_{\beta,i}$$

and LHS can be obtained from the above by interchanging the indices i and j. Comparison of each of the coefficients of $x_{\alpha_i}, x_{\alpha_j+\alpha_i}, x_{\alpha_j}$ gives

$$T_{i,\beta}h_{\beta,j} = T_{j,\beta}h_{\beta,i} \quad \text{if } (\alpha_i, \beta) = (\alpha_j, \beta) = 0 \text{ and } (\alpha_i, \alpha_j) = -1. \tag{31}$$

Since the other cases come down to similar computations, we only list the results.

Case $(\alpha_i, \beta) = 0$, $(\alpha_j, \beta) = -1$. Here we have

$$\mathrm{RHS} = x_{\alpha_i}(-mT_{i,\beta}h_{\beta,i} + T_{i,\beta+\alpha_j}h_{\beta,i}) + x_{\alpha_j}(-mT_{j,\beta}h_{\beta,i} + T_{i,\beta}) + x_{\alpha_i+\alpha_j}(T_{j,\beta}h_{\beta,i})$$

and

$$\begin{aligned}\mathrm{LHS} = {} & x_{\alpha_i}\left(m^2T_{i,\beta} + T_{j,\beta} - mT_{i,\beta+\alpha_j}\right) \\ & + x_{\alpha_j}(-mT_{j,\beta}h_{\beta,i} - mT_{j,\beta+\alpha_j} + T_{j,\beta+\alpha_j+\alpha_i}) \\ & + x_{\alpha_i+\alpha_j}(-mT_{i,\beta} + T_{i,\beta+\alpha_j}),\end{aligned}$$

which gives

$$T_{i,\beta+\alpha_j} = T_{j,\beta}h_{\beta,i} + mT_{i,\beta}, \tag{32}$$

$$T_{j,\beta+\alpha_j+\alpha_i} = T_{i,\beta} + mT_{j,\beta+\alpha_j}. \tag{33}$$

Table 1
Equations for $T_{i,\beta}$

$T_{i,\beta}$	condition	reference
0	$\beta = \alpha_j$ and $i \neq j$	(39)
1	$\beta = \alpha_i$	(29)
m	$\beta = \alpha_i + \alpha_j$	(30)
$h_{\alpha_i,j}^{-1}T_{i,\beta-\alpha_j}$	$(\alpha_j, \beta) = 1$ and $(\alpha_i, \alpha_j) = 0$	(28)
$T_{j,\beta-\alpha_i-\alpha_j} + mT_{i,\beta-\alpha_j}$	$(\alpha_i, \beta) = 0$ and $(\alpha_j, \beta) = 1$ and $(\alpha_i, \alpha_j) = -1$	(34)
$T_{j,\beta-\alpha_j}h_{\beta-\alpha_j,i} + mT_{i,\beta-\alpha_j}$	$(\alpha_i, \beta) = -1$ and $(\alpha_j, \beta) = 1$ and $(\alpha_i, \alpha_j) = -1$	(36)
$T_{j,\beta-\alpha_i}h_{\beta,j}^{-1}$	$(\alpha_i, \beta) = 1$ and $(\alpha_j, \beta) = 0$ and $(\alpha_i, \alpha_j) = -1$	(35)

Case $(\alpha_i, \beta) = 0$, $(\alpha_j, \beta) = 1$.

$$\mathrm{RHS} = x_{\alpha_i} T_{i,\beta-\alpha_j} h_{\beta,i} + x_{\alpha_j}(-mT_{j,\beta} h_{\beta,i} + T_{i,\beta}) + x_{\alpha_j+\alpha_i} T_{j,\beta} h_{\beta,i}$$

and

$$\mathrm{LHS} = x_{\alpha_i}(T_{j,\beta} - mT_{i,\beta-\alpha_j}) + x_{\alpha_j} T_{j,\beta-\alpha_j-\alpha_i} + x_{\alpha_i+\alpha_j} T_{i,\beta-\alpha_j}$$

whence

$$T_{i,\beta} = T_{j,\beta-\alpha_j-\alpha_i} + mT_{i,\beta-\alpha_j}, \tag{34}$$

$$T_{j,\beta} = T_{i,\beta-\alpha_j} h_{\beta,i}^{-1}. \tag{35}$$

Case $(\alpha_i, \beta) = 1$, $(\alpha_j, \beta) = -1$. Now

$$\mathrm{RHS} = x_{\alpha_i}(T_{i,\beta-\alpha_i} h_{\beta-\alpha_i,j}) + x_{\alpha_j}(T_{i,\beta} - mT_{j,\beta-\alpha_i}) + x_{\alpha_i+\alpha_j}(T_{j,\beta-\alpha_i})$$

and

$$\begin{aligned}\mathrm{LHS} = {} & x_{\alpha_i}\bigl(m^2 T_{i,\beta} - mT_{i,\beta+\alpha_j} + T_{j,\beta}\bigr) + x_{\alpha_j}(T_{j,\beta+\alpha_j} h_{\beta+\alpha_j,i} - mT_{j,\beta-\alpha_i}) \\ & + x_{\alpha_i+\alpha_j}(T_{i,\beta+\alpha_j} - mT_{i,\beta})\end{aligned}$$

whence

$$T_{j,\beta} = T_{i,\beta-\alpha_i} h_{\beta-\alpha_i,j} + mT_{j,\beta-\alpha_i}, \tag{36}$$

$$T_{j,\beta+\alpha_j} = T_{i,\beta} h_{\beta+\alpha_j,i}^{-1}, \tag{37}$$

$$T_{i,\beta+\alpha_j} = T_{j,\beta-\alpha_i} + mT_{i,\beta}. \tag{38}$$

We now consider the coefficients of l^{-2} and of l^{-3} in Eqs. (B1), (B2) for σ_i. We claim that, given (28)–(38), a necessary condition for the corresponding equations to hold is

$$T_{k,\alpha_j} = 0 \quad \text{if } k \neq j. \tag{39}$$

To see this, note that, if $k \not\sim j$, the coefficient of l^{-2} gives $T_k T_j = T_j T_k$ which, applied to x_{α_j}, yields (39). If $k \sim j$, note

$$T_k \tau_j x_{\alpha_k} = T_k(-mx_{\alpha_k} + x_{\alpha_k+\alpha_j}) = 0$$

as $T_{k,\alpha_k+\alpha_j} = mz = mT_{k,\alpha_k}$ by (30). Now use the action of

$$T_j \tau_k T_j + \tau_j T_k T_j + T_j T_k \tau_j = T_k \tau_j T_k + \tau_k T_j T_k + T_k T_j \tau_k$$

on x_{α_j}. We see only the middle terms do not vanish because of the relation above and so

$$\tau_j x_{\alpha_k} T_{k,\alpha_j} z = \tau_k x_{\alpha_j} T_{j,\alpha_k} T_{k,\alpha_j}.$$

By considering the coefficient of x_{α_k}, which occurs only on the left-hand side, we see that (39) holds.

A consequence of this is that $T_i T_j = 0$ if $i \neq j$. Now all the equations for the l^{-2} and l^{-3} coefficients are easily satisfied. In the non-commuting case of l^{-2}, the first terms on either side are 0 by the relation above and the other terms are 0 as $T_j T_k = 0$.

We have seen that, in order for $s_i \mapsto \sigma_i$ to determine a representation, the $T_{i,\beta}$ have to satisfy Eqs. (27)–(39). This system of equations, however, is redundant. Indeed, when the root in the index of the left-hand side of (32) is set to γ, we obtain (36) for γ instead of β. Similarly, (33) is equivalent to (34), while (37) is equivalent to (35), and (38) is equivalent to (34). Consequently, in order to finish the proof that Table 1 contains a sufficient set of relations, we must show that (31) and (27) follow from those of the table. These proofs are given in Lemmas 6.5 and 6.7 below.

It remains to establish that the matrices σ_k are invertible. To prove this, we observe that the linear transformation $\sigma_k^2 + m\sigma_k - 1$ maps V onto the submodule spanned by x_{α_k} and that the image of x_{α_k} under σ_k is $x_{\alpha_k} l^{-1}$. This is easy to establish and will be shown in Lemma 6.10 below. □

Corollary 6.4. *If the $T_{i,\beta} \in Z_0^{(0)}$ satisfy the equations in Table* 1*, then these obey the following rules.*

(i) $T_{i,\beta} = 0$ *whenever* $i \notin \mathrm{Supp}(\beta)$.
(ii) *If* $(\alpha_i, \beta) = 1$*, then* $T_{i,\beta} = m d_{\alpha_i}^{-1} s_\beta^{-1} s_i s_\beta d_\beta$.

Proof. (i) follows from (39) by use of (28) and (36). Observe that, if $i \notin \mathrm{Supp}(\beta)$ and $(\alpha_j, \beta) = 1$ for some $j \sim i$, then $j \notin \mathrm{Supp}(\beta - \alpha_j)$.

(ii) By induction on $\mathrm{ht}(\beta)$. The assertion is vacuous when $\mathrm{ht}(\beta) = 1$. Suppose $\mathrm{ht}(\beta) = 2$. Then $s_\beta = s_j s_i s_j$ for some node j adjacent to i in M. Therefore,

$$m d_{\alpha_i}^{-1} s_\beta^{-1} s_i s_\beta d_\beta = m d_{\alpha_i}^{-1} s_j^{-1} s_i^{-1} s_j^{-1} s_i s_j s_i s_j d_\beta = m d_\beta^{-1} s_j^{-1} s_j^{-1} s_i^{-1} s_j^{-1} s_i s_j s_i s_j d_\beta = m$$

and, by (30) $T_{i\,\beta} = m$, as required.

Now suppose $\mathrm{ht}(\beta) > 2$.

If j is a node distinct from i such that $(\alpha_j, \beta) = 1$, then, necessarily, $i \not\sim j$ (for otherwise $(\alpha_i, \beta - \alpha_j) = 2$, so $\beta = \alpha_i + \alpha_j$, contradicting $\mathrm{ht}(\beta) > 2$). Now (28) applies, giving

$$\begin{aligned} T_{i,\beta} &= h_{\alpha_i,j}^{-1} T_{i,\beta-\alpha_i} \quad \text{by (28)} \\ &= m d_{\alpha_i}^{-1} s_j^{-1} s_{\beta-\alpha_j}^{-1} s_i s_{\beta-\alpha_j} s_j d_\beta \quad \text{by induction} \\ &= m d_{\alpha_i}^{-1} s_\beta^{-1} s_j s_i s_j^{-1} s_\beta d_\beta \quad \text{by definition of } s_\beta \\ &= m d_{\alpha_i}^{-1} s_\beta^{-1} s_i s_\beta d_\beta \quad \text{as } s_i s_j = s_j s_i, \end{aligned}$$

as required.

Suppose l is a node distinct from i such that $(\alpha_l, \beta) = 0$ and $i \sim l$. Then (35) applies, giving

$$\begin{aligned}
T_{i,\beta} &= T_{l,\beta-\alpha_i} h_{\beta,l}^{-1} \quad \text{by (35)}\\
&= m d_{\alpha_l}^{-1} \big(s_{\beta-\alpha_i}^{-1} s_l\big) s_{\beta-\alpha_i} \big(d_{\beta-\alpha_i} d_\beta^{-1}\big) s_l^{-1} d_\beta \quad \text{by induction}\\
&= m d_{\alpha_i}^{-1} \big(s_l s_i^{-1} s_l^{-1}\big) s_\gamma^{-1} s_l s_\gamma \big(s_l s_i s_l^{-1}\big) d_\beta \quad \text{by definition of } d_\beta \text{ and } s_\beta\\
&= m d_{\alpha_i}^{-1} s_i^{-1} s_l^{-1} s_i s_\gamma^{-1} s_l s_\gamma s_i^{-1} s_l s_i d_\beta \quad \text{by the braid relation}\\
&= m d_{\alpha_i}^{-1} s_i^{-1} s_l^{-1} s_\gamma^{-1} s_i s_l s_i^{-1} s_\gamma s_l s_i d_\beta \quad \text{by Lemma 3.9}\\
&= m d_{\alpha_i}^{-1} \big(s_i^{-1} s_l^{-1} s_\gamma^{-1} s_l^{-1}\big) (s_i s_l s_\gamma s_l s_i) d_\beta \quad \text{by the braid relation}\\
&= m d_{\alpha_i}^{-1} s_\beta^{-1} s_i s_\beta d_\beta \quad \text{by definition of } s_\beta,
\end{aligned}$$

as required. □

Lemma 6.5. *The equations for β in* (31) *are consequences of the relations of Table* 1 *and those of* (31) *and* (27) *for positive roots of height less than* $\mathrm{ht}(\beta)$.

Proof. The equation says that $T_{k,\beta} h_{\beta,j} = T_{j,\beta} h_{\beta,k}$ whenever $(\alpha_k, \beta) = (\alpha_j, \beta) = 0$ and $k \sim j$. The initial case of β having height 1 is direct from (39). Suppose therefore, $\mathrm{ht}(\beta) > 1$. There exists $m \in \{1, \ldots, n\}$ such that $(\alpha_m, \beta) = 1$. If $(\alpha_m, \alpha_k) = (\alpha_m, \alpha_j) = 0$, then, by the induction hypothesis and (18), $T_{k,\beta-\alpha_m} h_{\beta,j} = T_{k,\beta-\alpha_m} h_{\beta-\alpha_m,j} = T_{j,\beta-\alpha_m} h_{\beta-\alpha_m,k} T_{j,\beta-\alpha_m} h_{\beta,k}$, so, applying (28) twice, we find

$$T_{k,\beta} h_{\beta,j} = h_{\alpha_k,m}^{-1} T_{k,\beta-\alpha_m} h_{\beta,j} = h_{\alpha_j,m}^{-1} T_{j,\beta-\alpha_m} h_{\beta,k} = T_{j,\beta} h_{\beta,k},$$

as required.

Therefore, interchanging k and j if necessary, we may assume that $j \sim m$, whence $k \not\sim m$ (as the Dynkin diagram contains no triangles). Now $\delta = \beta - \alpha_m - \alpha_j$ and $\gamma = \delta - \alpha_k$ are positive roots and $(\alpha_k, \delta) = 1$, so (28) gives $T_{m,\gamma} = h_{\alpha_m,k} T_{m,\delta}$, which, by induction on height, and (22), leads to

$$h_{\alpha_k,m}^{-1} T_{j,\gamma} = h_{\alpha_m,k}^{-1} T_{m,\gamma} h_{\gamma,j} h_{\gamma,m}^{-1} = T_{m,\delta} h_{\gamma,j} h_{\gamma,m}^{-1}.$$

Observing that, by straightforward application of the braid relations and the definition of $h_{\beta,k}$, we also have

$$\begin{aligned}
h_{\gamma,j} h_{\gamma,m}^{-1} h_{\beta,j} &= h_{\beta,k},\\
h_{\delta,m}^{-1} h_{\beta,j} &= h_{\beta-\alpha_m,k}^{-1} h_{\beta,k},
\end{aligned}$$

we derive

$$\begin{aligned}
T_{k,\beta}h_{\beta,j} &= h_{\alpha_k,m}^{-1}T_{k,\beta-\alpha_m}h_{\beta,j} \quad \text{by (28)}\\
&= h_{\alpha_k,m}^{-1}(T_{j,\gamma}+mT_{k,\delta})h_{\beta,j} \quad \text{by (34)}\\
&= T_{m,\delta}h_{\gamma,j}h_{\gamma,m}^{-1}h_{\beta,j}+mT_{k,\delta}h_{\delta,m}^{-1}h_{\beta,j} \quad \text{by the above and (27) for } \gamma,\delta\\
&= T_{m,\delta}T_{m,\delta}h_{\beta,k}+mT_{k,\delta}h_{\beta-\alpha_m,k}^{-1}h_{\beta,k} \quad \text{by the above}\\
&= (T_{m,\delta}+mT_{j,\beta-\alpha_m})h_{\beta,k} \quad \text{by (35)}\\
&= T_{j,\beta}h_{\beta,k} \quad \text{by (34)},
\end{aligned}$$

as required. □

The relation (27) is new compared to [7]. But it is superfluous. In order to see this, we first prove some auxiliary claims.

Lemma 6.6. *Let h, k be generators (or conjugates thereof) in the Hecke algebra $Z_0^{(0)}$. Then, for any $t\in Z_0^{(0)}$,*

(i) $h^{-1}t-tk^{-1}=ht-tk$,
(ii) $h^{-1}(t+h^{-1}tk^{-1})k=t+h^{-1}tk^{-1}$.

Proof. (i) Expand the left-hand side and use that $z^{-1}=z+m$ for every conjugate of a generator.

(ii) By (i), $tk+h^{-1}t=ht+tk^{-1}$. Multiplying both sides from the left by h^{-1} and pulling out a factor k at the right of the left-hand side, we find the required relation. □

Lemma 6.7. *The equations for β in* (27) *are consequences of the relations of Table* 1 *and those of* (31) *and* (27) *for positive roots of height less than* $\mathrm{ht}(\beta)$.

Proof. Suppose that the positive root β and the distinct nodes l, i satisfy $(\alpha_l,\beta)=0$ and $i\not\sim l$. By Corollary 6.4(i), we know that $T_{i,\beta}=0$ if $i\notin\mathrm{Supp}(\beta)$, so we need only consider cases where $i\in\mathrm{Supp}(\beta)$.

If $\mathrm{ht}(\beta)=1$, then, by (29) and (39), $T_{i,\beta}=0$ and there is nothing to prove unless $\beta=\alpha_i$. In the latter case $T_{i,\beta}=1$ and

$$h_{\alpha_i,l}^{-1}T_{i,\beta}h_{\beta,l}=h_{\alpha_i,l}^{-1}h_{\alpha_i,l}=1,$$

so (27) is satisfied.

If $\mathrm{ht}(\beta)=2$, then $\beta=\alpha_i+\alpha_j$ for some j and $T_{i,\beta}=m$ by (30). As α_l is orthogonal to both β and α_i, it must be orthogonal to α_j as well. Now

$$h_{\alpha_i,l}^{-1}T_{i,\beta}h_{\beta,l}=mh_{\alpha_i,l}^{-1}h_{\alpha_i+\alpha_j,l}=md_{\alpha_i}^{-1}s_l^{-1}d_{\alpha_i}d_{\alpha_i}^{-1}s_j^{-1}s_ls_jd_{\alpha_i}=m,$$

as required.

Case (28): there is a node j with $(\alpha_j, \beta) = 1$ and $(\alpha_i, \alpha_j) = 0$. Then $T_{i,\beta} = h^{-1}_{\alpha_i,j} T_{i,\beta-\alpha_j}$. If $j \not\sim l$, we find

$$\begin{aligned} h^{-1}_{\alpha_i,l} T_{i,\beta} h_{\beta,l} &= h^{-1}_{\alpha_i,l} h^{-1}_{\alpha_i,j} T_{i,\beta-\alpha_j} h_{\beta,l} \quad \text{by (28)} \\ &= h^{-1}_{\alpha_i,j} h^{-1}_{\alpha_i,l} T_{i,\beta-\alpha_j} h_{\beta-\alpha_j,l} \quad \text{by (16) and (18)} \\ &= h^{-1}_{\alpha_i,j} T_{i,\beta-\alpha_j} \quad \text{by induction} \\ &= T_{i,\beta} \quad \text{by (28).} \end{aligned}$$

If $j \sim l$, we find

$$\begin{aligned} h^{-1}_{\alpha_i,l} T_{i,\beta} h_{\beta,l} &= h^{-1}_{\alpha_i,l} h^{-1}_{\alpha_i,j} T_{i,\beta-\alpha_j} h_{\beta,l} \quad \text{by (28)} \\ &= h^{-1}_{\alpha_i,l} h^{-1}_{\alpha_i,j} h^{-1}_{\alpha_i,l} T_{i,\beta-\alpha_j-\alpha_l} h_{\beta,l} \quad \text{by (28)} \\ &= h^{-1}_{\alpha_i,j} h^{-1}_{\alpha_i,l} h^{-1}_{\alpha_i,j} T_{i,\beta-\alpha_j-\alpha_l} h_{\beta-\alpha_j-\alpha_l,j} \quad \text{by (17) and (20)} \\ &= h^{-1}_{\alpha_i,j} h^{-1}_{\alpha_i,l} T_{i,\beta-\alpha_j-\alpha_l} \quad \text{by induction} \\ &= T_{i,\beta} \quad \text{by (28) applied twice.} \end{aligned}$$

This ends case (28).

Case (34): $(\alpha_i, \beta) = 0$ and there is a node $j \sim i$ with $(\alpha_j, \beta) = 1$. Then $T_{i,\beta} = T_{j,\beta-\alpha_i-\alpha_j} + m T_{i,\beta-\alpha_j}$. Now

$$h^{-1}_{\alpha_i,l} T_{i,\beta} h_{\beta,l} = h^{-1}_{\alpha_i,l} (T_{j,\beta-\alpha_i-\alpha_j} + m T_{i,\beta-\alpha_j}) h_{\beta,l}.$$

If $j \not\sim l$, we find

$$\begin{aligned} h^{-1}_{\alpha_i,l} T_{i,\beta} h_{\beta,l} &= h^{-1}_{\alpha_i,l} (T_{j,\beta-\alpha_i-\alpha_j} + m T_{i,\beta-\alpha_j}) h_{\beta,l} \quad \text{by (34)} \\ &= h^{-1}_{\alpha_i,l} T_{j,\beta-\alpha_i-\alpha_j} h_{\beta-\alpha_i-\alpha_j,l} + m h^{-1}_{\alpha_i,l} T_{i,\beta-\alpha_j} h_{\beta-\alpha_j,l} \quad \text{by (18)} \\ &= T_{j,\beta-\alpha_i-\alpha_j} + m T_{i,\beta-\alpha_j} \quad \text{by induction} \\ &= T_{i,\beta} \quad \text{by (34).} \end{aligned}$$

If $j \sim l$, we claim

$$T_{i,\beta} = T_{l,\delta} + m\bigl(T_{j,\gamma} + h^{-1}_{\alpha_i,l} T_{j,\gamma} h^{-1}_{\beta,l}\bigr), \tag{40}$$

where $\gamma = \beta - \alpha_i - \alpha_j - \alpha_l$ and where $\delta = \gamma - \alpha_j$ are positive roots. For

$$\begin{aligned}
T_{i,\beta} &= T_{j,\beta-\alpha_i-\alpha_j} + mT_{i,\beta-\alpha_j} \quad \text{by (34)} \\
&= (T_{l,\delta} + mT_{j,\gamma}) + mh_{\alpha_i,l}^{-1} T_{i,\beta-\alpha_j-\alpha_l} \quad \text{by (34) and (28)} \\
&= T_{l,\delta} + mT_{j,\gamma} + mh_{\alpha_i,l}^{-1} T_{j,\gamma} h_{\beta-\alpha_j-\alpha_l,j}^{-1} \quad \text{by (35)} \\
&= T_{l,\delta} + mT_{j,\gamma} + mh_{\alpha_i,l}^{-1} T_{j,\gamma} h_{\beta,l}^{-1} \quad \text{by (20).}
\end{aligned}$$

By (20), we have $h_{\beta,l} = h_{\beta-\alpha_j-\alpha_l,j} = h_{\delta,i}$, so, by induction we find

$$h_{\alpha_i,l}^{-1} T_{l,\delta} h_{\beta,l} = \left(T_{l,\delta} h_{\delta,i}^{-1}\right) h_{\beta,l} = T_{l,\delta}.$$

So the first summand of (40) is invariant under simultaneous left multiplication by $h_{\alpha_i,l}^{-1}$ and right multiplication by $h_{\beta,l}$. The same holds for the second summand, $m(T_{j,\gamma} + h_{\alpha_i,l}^{-1} T_{j,\gamma} h_{\beta,l}^{-1})$ by Lemma 6.6 applied with $h = h_{\alpha_i,l}$, $k = h_{\beta,l}$, and $t = T_{j,\gamma}$. Consequently (27) holds for $T_{i,\beta}$ in case (34).

Case (36): $(\alpha_i, \beta) = -1$ and there is a node $j \sim i$ with $(\alpha_j, \beta) = 1$. Then $T_{i,\beta} = T_{j,\beta-\alpha_j} h_{\beta-\alpha_j,i} + mT_{i,\beta-\alpha_j}$. Now

$$h_{\alpha_i,l}^{-1} T_{i,\beta} h_{\beta,l} = h_{\alpha_i,l}^{-1} (T_{j,\beta-\alpha_j} h_{\beta-\alpha_j,i} + mT_{i,\beta-\alpha_j}) h_{\beta,l}.$$

If $j \not\sim l$, we find

$$\begin{aligned}
h_{\alpha_i,l}^{-1} T_{i,\beta} h_{\beta,l} &= h_{\alpha_i,l}^{-1} (T_{j,\beta-\alpha_j} h_{\beta-\alpha_j,i} + mT_{i,\beta-\alpha_j}) h_{\beta,l} \quad \text{by (34)} \\
&= h_{\alpha_i,l}^{-1} T_{j,\beta-\alpha_j} h_{\beta-\alpha_j,l} h_{\beta-\alpha_j,i} + mh_{\alpha_i,l}^{-1} T_{i,\beta-\alpha_j} h_{\beta-\alpha_j,l} \quad \text{by (18) and (16)} \\
&= T_{j,\beta-\alpha_j} h_{\beta-\alpha_j,i} + mT_{i,\beta-\alpha_j} \quad \text{by induction} \\
&= T_{i,\beta} \quad \text{by (34).}
\end{aligned}$$

If $j \sim l$, we claim

$$T_{i,\beta} = T_{l,\gamma} h_{\gamma,j} h_{\beta-\alpha_j,i} + m\left(T_{j,\gamma} h_{\beta-\alpha_j,i} + h_{\alpha_i,l}^{-1} T_{i,\gamma}\right), \tag{41}$$

where $\gamma = \beta - \alpha_j - \alpha_l$ is a positive root. For

$$\begin{aligned}
T_{i,\beta} &= T_{j,\beta-\alpha_j} h_{\beta-\alpha_j,i} + mT_{i,\beta-\alpha_j} \quad \text{by (36)} \\
&= T_{l,\gamma} h_{\gamma,j} h_{\beta-\alpha_j,i} + mT_{j,\gamma} h_{\beta-\alpha_j,i} + mh_{\alpha_i,l}^{-1} T_{i,\gamma} \quad \text{by (36) and (28).}
\end{aligned}$$

By Lemma 24, we have

$$\begin{aligned}
h_{\gamma,i}^{-1} h_{\gamma,j} h_{\beta-\alpha_j,i} h_{\beta,l} &= d_\beta^{-1} \left(s_j^{-1} s_l^{-1} s_i^{-1} s_l s_j\right)\left(s_j^{-1} s_l^{-1} s_j s_l s_j\right)\left(s_j^{-1} s_i s_j\right)(s_l) d_\beta \\
&= d_\beta^{-1} \left(s_j^{-1} s_i^{-1} s_j\right)\left(s_j^{-1} s_l^{-1} s_j s_l s_j\right)\left(s_i s_j s_i^{-1}\right)(s_l) d_\beta \\
&= d_\beta^{-1} s_j^{-1} s_i^{-1} s_l^{-1} s_j s_l s_i s_j s_l d_\beta
\end{aligned}$$

$$
\begin{aligned}
&= d_\beta^{-1} s_j^{-1} s_l^{-1} s_i^{-1} s_j s_i s_l s_j s_l d_\beta \\
&= d_\beta^{-1} s_j^{-1} s_l^{-1} s_j s_i s_j^{-1} s_l s_j s_l d_\beta \\
&= d_\beta^{-1} s_l s_j^{-1} s_l^{-1} s_i s_l s_j d_\beta \\
&= d_\beta^{-1} s_l s_j^{-1} s_i s_j d_\beta \\
&= h_{\gamma,j} h_{\beta-\alpha_j}.
\end{aligned}
$$

Hence, using induction, we find for the first summand of (41)

$$
h_{\alpha_i,l}^{-1}(T_{l,\gamma} h_{\gamma,j} h_{\beta-\alpha_j,i}) h_{\beta,l} = T_{l,\gamma} h_{\gamma,i}^{-1} h_{\gamma,j} h_{\beta-\alpha_j,i} h_{\beta,l} = T_{l,\delta} h_{\gamma,j} h_{\beta-\alpha_j},
$$

proving that it is invariant under simultaneous left multiplication by $h_{\alpha_i,l}^{-1}$ and right multiplication by $h_{\beta,l}$.

The same holds for the second summand, $m(T_{j,\gamma} h_{\beta-\alpha_j,i} + h_{\alpha_i,l}^{-1} T_{i,\gamma})$ as we shall establish next. First of all, note that $h_{\gamma,j} = h_{\beta,l}$ by (20) and that $h_{\gamma,i} = h_{\beta-\alpha_j,i}$ by (18). Moreover, by (31) for γ, we have $T_{i,\gamma} h_{\gamma,j} = T_{j,\gamma} h_{\gamma,i}$. Substituting all this in the second summand, we obtain

$$
\begin{aligned}
m\bigl(T_{j,\gamma} h_{\beta-\alpha_j,i} + h_{\alpha_i,l}^{-1} T_{i,\gamma}\bigr) &= m\bigl(T_{j,\gamma} h_{\gamma,i} + h_{\alpha_i,l}^{-1} T_{i,\gamma}\bigr) = m\bigl(T_{i,\gamma} h_{\gamma,j} + h_{\alpha_i,l}^{-1} T_{i,\gamma}\bigr) \\
&= m\bigl(T_{i,\gamma} h_{\beta,l} + h_{\alpha_i,l}^{-1} T_{i,\gamma}\bigr).
\end{aligned}
$$

Again, using Lemma 6.6 applied with $h = h_{\alpha_i,l}$, $k = h_{\beta,l}$, and $t = T_{i,\gamma}$, we find the required invariance. Consequently (27) holds for $T_{i,\beta}$ in case (34).

Case (35): $(\alpha_i, \beta) = 1$ and there is a node $j \sim i$ with $(\alpha_j, \beta) = 0$. Then $T_{i,\beta} = T_{j,\beta-\alpha_i} h_{\beta,j}^{-1}$. Now

$$
h_{\alpha_i,l}^{-1} T_{i,\beta} h_{\beta,l} = h_{\alpha_i,l}^{-1} T_{j,\beta-\alpha_i} h_{\beta,j}^{-1} h_{\beta,l}.
$$

If $j \not\sim l$, we find

$$
\begin{aligned}
h_{\alpha_i,l}^{-1} T_{i,\beta} h_{\beta,l} &= h_{\alpha_i,l}^{-1} T_{j,\beta-\alpha_i} h_{\beta,j}^{-1} h_{\beta,l} \quad \text{by (35)} \\
&= h_{\alpha_i,l}^{-1} T_{j,\beta-\alpha_i} h_{\beta-\alpha_i,l} h_{\beta,j}^{-1} \quad \text{by (16) and (18)} \\
&= T_{j,\beta-\alpha_i} h_{\beta,j}^{-1} \quad \text{by induction} \\
&= T_{i,\beta} \quad \text{by (35).}
\end{aligned}
$$

If $j \sim l$, observe that $h_{\beta-\alpha_i,l}^{-1} h_{\beta,j}^{-1} h_{\beta,l} = h_{\beta-\alpha_i-\alpha_j,i} h_{\beta-\alpha_i,l}^{-1} h_{\beta,j}^{-1}$ in view of (18), (20), and (17). Also, $h_{\alpha_i,l} = h_{\alpha_l,i}$ by a double application of (21). Therefore,

$$\begin{aligned}
h_{\alpha_i,l}^{-1} T_{i,\beta} h_{\beta,l} &= h_{\alpha_i,l}^{-1} T_{l,\beta-\alpha_i-\alpha_j} h_{\beta-\alpha_i,l}^{-1} h_{\beta,j}^{-1} h_{\beta,l} \quad \text{by (35) twice} \\
&= h_{\alpha_l,i}^{-1} T_{l,\beta-\alpha_i-\alpha_j} h_{\beta-\alpha_i-\alpha_j,i} h_{\beta-\alpha_i,l}^{-1} h_{\beta,j}^{-1} \quad \text{by the above} \\
&= T_{l,\beta-\alpha_i-\alpha_j} h_{\beta-\alpha_i,l}^{-1} h_{\beta,j}^{-1} \quad \text{by induction} \\
&= T_{i,\beta} \quad \text{by (35) twice.} \qquad \square
\end{aligned}$$

The proposition enables us to describe an algorithm computing the $T_{i,\beta}$.

Algorithm 6.8. The Hecke algebra elements $T_{i,\beta}$ of Theorem 6.1 can be computed as follows by using Table 1.

(i) If $i \notin \text{Supp}(\beta)$, then, in accordance with (39), set $T_{i,\beta} = 0$.

From now on, assume $i \in \text{Supp}(\beta)$.

(ii) If $\text{ht}(\beta) \leqslant 2$, Eqs. (29) and (30), that is, the second and third lines of Table 1, determine $T_{i,\beta}$.

From now on, assume $\text{ht}(\beta) > 2$. We proceed by recursion, expressing $T_{i,\beta}$ as a $Z_0^{(0)}$-bilinear combination of $T_{k,\gamma}$'s with $\text{ht}(\gamma) < \text{ht}(\beta)$.

(iii) If $(\alpha_i, \beta) = 1$, in accordance with Corollary 6.4(ii), set $T_{i,\beta} = md_{\alpha_i}^{-1} s_\beta^{-1} s_i s_\beta d_\beta$.

From now on, assume $(\alpha_i, \beta) \in \{0, -1\}$.

(iv) Search for a $j \in \{1, \ldots, n\}$ such that $(\alpha_i, \alpha_j) = 0$ and $(\alpha_j, \beta) = 1$. If such a j exists, then $\beta - \alpha_j \in \Phi$ and (28) expresses $T_{i,\beta}$ as a multiple of $T_{i,\beta-\alpha_j}$.

(v) So, suppose there is no such j. There is a j for which $\beta - \alpha_j$ is a root, so $(\alpha_j, \beta) = 1$. As $(\alpha_i, \beta) \neq 1$, we must have $i \sim j$. According as $(\alpha_i, \beta) = 0$ or -1, the identities (34) or (36) express $T_{i,\beta}$ as a $Z_0^{(0)}$-bilinear combination of $T_{i,\beta-\alpha_j}$ and some $T_{j,\gamma}$ with $\text{ht}(\gamma) < \text{ht}(\beta)$.

This ends the algorithm. Observe that all lines of Table 1 have been used, with (35) implicitly in (iii).

The algorithm computes a Hecke algebra element for each i, β based on Table 1, showing that there is at most one solution to the set of equations. The next result shows that the computed Hecke algebra elements do indeed give a solution.

Proposition 6.9. *The equations of Table* 1 *have a unique solution.*

Proof. We will first show that the Hecke algebra elements $T_{i,\beta}$ defined by Algorithm 6.8 are well defined by the algorithm and then that they satisfy the equations of Table 1. Both assertions are proved by induction on $\text{ht}(\beta)$, the height of β.

If β has height 1 or 2, $T_{i,\beta}$ is chosen in step (i) if $\beta = \alpha_j$ with $j \neq i$ and in step (ii) otherwise. Indeed there is a unique solution.

Now assume $\mathrm{ht}(\beta) \geqslant 3$. Suppose first that $T_{i,\beta}$ is determined in step (iii). This means that $(\alpha_i, \beta) = 1$. This is unique as it is a closed form.

We now suppose that $T_{i,\beta}$ is chosen in step (iv). This means there is a j for which $(\alpha_i, \alpha_j) = 0$ and $(\alpha_j, \beta) = 1$. We must show that if there are two such j the result is the same. Suppose there are distinct j and j' for which $(\alpha_j, \beta) = (\alpha_{j'}, \beta) = 1$ and $(\alpha_j, \alpha_i) = (\alpha_{j'}, \alpha_i) = 0$. Then by our definition $T_{i,\beta} = h^{-1}_{\alpha_i, j'} T_{i,\beta-\alpha_{j'}}$ and we must show that

$$T_{i,\beta} = h^{-1}_{\alpha_i, j} T_{i,\beta-\alpha_j}.$$

If $j \sim j'$, then $(\beta - \alpha_j, \alpha_{j'}) = 2$ and $\beta = \alpha_j + \alpha_{j'}$ has height 2. This means we can assume $j \not\sim j'$. Then $(\beta - \alpha_j, \alpha_{j'}) = 1$ and $(\beta - \alpha_{j'}, \alpha_j) = 1$. In particular, $\beta - \alpha_j - \alpha_{j'}$ is also a root. Now apply (28) and the induction hypothesis to see $T_{i,\beta-\alpha_j} = h^{-1}_{\alpha_i, j'} T_{i,\beta-\alpha_j-\alpha_{j'}}$ and $T_{i,\beta-\alpha_{j'}} = h^{-1}_{\alpha_i, j} T_{i,\beta-\alpha_j-\alpha_{j'}}$, and so by (16), we find

$$h^{-1}_{\alpha_i, j} T_{i,\beta-\alpha_j} = h^{-1}_{\alpha_i, j'} T_{i,\beta-\alpha_{j'}}.$$

This shows the definitions are the same with either choice.

We may now assume that $T_{i,\beta}$ was chosen in step (v). If j is the one chosen in step (v), then $T_{i,\beta}$ was chosen to satisfy (34) or (36). Suppose now that there is another index j' which was used in step (v) to define $T_{i,\beta}$. For these the conditions are $(\alpha_j, \beta) = (\alpha_{j'}, \beta) = 1$ and $(\alpha_i, \alpha_j) = (\alpha_i, \alpha_{j'}) = -1$. Clearly $j \not\sim j'$ for otherwise there would be a triangle in the Dynkin diagram M. Therefore, $(\alpha_{j'}, \beta - \alpha_j) = 1$, and so $\beta - \alpha_j - \alpha_{j'}$ is a root. We distinguish according to the two possibilities for (α_i, β).

Assume first $(\alpha_i, \beta) = 0$. Then, $(\alpha_i, \beta - \alpha_j - \alpha_{j'}) = 2$, and so $\beta = \alpha_i + \alpha_j + \alpha_{j'}$. By using (34), with either j or with j', we find $T_{i,\beta} = m^2$, independent of the choice of j or j'.

Next assume $(\alpha_i, \beta) = -1$. Then $(\alpha_i, \beta - \alpha_j - \alpha_{j'}) = 1$, so $\gamma = \beta - \alpha_j - \alpha_{j'} - \alpha_i$ is a root. We need to establish that the result of application of (36) to $T_{i,\beta}$ does not depend on the choice j or j'. We do so by showing that the result can be expressed in an expression symmetric in j and j'. Observe that γ is an expression symmetric in j and j'. The expression of $T_{i,\beta}$ obtained by applying (36) to j is

$$T_{j,\beta-\alpha_j} h_{\beta-\alpha_j,i} + m T_{i,\beta-\alpha_j}. \tag{42}$$

By (34), the second summand of the right-hand side equals

$$m T_{i,\beta-\alpha_j} = m T_{j',\gamma} + m^2 T_{i,\beta-\alpha_j-\alpha_{j'}}.$$

For the first summand of (42) we find

$$\begin{aligned} T_{j,\beta-\alpha_j} h_{\beta-\alpha_j,i} &= h^{-1}_{\alpha_j, j'} T_{j,\beta-\alpha_j-\alpha_{j'}} h_{\beta-\alpha_j,i} \quad \text{by (28)} \\ &= h^{-1}_{\alpha_j, j'} (T_{i,\gamma} h_{\gamma,j} + m T_{j,\gamma}) h_{\beta-\alpha_j,i} \quad \text{by (36)}. \end{aligned}$$

Expanding (42) with these expressions, we find by use of $h_{\alpha_j,j'} = h_{\alpha_{j'},j}$ (see (21)), $h_{\gamma,j'} = h_{\beta-\alpha_j,i}$ (see (22)), and (27),

$$h^{-1}_{\alpha_j,j'}T_{i,\gamma}h_{\gamma,j}h_{\beta-\alpha_j,i} + m\bigl(h^{-1}_{\alpha_j,j'}T_{j,\gamma}h_{\beta-\alpha_j,i} + T_{j',\gamma}\bigr) + m^2T_{i,\beta-\alpha_j-\alpha_{j'}}$$
$$= h^{-1}_{\alpha_j+\alpha_{j'}+\alpha_i,i}T_{i,\gamma}h_{\gamma,j}h_{\gamma,j'} + m(T_{j,\gamma} + T_{j',\gamma}) + m^2T_{i,\beta-\alpha_j-\alpha_{j'}}.$$

Since $h_{\gamma,j}$ and $h_{\gamma,j'}$ commute, cf. (16), the result is indeed symmetric in j and j'. This shows that the algorithm gives unique Hecke algebra elements $T_{i,\beta}$.

We now show that the relations of Table 1 all hold for $T_{i,\beta}$ as computed by the algorithm. If the height of β is one or two the values are given by (39) and (29) of the table and none of the other relations hold as there are no applicable j.

We consider each of the remaining relations, one at a time, and show that each holds by assuming the relations all hold for roots of lower height.

If $(\alpha_i, \beta) = 1$ the value of $T_{i,\beta}$ is given in step (iii). The relevant equations are (28) and (35). The proof of Corollary 6.4(ii) shows that both equations are satisfied by the closed formula which is the outcome of our algorithm.

We have yet to check (34) and (36) in which case (β, α_i) is 0 or -1. Notice (28) and (35) require $(\alpha_i, \beta) = 1$ and do not apply here. In these cases $T_{i,\beta}$ is chosen in step (iv) or step (v).

Suppose first $T_{i,\beta}$ was chosen by step (iv). In this case there is a j' with $(\alpha_{j'}, \beta) = 1$, $(\alpha_i, \alpha_{j'}) = -1$. As $T_{i,\beta}$ is determined by step (iv) of the algorithm,

$$T_{i,\beta} = h^{-1}_{\alpha_i,j'}T_{i,\beta-\alpha_{j'}}.$$

We have already seen that this is independent of the choice of j' and so if there is another j for which $(\alpha_j, \beta) = 1$ with $(\alpha_i, \alpha_j) = 1$, (28) holds. To check (34) we suppose there is a j for which $(\alpha_i, \beta) = 1$ with $(\alpha_i, \alpha_j) = -1$. We must have $j \not\sim j'$, for otherwise we would again be in the height 2 case. In order to obtain (34) we must show that

$$h^{-1}_{\alpha_i,j'}T_{i,\beta-\alpha_{j'}} = T_{j,\beta-\alpha_i-\alpha_j} + mT_{i,\beta-\alpha_j}.$$

As for the left-hand side, $(\beta - \alpha_{j'}, \alpha_j) = 1$ and $(\alpha_i, \alpha_j) = -1$, so by (34), we have

$$h^{-1}_{\alpha_i,j'}T_{i,\beta-\alpha_{j'}} = h^{-1}_{\alpha_i,j'}T_{j,\beta-\alpha_{j'}-\alpha_j-\alpha_i} + mh^{-1}_{\alpha_i,j'}T_{i,\beta-\alpha_{j'}-\alpha_j}.$$

As for the right-hand side, as $(\alpha_j, \alpha_{j'}) = 0$, we can use (28) to obtain

$$T_{j,\beta-\alpha_i-\alpha_j} = h^{-1}_{\alpha_j,j'}T_{j,\beta-\alpha_j-\alpha_i-\alpha_{j'}} \quad \text{and} \quad T_{i,\beta-\alpha_j} = h^{-1}_{\alpha_i,j'}T_{i,\beta-\alpha_j-\alpha_{j'}},$$

and so the right-hand side equals the left-hand side if $h_{\alpha_j,j'} = h_{\alpha_i,j'}$. But this is (23).

We have yet to consider the case $(\alpha_i, \beta) = -1$, when $T_{i,\beta}$ is chosen in step (iv). Suppose j' is the choice used in step (iv). As we saw in the case $(\alpha_i, \beta) = 0$, (28) holds for any j with $(\alpha_j, \beta) = 1$ and with $(\alpha_i, \alpha_j) = 0$ by the uniqueness of the definition of $T_{i,\beta}$. We need to treat the case $(\alpha_j, \beta) = 1$ with $(\alpha_i, \alpha_j) = -1$ and show (36) holds. In particular we need to show

$$h^{-1}_{\alpha_i, j'} T_{i,\beta-\alpha_{j'}} = T_{j,\beta-\alpha_j} h_{\beta-\alpha_j,i} + mT_{i,\beta-\alpha_j}.$$

Use (36) on the left-hand side to get

$$h^{-1}_{\alpha_i, j'} T_{j,\beta-\alpha_{j'}-\alpha_j} h_{\beta-\alpha_j-\alpha_{j'},i} + mh^{-1}_{\alpha_i, j'} T_{i,\beta-\alpha_j-\alpha_{j'}}.$$

On the right-hand side use (28) to get

$$h^{-1}_{\alpha_j, j'} T_{j,\beta-\alpha_j-\alpha_{j'}} h_{\beta-\alpha_j,i} + mh^{-1}_{\alpha_i, j'} T_{i,\beta-\alpha_j-\alpha_j}.$$

The needed equation will hold provided $h_{\alpha_i,j'} = h_{\alpha_j,j'}$ and $h_{\beta-\alpha_j-\alpha_{j'},i} = h_{\beta-\alpha_j,i}$. The first is (23) and the second is (18).

This shows that all the equations are satisfied if $T_{i,\beta}$ is chosen in step (iv). But if $T_{i,\beta}$ was chosen in step (v) we have already checked any two choices of j give the same answer for (36) and so this equation is satisfied also. We have now shown all the relations in Table 1 hold. □

At this point we have established the existence of a linear representation σ of A on $V^{(0)}$. We need some properties of projections which have already arisen in [7]. In particular let $f_i = ml^{-1}e_i$. The following lemma shows these elements are multiples of projections.

Lemma 6.10. *The endomorphisms $\sigma(f_i)$ of $V^{(0)}$ satisfy*

$$\sigma(f_i)x_\beta = \begin{cases} (l^{-2} + ml^{-1} - 1)x_{\alpha_i} & \text{if } (\alpha_i, \beta) = 2, \\ l^{-1}x_{\alpha_i} T_{i,\beta}(h_{\beta,i} + m + l^{-1}) & \text{if } (\alpha_i, \beta) = 0, \\ l^{-1}x_{\alpha_i}(T_{i,\beta+\alpha_i} + l^{-1}T_{i,\beta}) & \text{if } (\alpha_i, \beta) = -1, \\ l^{-1}x_{\alpha_i}(T_{i,\beta-\alpha_i} + (m + l^{-1})T_{i,\beta}) & \text{if } (\alpha_i, \beta) = 1. \end{cases}$$

In particular, $\sigma(f_i)x_\beta \in x_{\alpha_i} l^{-1} Z_0^{(1)}[l^{-1}]$ if $\beta \neq \alpha_i$ and $\sigma(f_i)x_{\alpha_i} \in x_{\alpha_i}(-1 + l^{-1}Z_0^{(1)}[l^{-1}])$.

Proof. Suppose first $(\alpha_i, \beta) = 2$ in which case $\beta = \alpha_i$. Using the definition of σ and (29) gives $\sigma_i x_{\alpha_i} = l^{-1}x_{\alpha_i}$. Now $\sigma(f_i)x_{\alpha_i} = (l^{-2} + ml^{-1} - 1)x_{\alpha_i}$.

Suppose $(\alpha_i, \beta) = 0$. Then $\sigma_i x_\beta = x_\beta h_{\beta,i} + l^{-1}x_{\alpha_i} T_{i,\beta}$. Now

$$\sigma_i^2 x_\beta = x_\beta h^2_{\beta,i} + l^{-1}x_{\alpha_i} T_{i,\beta} h_{\beta,i} + l^{-2}x_{\alpha_i} T_{i,\beta}.$$

Evaluating $\sigma(f_i)$ on x_{α_i} and using the Hecke algebra quadratic relation for $h_{\beta,i}$ gives that the coefficient of x_β is 0. Adding the other terms gives $l^{-1}x_{\alpha_i} T_{i,\beta}(h_{\beta,i} + m + l^{-1})$ as stated.

Suppose $(\alpha_i, \beta) = -1$. Now $\sigma_i x_\beta = x_{\beta+\alpha_i} - mx_\beta + l^{-1}x_{\alpha_i} T_{i,\beta}$. Applying σ_i again gives $\sigma_i^2 x_\beta = x_\beta + l^{-1}x_{\alpha_i} T_{i,\beta+\alpha_i} - m(x_{\beta+\alpha_i} - mx_\beta + l^{-1}x_{\alpha_i} T_{i,\beta}) + l^{-2}x_{\alpha_i} T_{i,\beta}$. Again adding gives the result.

If $(\alpha_i, \beta) = 1$, $\sigma_i x_\beta = x_{\beta-\alpha_i} + l^{-1} x_{\alpha_i} T_{i,\beta}$. Now $\sigma_i^2 x_\beta = x_\beta - m x_{\beta-\alpha_i} + l^{-1} x_{\alpha_i} T_{i,\beta-\alpha_i} + l^{-2} x_{\alpha_i} T_{i,\beta}$. Adding and again using the quadratic relation gives the result.

The final statement follows from the fact that the $T_{i,\gamma}$ and $h_{\beta,i}$ belong to $Z_0^{(1)}[l^{-1}]$ (that is, there is no l involved). □

Proof of Theorem 6.1. In view of Proposition 6.3 we need only check (D1), (R1), (R2), and that $\sigma(e_i e_j) = 0$ for $i \not\sim j$. But (D1) is just the definition. By Lemma 6.10 we know $\sigma(e_i)x_\beta$ is in the space spanned by x_{α_i}. Now (R1) follows as $\sigma_i x_{\alpha_i} = l^{-1} x_{\alpha_i}$. For $i \not\sim j$ we know $\sigma(e_i e_j) = \sigma(e_j e_i)$. By Lemma 6.10 this is in $x_{\alpha_i} Z_0^{(0)}$ and also in $x_{\alpha_j} Z_0^{(0)}$, and so it is 0. As for (R2) again $\sigma(e_i)x_\beta$ is a multiple of x_{α_i}. Now $\sigma_j x_{\alpha_i} = x_{\alpha_i+\alpha_j} - m x_{\alpha_i}$. Lemma 6.10 gives

$$\sigma(f_i)(x_{\alpha_i+\alpha_j} - m x_{\alpha_i}) = x_{\alpha_i}\bigl(l^{-1}\bigl(m + l^{-1}\bigr)m - \bigl(l^2 + ml^{-1} - 1\bigr)m\bigr) = m x_{\alpha_i}.$$

Now scaling to get $\sigma(e_i)$ gives the result. We have shown that Theorem 6.1 holds. □

We now show how to construct irreducible representations of B which have I_2 in the kernel.

Lemma 6.11. *For each node i of M, we have $\sigma(Z_i^{(0)})x_{\alpha_i} = x_{\alpha_i} Z_0^{(0)}$.*

Proof. For j and i adjacent nodes, the following computation shows $\sigma_i \sigma_j x_{\alpha_i} = x_{\alpha_j}$.

$$\begin{aligned}\sigma_i \sigma_j x_{\alpha_i} &= \sigma_i (x_{\alpha_i+\alpha_j} - m x_{\alpha_i}) = x_{\alpha_j} + l^{-1} T_{i,\alpha_i+\alpha_j} x_{\alpha_i} - m l^{-1} x_{\alpha_i} \\ &= x_{\alpha_j} + l^{-1} x_{\alpha_i} m - m l^{-1} x_{\alpha_i} = x_{\alpha_j}.\end{aligned}$$

By induction on the length of a path from i to k in M, this gives

$$\sigma\bigl(\widehat{w_{ik}}\bigr)x_{\alpha_i} = x_{\alpha_k}. \tag{43}$$

Therefore, for j and k distinct non-adjacent nodes of M,

$$x^{-1}\sigma\bigl(\widehat{w_{ki}}\,\hat{j}\,\widehat{w_{ik}}e_i\bigr)x_{\alpha_i} = \sigma\bigl(\widehat{w_{ki}}\,\hat{j}\bigr)x_{\alpha_k} = \sigma\bigl(\widehat{w_{ki}}\bigr)\sigma_j x_{\alpha_k} = \sigma\bigl(\widehat{w_{ki}}\bigr)x_{\alpha_k} h_{\alpha_k,j} = x_{\alpha_i} h_{\alpha_k,j}.$$

As $\sigma(Z_i^{(0)})$ is generated by elements of the form $\sigma(\widehat{w_{ki}}\,\hat{j}\,\widehat{w_{ik}}e_i)$, it follows that

$$\sigma\bigl(Z_i^{(0)}\bigr)x_{\alpha_i} \subseteq x_{\alpha_i} Z_0^{(0)}.$$

Note it follows from Lemma 6.10 that $x^{-1}\sigma(e_i)x_{\alpha_i} = x_{\alpha_i}$.

As for the converse, this follows from Lemma 3.8(ii), which implies that $Z_0^{(0)}$ is generated by $h_{\alpha_k,i}$, for $i \not\sim k$, $i \neq k$. (For, by definition, $Z_0^{(0)}$ is generated by $\widehat{C}$ mod I_2.) □

Suppose θ is any representation of Z_0, acting on a vector space U over K, where $K = \mathbb{Q}(r)$, or an algebraic extension thereof. Then we can form a representation of B on the vector space $V \otimes_{Z_0} U$ over $K(l)$ which is the direct sum of vector spaces $x_\beta U$ where each is a vector space isomorphic to U. Let V be the representation space of Theorem 6.1. For each i define an action of σ_i on $V \otimes_{Z_0} U$ by letting elements of Z_0 act directly on U. In particular, $\sigma_i x_{\alpha_i} u = l^{-1} x_{\alpha_i} u$; if $(\alpha_i, \beta) = 0$, then $\sigma_i x_\beta u = x_\beta \theta(h_{\beta,i}) u + l^{-1} x_{\alpha_i} \theta(T_{i,\beta}) u$; for $(\alpha_i, \beta) = 1$ we have $\sigma_i(x_\beta u) = x_{\beta-\alpha_i} u + l^{-1} x_{\alpha_i} \theta(T_{i,\beta}) u$ and if $(\alpha_i, \beta) = -1$ we have $\sigma_i x_\beta u = x_{\beta+\beta_i} u - m x_\beta u + l^{-1} x_{\alpha_i} \theta(T_{i,\beta}) u$. This is a representation by Theorem 6.1. Denote it Γ_θ.

Lemma 6.12. *If θ is an irreducible representation of $Z_0^{(0)}$, then the representation Γ_θ is also irreducible. For inequivalent representations θ, θ', the resulting representations Γ_θ and $\Gamma_{\theta'}$ are also inequivalent.*

Proof. Suppose V_1 is a proper non-trivial invariant subspace of $V \otimes_{Z_0} U$. We show first that $\sigma(f_i) V_1 = 0$ for all nodes i of M. By Lemma 6.10, $\sigma(f_i) V \otimes_{Z_0} U$ is in $x_{\alpha_i} \theta(Z_0^{(0)}) U$ which is in $x_{\alpha_i} U$. This means that $\sigma(f_i) V_1$ is in $x_{\alpha_i} U$. Suppose there is a node i with $\sigma(f_i) V_1$ non-zero. This means there is a non-zero element of $u \in U$ such that $x_{\alpha_i} u \in V_1$. In Lemma 6.11, we have seen that $Z_i^{(0)} x_{\alpha_i} = x_{\alpha_i} Z_0^{(0)}$. Hence

$$x_{\alpha_i} \theta(Z_0^{(0)}) u = Z_i^{(0)} x_{\alpha_i} \subseteq V_1.$$

But θ is irreducible and so all of $x_{\alpha_i} U$ is contained in V_1.

By Lemma 6.11, $x_{\alpha_k} U$ is in V_1 for all k. We show by induction on the height of a positive root $\mathrm{ht}(\beta)$ that $x_\beta U$ is in V_1. Assume $\mathrm{ht}(\beta) \geqslant 2$. Choose a node j with $\beta = r_j(\beta - \alpha_j)$. By induction, $x_{\beta-\alpha_j} U$ is in V_1. But for each $u \in U$, the vector $\sigma_j x_{\beta-\alpha_j} u$ is a sum of $x_\beta u$ and vectors already known to be in V_1 and so $x_\beta U$ is in V_1. But this means all of $V \otimes_{Z_0} U$ is in V_1, contradicting that V_1 is proper. This shows $\sigma(f_i) V_1 = 0$ for each node i.

As V_1 is invariant, its image $\sigma(\widehat{w_{\beta,j}} f_j \widehat{w_{\beta,j}}^{-1}) V_1$ under a conjugate of $\sigma(f_i)$ is also trivial. We will derive from this that V_1 is 0. To this end, choose an order on Φ^+ that is consistent with height. For each β choose a node $j(\beta)$ in the support of β. Notice that Lemma 6.10 shows that the image of $\sigma(f_i)$ is in $x_{\alpha_i} Z_0^{(0)}$. Let L be the matrix whose rows and columns are indexed by Φ^+ in the fixed order and whose β, γ entry is the coefficient of x_β in $\sigma(\widehat{w_{\beta,j(\beta)}} f_{j(\beta)} \widehat{w_{\beta,j(\beta)}}^{-1}) x_\gamma$. This means the entries are elements of $\theta(Z_0^{(0)})$. As each $\sigma(\widehat{w_{\beta,j(\beta)}} f_{j(\beta)} \widehat{w_{\beta,j(\beta)}}^{-1}) V_1 = 0$, we have $L V_1 = 0$.

Observe that L can be viewed as a matrix with entries in $K[l^{-1}]$ by interpreting the entries from $\theta(Z_0^{(0)})$ as submatrices over $K[l^{-1}]$. We claim that L is non-singular. By the Lawrence–Krammer action rules, the β, γ entry of L mod l^{-1} is readily seen to be the coefficient of $x_{\alpha_{j(\beta)}}$ in $\sigma(f_{j(\beta)} \widehat{w_{\beta,j(\beta)}}^{-1}) x_\gamma$. If $\beta = \gamma$, then this coefficient is equal to -1 modulo l^{-1}, and if β is less than γ in the given order, then there is no summand $x_{\alpha_{j(\beta)}}$ present in the expansion of $\sigma(\widehat{w_{\beta,j(\beta)}}^{-1}) x_\gamma$ and so the β, γ coefficient of L is 0. This means L modulo l^{-1} is lower-triangular with -1 on the diagonal, whence non-singular.

Therefore, the equality $L V_1 = 0$ implies $V_1 = 0$. We conclude that there is no invariant subspace and the representation is irreducible.

Finally, we argue that inequivalent θ lead to inequivalent Γ_θ. To this end we consider the trace of each element $\widehat{w_{ki}} \hat{z} \widehat{w_{ik}} e_i$ of Z_i in Γ_θ, where z is in $W_{k^\perp}$. By Lemma 6.10, the only contributions to the trace occur for vectors in $x_\alpha\theta(Z_0)$, and, in view of Lemma 6.11, this contribution is $m^{-1}(l^{-1}+m-l^{-1})\,\mathrm{tr}(\theta(d_{\alpha_k}^{-1}\hat{z}d_{\alpha_k}))$. Since $d_{\alpha_k}^{-1}\hat{z}d_{\alpha_k}$, for k a node of M and $z \in W_{k^\perp}$, span Z_0 over $K(l)$, these values uniquely determine θ. □

With these results in hand we are now ready to show that the dimension of I_1/I_2 is at least the dimension we need for Theorem 1.2.

Proof of Theorem 1.2. In Theorem 6.12 we have constructed irreducible representations Γ_θ of B/I_2 of dimension $|\Phi^+|\dim\theta$ for any irreducible representation θ of Z_0. Since I_1 is not in the kernel of these representations, they are irreducible representations of I_1/I_2. Moreover, Z_0, being a Hecke algebra over $\mathbb{Q}(l,m)$ of spherical type, is semi-simple, so summing the squares of the dimensions of the irreducibles of Z_0 gives $\dim(Z_0)$. Hence the dimension of I_1/I_2 is at least $|\Phi^+|^2\dim(Z_0)$. By Theorem 5.6, this is also an upper bound for the dimension, whence equality. The semisimplicity follows as B/I_1, being the Hecke algebra of type M, is semisimple, and the sum of the squares of the irreducible representations of I_1/I_2 is the dimension of I_1/I_2. □

To end this section, we observe that the usual Lawrence–Krammer representation is the representation Γ_θ, where θ is the linear character of Z_0 determined by $\theta(h_{\beta,i}) = r^{-1}$ for all pairs $(\beta,i) \in \Phi^+ \times M$ with $(\alpha_i,\beta) = 0$.

7. Consequences and conjectures

This section gives some consequences of the main results of the previous sections, as well as some of our ideas about the general structure of BMW algebras.

7.1. Global structure of BMW algebras

Indications for the validity of our theorems were first found by experimental computations in GBNP [6]. However, the sheer size of the algebras involved makes the computations difficult. For instance, the dimension of I_1/I_2 in $B(\mathrm{E}_8)$ is equal to 41 803 776 000.

Nevertheless, some experimenting with $B(\mathrm{D}_4)$ and knowledge of the classical BMW algebra $B(\mathrm{A}_n)$ lead us to conjecture that, if J is a coclique of M of size $i > 1$, then I_J is an ideal properly contained in I_{i-1}.

If J and K are conjugate by an element $w \in W$, then as we have seen in Proposition 4.2(ii), the ideals I_J and I_K coincide. Computations in B of type D_4 show that for J and K of size 2 but in distinct orbits, we find distinct ideals $I_J = Be_JB$, $I_K = Be_KB$. Also the pattern that, for each coclique J of size i, we have $I_J/I_{i+1} = Be_JB/I_{i+1} = \widehat{D_J}Z_J\widehat{D_J}^{\mathrm{op}}/I_{i+1}$ for a suitable set D_J of coset representatives of the stabilizer of $\{r_j \mid j \in J\}$ in W and a subalgebra Z_J of B isomorphic to a suitable subtype C_J of M. Thus, we expect that $\dim(I_J/I_{i+1})$ is a multiple of N^2 by the order of a Coxeter group of some subtype C_J of M, where N is the length of the W-orbit of $\{r_j \mid j \in J\}$. This would imply that the dimension of B be equal to

$$\sum_J N_J^2 \left|W(C_J)\right|.$$

Here J runs over the W-equivalence classes of cocliques in M, including the empty set, with $C_\emptyset = M$ and $N_\emptyset = 1$, so that the contribution for $J = \emptyset$ equals $|W|$, the dimension of B/I_1, the Hecke algebra of type M.

The conjecture holds for $B(\mathrm{A}_n)$. Here W is known to have a single orbit on cocliques in M of any given size $i \in \{1, \ldots, \lceil n/2 \rceil\}$; for $J = \{1, 3, \ldots, 2i-1\}$, the type C_J is the Coxeter type of the centralizer in W of $\{\alpha_j \mid j \in J\}$, that is, $C_J = \mathrm{A}_{n-2i}$, and

$$\dim(I_i/I_{i+1}) = N_i^2(n+1-2i) \quad \text{with } N_i = \binom{n+1}{\underbrace{2, 2, \ldots, 2}_{i\ \times}}.$$

These formulas also hold for $i = 0$ if we write $I_0 = B$ and $N_0 = 1$. We then find $\dim(B(\mathrm{A}_n)) = \sum_i \dim(I_i/I_{i+1}) = (2n+1)(2n-1)(2n-3)\cdots 1$, which is known from [17].

Our conjecture also holds for $B(\mathrm{D}_4)$. In $B(\mathrm{D}_4)$, there are three ideals of the form I_J for J of size 2, namely for $J = \{1, 3\}$, $\{1, 4\}$, $\{3, 4\}$. Each quotient I_J/I_3 has dimension $N_J^2 \cdot 2$, where $N_J = 6$. Thus C_J is of type A_1, rather than $\mathrm{A}_1\mathrm{A}_1$, the parabolic type of the centralizer of two orthogonal roots. This means that a complication with respect to the type A_n occurs in that the type C_J is not just the full type of the centralizer of $\{\alpha_j \mid j \in J\}$ in W. Similarly, $N_{\{1,2,3\}} = 3$, $C_{\{1,2,3\}} = \emptyset$, and $I_3 = I_{\{1,3,4\}}$ has dimension $N_{\{1,2,3\}}^2 \cdot 1 = 9$. In conclusion,

$$\begin{aligned}\dim\bigl(B(\mathrm{D}_4)\bigr) &= |W| + N_1^2\bigl|W\bigl(\mathrm{A}_1^3\bigr)\bigr| + 3 \times N_{\{1,3\}}^2\bigl|W(\mathrm{A}_1)\bigr| + N_{\{1,3,4\}}^2\bigl|W(\emptyset)\bigr| \\ &= 192 + 12^2 \cdot 8 + 3 \cdot 6^2 \cdot 2 + 3^2 = 1569.\end{aligned}$$

The shrink of C_J for J of size 2 extends to all types D_n for $n \geqslant 4$. In $B(\mathrm{D}_n)$ ($n \geqslant 5$), there are two conjugacy classes, one of which has representative $\{n-1, n\}$. In this case, or rather, in any case where J contains these two end nodes, the representation of $B(\mathrm{D}_n)$ on I_J factors through a representation of $B(\mathrm{A}_{n-1})$. We prove this as follows. To begin, we can take $J = \{n-1, n\}$. We claim that g_n acts precisely as g_{n-1}. First of all $g_n e_J = l^{-1} e_J = g_{n-1} e_J$. We proceed to show $g_n \hat{u} e_J = g_{n-1} \hat{u} e_J$ by induction on the length of $u \in W_{\{1,\ldots,n-1\}}$. Without loss of generality, we may assume $u \in D_{n^\perp, n^\perp}$ (observe that $n^\perp \cap \{1, \ldots, n-1\} = J^\perp \cup \{n-1\}$ in this case, so

$$\widehat{g_n a u b} e_J = \hat{a} \widehat{g_n} \hat{u} e_J \hat{b}$$

for $a, b \in n^\perp \cap \{1, \ldots, n\}$). But then, by known properties of the Coxeter group, we have either $\hat{u} = g_{n-2}$ or $\hat{u} = g_{n-2} g_{n-1} g_{n-3} g_{n-2}$. As all indices are in $\{n-3, \ldots, n\}$, the identity

$g_n\hat{u}e_J = g_{n-1}\hat{u}e_J$ can be verified in $B(\mathrm{D}_4)$ (after specialization to $n = 4$), where it is easily seen to hold. So in all cases, g_n acts exactly like g_{n-1}, proving that the $B(\mathrm{D}_n)$ representation on I_J factors through the quotient obtained by identifying g_n and g_{n-1}, and so through a BMW algebra of type $B(\mathrm{A}_{n-1})$. On the basis of observations like these, we conjecture that the dimension of $B(\mathrm{D}_n)$ is equal to $(2^n + 1)(2n - 1)!! - (2^{n-1} + 1)n!$

7.2. Parabolic subalgebras and restrictions

Let J be a set of nodes of M. We will discuss B_J, the subalgebra of B generated by all g_j with $j \in J$. Clearly, there is a surjective homomorphism from $B(J)$, the BMW algebra of type $M|_J$ onto B_J. We conjecture however, at least for M of spherical type, that this map is an isomorphism. It is an easy consequence of Theorem 1.2 that this assertion holds modulo I_2, in the sense that $B_J/(I_2 \cap B_J)$ is isomorphic to the quotient of $B(J)$ by its ideal I_2.

The restriction of the generalized Lawrence–Krammer representation for B on V over Z_0 to B_J is easy to analyze. For $a : \mathrm{M} \setminus J \to \mathbb{N}$, put $\Phi^+_{J,a} = \{\beta \in \Phi^+ \mid C_{\beta,k} = a_k \text{ for } k \in M \setminus J\}$ and let $V_{J,a}$ be the subspace of V generated by x_β with $\beta \in \Phi^+_{J,a}$. Then it is easily seen from the Lawrence–Krammer action rules that $V_{J,0}$ is a B_J-invariant subspace of V, which is isomorphic to the Lawrence–Krammer representation of $B(J)$, up to an extension of scalars. Moreover, the subspace $V_{J,a} + V_{J,0}$ is B_J-invariant for any choice of a. In view of Lemma 6.10, the action of B_J on the quotient $(V_{J,a} + V_{J,0})/V_{J,0}$ factors through the Hecke algebra $B_J/(I_1 \cap B_J)$. We expect that the particular representations for B_J on $(V_{J,a} + V_{J,0})/V_{J,0}$ can be found by combinatorics of the root system, similar to the case of type A_n, discussed in [17].

To see how this works in a specific example we consider $B(\mathrm{D}_n)$ with $n \geqslant 5$ and $J = \{2, 3, \ldots, n\}$, so we will consider the action of B_J on $V_{J,i}$ for $i = 0, 1$. Here $\Phi^+_{J,0}$ is the set of roots $\varepsilon_i \pm \varepsilon_j$ for $2 \leqslant i \leqslant j \leqslant n$ and $\Phi^+_{J,1}$ is the set of roots $\varepsilon_1 \pm \varepsilon_j$ for $2 \leqslant j \leqslant n$, where $(\varepsilon_i)_{1\leqslant i\leqslant n}$ is an orthonormal basis of Euclidean n-space. B_J maps the span of $\{x_\beta \mid \beta \in \Phi^+_{J,0}\}$, which is $V_{J,0}$, to itself by the construction for $B(J) \cong B(\mathrm{D}_{n-1})$. Also the Hecke algebra Z_0 for $B(\mathrm{D}_{n-1})$, which is $\langle g_2\rangle \times \langle g_4, \ldots, g_n\rangle$, can be embedded into the Hecke algebra Z_0 for $B(\mathrm{D}_n)$, which is $\langle g_1\rangle \times \langle g_3, g_4, \ldots, g_n\rangle$, by mapping g_2 to g_1 and fixing $\langle g_4, \ldots, g_n\rangle$. Furthermore, if θ_{res} is θ restricted to Z_0 for B_J with this embedding, the resulting representation of $B(\mathrm{D}_{n-1})$ is $\Gamma_{\theta_{\mathrm{res}}}$. As mentioned above, the action of $B(\mathrm{D}_{n-1})$ on the quotient vector space $(V_{J,1} + V_{J,0})/V_{J,0}$ factors through the Hecke algebra of type D_{n-1}. The representation then breaks into these two actions with the action on the quotient being a Hecke algebra action. The span $V_{J,1}$ of the x_β for $\beta \in \Phi^+_{J,1}$, is not invariant but using semisimplicity there is an invariant subspace giving this representation. This gives a branching rule from $B(\mathrm{D}_n)$ to $B(\mathrm{D}_{n-1})$.

7.3. The Brauer algebra

Let E be the subring $\mathbb{Q}(x)[l^\pm]$ of $\mathbb{Q}(l, x)$. We conjecture that there is a subalgebra $B^{(0)}$ of B defined over E containing a spanning set of B with the property that after transition modulo $(l - 1)$ we obtain a monomial algebra whose basis can be described in terms of the root system of type M. For B of type A_n it is the well-known Brauer algebra, introduced in [4]. We expect the conjectured basis $\bigcup_J \widehat{D_J}\,\widehat{W_{C_J}}\,\widehat{D_J}^{\mathrm{op}}$ of B discussed in Section 7.1, to be a monomial basis mod E for the Brauer algebra. Its elements should correspond to pictures, which consist of triples consisting of two sets of orthogonal roots, both W-conjugate to $\{\alpha_j \mid j \in J\}$, and an element of $W(C_J)$, a Coxeter group in a quotient of the centralizer of J in W. This correspondence is well known for type A_n. The basis of I_1/I_2 found in Theorem 1.2 can be used to establish the validity of this conjecture for B/I_2.

7.4. Conclusion

For Coxeter diagrams that are not simply laced, we expect a natural BMW algebra to exist as well. For type B_n, an approach is given in [12]. More generally, by means of a folding $\phi : M \to M'$ of Coxeter diagrams, a BMW algebra of spherical type M' could be constructed as the subalgebra of $B(M)$ generated by suitable products of g_i for $g_i \in \phi^{-1}(a)$, one for each $a \in M'$, in much the same way the Artin group of type M' is embedded into the one of type M, see [8]. However, further research is needed to see if this definition is independent (up to isomorphism) of the choice of ϕ for fixed M', as well as to find an intrinsic definition of this algebra.

The BMW algebras of type A_n play a role in algebraic topology, in particular, in the theory of knots. The versions of spherical type ADE are related to the topology of the quotient space by W of the complement of the union of all reflection hyperplanes in the complexified space of the reflection representation of (W, R). After all, by [5], the Artin group A is the fundamental group of this space. A direct relationship, for instance, a definition of the BMW algebra in terms of this topology, would be of interest.

Brauer algebras play a role in tensor categories for the representations of classical Lie groups, and the corresponding BMW algebras seem to play a similar role for the related quantum groups. It is conceivable that the new BMW algebras constructed here play a similar role for the tensor categories of representations of quantum groups for the other types.

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