Studies on consistency measure of hesitant fuzzy preference relations

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Abstract

Hesitant fuzzy sets (HFSs), as an extension of fuzzy sets, consider the degrees of membership by a set of possible values rather than a single one. For further applications of HFSs to decision making, we develop a concept of hesitant fuzzy preference relations (HFPRs) as a tool to collect and present decision makers’ (DMs) preferences. Due to the importance of consistency measure for HFPRs to ensure that DMs are being neither random nor illogical, we develop a regression method to transform HFPRs to fuzzy preference relations (FPRs) with the highest consistency level. Some examples are given for illustration.

1. Introduction

Hesitant fuzzy sets (HFSs) [1] whose membership functions represented by a set of possible values, are a new effective tool to express human’s hesitancy in daily life. The motivation for introducing HFSs is that it is sometimes difficult to determine the membership of an element into a set, and in some circumstances, this difficulty is caused by a doubt between a few different values. For example, two decision makers (DMs) discuss the membership of \( x \) into \( A \), and one wants to assign 0.5 and the other 0.6. Recently, HFSs have become a hot topic and receive more and more attentions [2], [3], [4], [5].

In order to extend HFSs to decision making, such as multi-criteria decision making, we should consider how to present preferences provided by the DMs. In daily life, decision making is the common phenomenon, which is to select the optimal alternative from several alternatives or to get their ranking by aggregating the performances of each alternative. To deal with decision making, Analytic Hierarchy Process [6] (AHP) is one of the most popular and powerful techniques, which has been widely used for the decision making problems to rank, select, evaluate and benchmark the decision alternatives. In AHP, one step incorporates the individual preferences that reflect the relative importance of elements at a level of hierarchy. The preferences provided by the DMs are represented by preference relations [7], [8], [9], [10], [11] which are the most common tool to collect and present DMs’ preferences in decision making.

Therefore, in this paper, we propose a new concept of the hesitant fuzzy preference relation (HFPR), and develop consistence measures for HFPRs, which are helpful for applications of HFSs in decision making. Basic elements in HFSs, called hesitant fuzzy elements (HFEs), which include several possible membership degrees constitute the HFPR. And since generally, different HFEs have different number of membership degrees, we should normalize HFEs first for further studies on consistency measure of HFPRs. To make HFEs have same number of membership degrees, we consider two opposite principles for such normalization: 1) \( \alpha \)-normalization, by removing elements of HFEs and 2) \( \beta \)-normalization, by adding elements of HFEs. Using \( \alpha \)-normalization, we can transform HFPRs into...
FPRs; using $\beta$-normalization, we can get a HFPR with all HFEs have the same number of membership degrees. Since FPRs have been proven to be an effective tool used in decision making [12], [13], [14], [15], we concentrate on the $\alpha$-normalization in this paper.

In order to do this, the rest of the paper is organized as follows: Section 2 gives an introduction to HFSs, FPRs and consistency measure. In Section 3, we develop the HFPR, and a regression method to transform HFPRs into FPRs with some illustrative examples. Section 4 ends the paper with the concluding remarks.

2. Preliminaries

2.1. Hesitant fuzzy sets and fuzzy preference relations

Torra [1] originally developed HFSs which cover arguments with a set of possible values:

**Definition 1**: Let $X$ be a fixed set, a hesitant fuzzy set (HFS) on $X$ is in terms of a function that when applied to $X$ returns a subset of $[0,1]$, which can be represented as the following mathematical symbol:

$$ E = \{< x, h(x) > | x \in X \} $$

where $h(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set $E$. For convenience, we call $h$ a hesitant fuzzy element (HFE).

For an HFE $h$, Xia and Xu [3] developed some operations as follows:

1) $h^\lambda = \bigcup_{\gamma \in h} \{\gamma^\lambda\}$, $\lambda > 0$;

2) $\lambda h = \bigcup_{\gamma \in h} \{1 - (1 - \gamma)^\lambda\}$, $\lambda > 0$.

Fuzzy preference relations (FPRs) [16] are an effective tool used to model the decision making process in decision making. The definition of FPRs can be represented as follows:

**Definition 2**: A fuzzy preference relation (FPR) $P$ on a set of alternatives, $X$, is a fuzzy set on the product set $X \times X$, that is characterized by a membership function $\mu_p : X \times X \rightarrow [0,1]$.

When the cardinality of $X$ is small, the preference relation may be conveniently represented by the $n \times n$ matrix $P = (p_{ij})$ being $p_{ij} = \mu_p(x_i, x_j)$, for all $i, j \in \{1,2,\ldots,n\}$. $p_{ij}$ is interpreted as the preference degree of the alternative $x_i$ over $x_j$: $p_{ij} = 0.5$ indicates indifference between $x_i$ and $x_j$ ($x_i \sim x_j$), $p_{ij} = 1$ indicates that $x_i$ is absolutely preferred $x_j$, and $p_{ij} > 0.5$ indicates that $x_i$ is preferred to $x_j$ ($x_i \succ x_j$). In this case, the preference matrix $P$, is usually assumed additive reciprocal, i.e., $p_{ij} + p_{ji} = 1$, for all $i, j \in \{1,2,\ldots,n\}$.

2.2. Consistency measure

The transitivity property is used to represent the idea that the preference degree obtained by directly comparing two alternatives should be equal to or greater than the preference degree between those two alternatives obtained using an indirect chain of alternatives. This property is desirable to avoid contradictions reflected in preference relations. Tanino (1984) introduced the additive fuzzy transitivity property (complete consistency) as follows:

$$ p_{ij} + p_{jk} = p_{ik} + 0.5 $$

3. Hesitant fuzzy preference relations and the regression method

Motivated by the definition of FPRs, we now develop the concept of the hesitant fuzzy preference relation (HFPR) as follows:
Definition 3. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a fixed set, then a hesitant fuzzy preference relation (HFPR) \( H \) on \( X \) is presented by a matrix \( H = (h_{ij})_{n \times n} \subset X \times X \), where \( h_{ij} = \{\gamma'_s, s = 1, 2, \ldots, l_{ij}\} \) is an HFE indicating all the possible preference degrees of the alternative \( x_i \) over \( x_j \). Moreover, \( h_{ij} \) should satisfy the following conditions:

\[
\gamma'^{(s)}_{ij} + \gamma'^{(1_s-l_{ij})}_{ji} = 1, \quad \gamma'^{s}_{ij} = \{0.5\}, \quad i, j = 1, 2, \ldots, n
\]  

where \( \gamma'^{(s)}_{ij} \) is the \( s \) th largest element in \( h_{ij} \).

Based on the principle of \( \alpha \)-normalization, we should select proper membership degrees in HFEs which are components of the HFPR so as to obtain a FPR called a reduced HFPR. Herrera-Viedma et al. [10] developed a method based on error analysis to measure the consistency level of a FPR. Motivated by this method, we use error analysis to deal with the selection process, then a reduce HFPR can be obtained with the highest consistency level.

Given a HFPR, represented by a matrix \( H = (h_{ij})_{n \times n} \subset X \times X \), where \( X = \{x_1, x_2, \ldots, x_n\} \) is a fixed set of alternatives, we first define some new necessary operations. According to Eq.(2), all possible preference degrees of the pair of the alternatives \((i, k)\) represented by an HFE \( h_{ik} \) \((i \neq k)\) can be estimated using an intermediate alternative \( j \) as follows:

\[
h_{ik}^j = h_{ij} + h_{jk} = 0.5
\]  

where the operations “\( + \)” and “\( \times \)” are defined as follows:

Definition 4. Let \( h, h_1 \) and \( h_2 \) be three HFEs, and \( a \) be a real number then we have

\[
h_1 + h_2 = \bigcup_{\gamma \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2\}
\]

\[
h \times a = \bigcup_{\gamma \in h} \{\gamma - a\}
\]

In order to use Eq.(4) to estimate \( h_{ik}^j \), the alternatives in \( H \) should generally be classified into several sets defined as follows:

\[
B = \{(i, k) \mid i, k \in \{1, 2, \ldots, n\} \land (i \neq k)\}
\]

\[
OV^B = \{(i, k) \in B \mid h_{ik} \text{ is an estimated HFE}\}
\]

\[
KV^B = (OV^B)^c
\]

\[
M^j_{ik} = \{j \neq i, k \mid (i, j), (j, k) \in KV^B\}
\]

where \( B \) is a set of all pairs of alternatives, \( OV^B \) is a set of pairs of alternatives whose preference degrees are represented by the HFE \( h_{ik} \), and we call it an estimated HFE; \( KV^B \) is the complement set of \( OV^B \) satisfying \( KV^B \cup OV^B = B \); \( M^j_{ik} \) is the set of the intermediate alternatives \( x_j \) \((j \neq i, k)\) that can be used to estimate \( h_{ik}^j \) by Eq.(4).

Consequently, by Eq.(4) we may get several HFEs, \( h_{ik}^j \) \((j = 1, 2, \ldots; j \neq i, k)\) indicating all possible estimated preference degrees of the pair of alternatives \((i, k)\). The regression method is to select proper preference degrees from the estimated HFE \( h_{ik}^j \) for all \( i, k = 1, 2, \ldots, n; i \neq k \) so as to obtain a reduced HFPR. For this purpose, we first calculate an average estimated preference degree defined as follows:
where $S_s$ is a function that indicates the summation of all elements in a set, $\# h_{ik}^j$ indicates the numbers of all possible preference degrees in $h_{ik}^j$.

Comparing the estimated HFE $h_{ik}$ with its average estimated preference degree $h_{ik}^A$, we define the error between them below:

**Definition 5.** An HFE indicating all possible errors between an estimated HFE $h_{ik}$ and its average estimated preference degree $A_{ikh}$ is defined as:

$$\varepsilon h_{ik} = \frac{2}{3} (\bigcup_{\varepsilon_x \in (h_{ik}^k)} \left| \varepsilon_{ik} \right|)$$

where the coefficient $2/3$ is used to make sure each error in $\varepsilon_{ik}$ belongs to the unit interval $[0,1]$.

To choose a preference degree in $h_{ik}$ associated with the lowest error according to Definition 5, we define a final preference degree $h_{ik}^*$ satisfying

$$\min(\varepsilon h_{ik}) = \frac{2}{3} (\bigcup_{\varepsilon_x \in (h_{ik}^k)} \left| \varepsilon_{ik} \right|)$$

Collect $h_{ik}^*$ for all $i, k = 1,2,\ldots,n; i \neq k$, we get a FPR called a reduced HFPR $\bar{H}$ transformed from the HFPR $H$. We now introduce the consistency measure to measure the consistency level of $\bar{H}$.

**Definition 6.** For the reduced HFPR $\bar{H}$, the consistency level associated to the final preference degree $h_{ik}^*$ is stated as:

$$cl_{ik} = 1 - \min(\varepsilon h_{ik})$$

The consistency level associated to a particular alternative is defined as follows:

**Definition 7.** For the reduced HFPR $\bar{H}$, the consistency level associated to a particular alternative $x_i$ is stated as:

$$cl_i = \frac{n}{2(n-1)} \sum_{i \neq k, k=1}^{n} (cl_{ik} + cl_{ki})$$

With respect to all alternatives, we have

**Definition 8.** The consistency level of $\bar{H}$ is stated as:

$$cl_{\bar{H}} = \frac{\sum_{i=1}^{n} cl_i}{n}$$

Clearly, the bigger value of $cl_{\bar{H}}$ ($cl_{\bar{H}} \in [0,1]$), the higher consistency level of $\bar{H}$.

Based on the analysis above, for the HFPR, $H = (h_{ij})_{n \times n} \subset X \times X$, where $X = \{x_1, x_2, \ldots, x_n\}$, the algorithm that transforms an HFPR into a reduced HFPR with the highest consistency level is developed as follows:
**Algorithm I**

**Step 1.** Randomly locate an HFE, $h_{ik} (i \neq k)$, as the estimated HFE. According to Eq.(4), calculate $h_{ik}^j$ for all $j \neq i, k$.

**Step 2.** Calculate the average estimated preference degree $h_{ik}^s$ by Eq.(11), and choose a final preference degree $h_{ik}^*$ by Eqs.(12) and (13).

**Step 3.** Repeat Steps 1 and 2 until all HFEs have been located as the estimated HFEs, then go to the next Step.

**Step 4.** Saving $h_{ik}^*$ for all $i, k \in 1, 2, ..., n; i \neq k$, we have a reduced HFPR $\tilde{H}$.

**Step 5.** Calculating the consistency level of $\tilde{H}$ according to Eqs.(14), (15) and (16), we have the consistency level of $\tilde{H}$.

**Step 6.** End.

**Example 1.** Assume an HFPR:

$$
H_1 = \begin{pmatrix}
\{0.5\} & \{0.4,0.5\} & \{0.6,0.7\} & \{0.6\} \\
\{0.5,0.6\} & \{0.5\} & \{0.8\} & \{0.4\} \\
\{0.3,0.4\} & \{0.2\} & \{0.5\} & \{0.1,0.2\} \\
\{0.4\} & \{0.6\} & \{0.8,0.9\} & \{0.5\}
\end{pmatrix}
$$

**Step 1.** Locate the HFE, $h_{12}$, as the estimated HFE. According to Eq.(4), we have

$$
h_{12}^3 = h_{13} + h_{12} = 0.5 = \{0.3,0.4\} ,
\ h_{12}^4 = h_{14} + h_{12} = 0.5 = \{0.7\}.
$$

**Step 2.** According to Eq.(11), we have

$$
h_{12}^s = \frac{S_j \left( \sum_{j \in M_{12}^j} h_{12}^j \right)}{\sum_{j \in M_{12}^j} (#h_{12}^j)} = \frac{(0.3 + 0.4) + 0.7}{2 + 1} = 0.467.
$$

By Eqs.(12) and (13), we have

$$
\varepsilon_{h_{12}} = \frac{2}{3} \left( \bigcup_{h_{12}^j \in (h_{12}^s)} | e_{12}^j \right) = \{0.044,0.022\},
\ min(\varepsilon_{h_{12}}) = 0.022 = \frac{2}{3} |0.5 - 0.467|.
$$

Thus, $h_{12}^* = 0.5$.

**Step 3.** Repeat Steps 1 and 2, we have

$$
h_{13}^* = 0.7 , \ min(\varepsilon_{h_{13}}) = 0.100 ;
\ h_{14}^* = 0.6 , \ min(\varepsilon_{h_{14}}) = 0.189 ;
\ h_{21}^* = 0.5 , \ min(\varepsilon_{h_{21}}) = 0.022 ;
\ h_{23}^* = 0.8 , \ min(\varepsilon_{h_{23}}) = 0.056 ;
\ h_{24}^* = 0.4 , \ min(\varepsilon_{h_{24}}) = 0.100 ;
\ h_{31}^* = 0.3 , \ min(\varepsilon_{h_{31}}) = 0.067 ;
\ h_{32}^* = 0.2 , \ min(\varepsilon_{h_{32}}) = 0.056 ;
\ h_{34}^* = 0.2 , \ min(\varepsilon_{h_{34}}) = 0.089 ;
\ h_{41}^* = 0.4 , \ min(\varepsilon_{h_{41}}) = 0.189 ;
\ h_{42}^* = 0.6 , \ min(\varepsilon_{h_{42}}) = 0.100 ;
\ h_{43}^* = 0.8 , \ min(\varepsilon_{h_{43}}) = 0.089 .
$$

**Step 4.** Saving $h_{ik}^*$ for all $i, k \ (i, k = 1,2,3,4; i \neq k)$, we can get the reduced HFPR $\tilde{H}_1$. 


\[
\hat{H}_1 = \begin{pmatrix}
0.5 & 0.5 & 0.7 & 0.6 \\
0.5 & 0.5 & 0.8 & 0.4 \\
0.3 & 0.2 & 0.5 & 0.2 \\
0.4 & 0.6 & 0.8 & 0.5 \\
\end{pmatrix}.
\]

**Step 5.** According to Eq.(14), we have

\[
cl_{12} = 0.978, \ cl_{21} = 0.978, \ cl_{13} = 0.900, \ cl_{31} = 0.933, \ cl_{14} = 0.811, \ cl_{41} = 0.811
\]

\[
cl_{23} = 0.944, \ cl_{32} = 0.944, \ cl_{24} = 0.900, \ cl_{42} = 0.900, \ cl_{34} = 0.911, \ cl_{43} = 0.911
\]

According to Eq.(15), we can get

\[
cl_1 = \frac{(cl_{12} + cl_{21}) + (cl_{13} + cl_{31}) + (cl_{14} + cl_{41})}{6} = 0.902
\]

\[
cl_2 = \frac{(cl_{21} + cl_{12}) + (cl_{23} + cl_{32}) + (cl_{24} + cl_{42})}{6} = 0.941
\]

\[
cl_3 = \frac{(cl_{31} + cl_{13}) + (cl_{32} + cl_{23}) + (cl_{34} + cl_{43})}{6} = 0.924
\]

\[
cl_4 = \frac{(cl_{41} + cl_{14}) + (cl_{42} + cl_{24}) + (cl_{43} + cl_{34})}{6} = 0.874
\]

Furthermore, by Eq.(16), the consistency level of an HFPR \( \hat{H}_1 \) is

\[
cl_{\hat{H}_1} = \frac{cl_1 + cl_2 + cl_3 + cl_4}{4} = 0.910
\]

with the consistency level 91.0% .

**Step 6.** End.

For an HFPR, \( H = (h_{ij})_{n\times n} \), since each possible preference degree \( h_{ij} \) can be a final choice, as a more straightforward method, an HFPR can be separated into all possible FPRs, and one of the FPRs with the highest consistency level can be found out by comparing consistency levels of all possible FPRs. In order to illustrate the computation complexity of this “separation method”, we give the following example:

**Example 2.** Based on the same HFPR, \( H \), we can generate eight possible FPRs from \( H \) denoted as follows:

\[
\begin{align*}
&f_{11}^H = \begin{pmatrix}
0.5 & 0.5 & 0.6 & 0.6 \\
0.5 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.1 \\
0.4 & 0.6 & 0.9 & 0.5 \\
\end{pmatrix},
&f_{21}^H = \begin{pmatrix}
0.5 & 0.5 & 0.7 & 0.6 \\
0.5 & 0.5 & 0.8 & 0.4 \\
0.3 & 0.2 & 0.5 & 0.1 \\
0.4 & 0.6 & 0.9 & 0.5 \\
\end{pmatrix},
&f_{31}^H = \begin{pmatrix}
0.5 & 0.5 & 0.6 & 0.6 \\
0.5 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.1 \\
0.4 & 0.6 & 0.9 & 0.5 \\
\end{pmatrix},
&f_{41}^H = \begin{pmatrix}
0.5 & 0.5 & 0.7 & 0.6 \\
0.5 & 0.5 & 0.8 & 0.4 \\
0.3 & 0.2 & 0.5 & 0.1 \\
0.4 & 0.6 & 0.8 & 0.5 \\
\end{pmatrix},
&f_{51}^H = \begin{pmatrix}
0.5 & 0.5 & 0.7 & 0.6 \\
0.5 & 0.4 & 0.6 & 0.6 \\
0.6 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.1 \\
\end{pmatrix},
&f_{61}^H = \begin{pmatrix}
0.5 & 0.5 & 0.6 & 0.6 \\
0.5 & 0.4 & 0.6 & 0.6 \\
0.6 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.1 \\
\end{pmatrix},
&f_{71}^H = \begin{pmatrix}
0.5 & 0.5 & 0.6 & 0.6 \\
0.5 & 0.4 & 0.7 & 0.6 \\
0.6 & 0.5 & 0.8 & 0.4 \\
0.3 & 0.2 & 0.5 & 0.1 \\
\end{pmatrix},
&f_{81}^H = \begin{pmatrix}
0.5 & 0.5 & 0.6 & 0.6 \\
0.4 & 0.6 & 0.9 & 0.5 \\
0.4 & 0.6 & 0.9 & 0.5 \\
0.4 & 0.6 & 0.9 & 0.5 \\
\end{pmatrix}.
\end{align*}
\]
According to the consistency measure of FPRs introduced by Herrera-Viedma et al. [10], we can get the consistency levels of $f_{pi}^{H_i}$ $(i = 1, 2,..., 8)$ denoted by

\[
\begin{align*}
cl_{f_{pi}^{H_i}} &= 89.63\%, \\
cl_{f_{p1}^{H_i}} &= 91.76\%, \\
cl_{f_{p2}^{H_i}} &= 90.56\%, \\
cl_{f_{p3}^{H_i}} &= 92.69\%, \\
cl_{f_{p4}^{H_i}} &= 88.52\%, \\
cl_{f_{p5}^{H_i}} &= 90.65\%, \\
cl_{f_{p6}^{H_i}} &= 89.44\%, \\
cl_{f_{p7}^{H_i}} &= 91.57\%.
\end{align*}
\]

Since

\[
cl_{f_{p1}^{H_i}} > cl_{f_{p2}^{H_i}} > cl_{f_{p3}^{H_i}} > cl_{f_{p4}^{H_i}} > cl_{f_{p5}^{H_i}} > cl_{f_{p6}^{H_i}} > cl_{f_{p7}^{H_i}} > cl_{f_{p8}^{H_i}},
\]

$f_{i}^{H_i}$ is the reduced HFPR with the highest consistency level.

Obviously, $f_{i}^{H_i} = \tilde{H}_1$, the same results can be got from the proposed method and the separation method. Comparing Examples 1 and 2, in order to obtain the reduced HFPR with the highest consistency level, the numbers of operational times needed by our regression method and the separation method are $2n(n-1)+n+1$ and $m(n(n-1)+n+1)$ ($m$ is the number of all possible FPRs separated from an HFPR), respectively. Since $m \geq 2$ (at least two FPRs can be obtained separated from an HFPR), we have $m(n(n-1)+n+1) > 2n(n-1)+n+1$. The advantage of the regression method is that it is a convenient method to find out a FPR from an HFPR with the highest consistency level quickly. Utilizing the Matlab software for computation, we find that the proposed method can save much more time and is much more effective than the separation method, and the bigger $m$, the more convenient is the regression method.

4. Conclusions

In this paper, we have proposed a new concept of hesitant fuzzy preference relations (HFPRs) which are a common tool to collect and present preferences provided by decision makers (DMs) in decision making. To ensure that DMs are being neither random nor illogical in decision making, we have studied the consistency measure of HFPRs. Based on a principle called $\alpha$-normalization for hesitant fuzzy elements (HFEs), a regression method is proposed to transform the HFPR into a fuzzy preference relation (FPR), called a reduced HFPR, with the highest consistency level. The proposed results in this paper should be helpful for further theory studies and practical applications of HFSs. And in future work, we will develop some consistency measure for HFPRs based on the principle of $\beta$-normalization.

References


