Stabilization of Solutions for a Class of Degenerate Equations in Divergence Form in One Space Dimension

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Received August 20, 1985; revised September 3, 1987

By extending the concept of the Liapunov functional and an idea due to Dafermos we establish an Invariance Principle for a class of (not necessarily strongly) monotone systems. This principle is then applied to continuous contraction, strongly monotone systems on a Hilbert space and to a class of $L^1$ contraction monotone semigroups generated by degenerate equations in divergence form in one space dimension.

INTRODUCTION

In this paper we develop a method for obtaining stabilization as $t \to +\infty$ for solutions of a class of (not necessarily strongly) monotone systems. Some of the main features of the applications considered are

(i) the nonexistence of conventional Liapunov functionals,
(ii) the existence of infinitely many equilibria,
(iii) a comparison principle.

In Section 1 we introduce the abstract setting of an ordered Banach space, $X$, with an order-preserving nonlinear semigroup $\{S(t)\}$ of operators. This allows us (following Matano $[M]$) to define supersolutions

* Partially supported by NSF Grant MCS-820L540

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0022-0396/88 $3.00
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and subsolutions as points of $X$ decreased and increased, respectively, by $S(t)$ for $t \geq 0$. We define two "Liapunov Operators":

$$\bar{V}(x) = \text{minimal supersolution above } x,$$

$$\underline{V}(x) = \text{maximal subsolution below } x.$$

In Section 2 we show that the first of these operators decreases and the second increases along trajectories. The terms "increase" and "decrease" are meant with respect to the order in $X$ and so $\bar{V}$ and $\underline{V}$ are bonafide extensions of the concept of Liapunov functions. We then impose further restrictions on $S(t)$, namely, we require that if $y \in \omega(x)$, the $\omega$-limit set of $x$ then $\omega(x) = \omega(y)$. In other words $\omega$-limit sets are minimal sets of the flow (see [NS]). One way of assuring this is to assume that trajectories are Liapunov stable, that is, for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|x_1 - x_2\| < \delta$ implies $\|S(t)x_1 - S(t)x_2\| < \epsilon$ for all $t > 0$. This is trivially the case for contraction semigroups.

Our main abstract result is the following extension of the LaSalle Invariance Principle:

**THEOREM.** Let $X$ and $S(t)$ be as above. Let $\alpha \leq \beta$ be sub- and supersolutions, respectively, and let $p$ be such that $\alpha \leq p \leq \beta$. Assume that the orbit $\gamma(p) = \{S(t)p: t \geq 0\}$ is relatively compact. Then

$\bar{V}$ takes only one value, $\bar{q}$, on $\omega(p)$ and

$\underline{V}$ takes only one value, $\underline{q}$ on $\omega(p)$.

Moreover, $\bar{q}$ and $\underline{q}$ are equilibria, that is, $S(t)\bar{q} = \bar{q}$ and $S(t)\underline{q} = \underline{q}$ for all $t \geq 0$.

This abstract result is suitable for establishing stabilization for a certain class of parabolic equations and in Section 3 we consider some applications. The first is to a uniformly parabolic equation (which generates a strongly monotone flow):

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + g(x, u),$$

in $\Omega$ for $t > 0$ \hspace{1cm} (1)

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega, \quad u(x, 0) = u_0(x) \text{ in } \Omega,$$

where $\Omega$ is a smoothly bounded domain in $\mathbb{R}^n$. Under hypotheses which guarantee that (1) generates a contraction semigroup on $L^2(\Omega)$ we show that bounded orbits stabilize as $t \to +\infty$, that is, the $\omega$-limit set of such an
orbit is a single equilibrium. More generally, we note that precompact orbits of strongly monotone contraction semigroups on a Hilbert space stabilize. This result was obtained by M. Hirsch [Hi] with a completely different method.

To introduce another class of applications we next consider the problem

\[ u_t = [(u_x)^m + u^a(u - 1)V_x]_x \quad \text{for } t \geq 0, \quad 0 < x < 1 \]  

(2)

with boundary conditions

\[ (u_x)^m + u^a(u - 1)V_x = 0 \quad \text{at } x = 0, 1. \]

Here \( V = V(x) \) is a specified potential (see Fig. 1). Using the abstract results and a technique developed for a general class of degenerate parabolic equations (not strongly monotone) we prove the following:

**Theorem.** Let \( u_0 \in L^\infty(\Omega), \Omega = (0, 1), u_0 \geq 0. \)

(i) If \( 0 \leq \alpha < 2m - 1 \) then \( u(., t) = S(t) u_0 \) converges in \( C^1(\Omega) \) to an equilibrium as \( t \to \infty. \)

(ii) If \( \alpha > 2m - 1 \) then \( \int q_u < \infty, \) where \( q_u \) is the minimal unbounded equilibrium, and any solution with \( \int u_0 \geq \int q_u \) becomes unbounded in \( L^\infty \) as \( t \to \infty. \)

Equation (2) has the following features that identify a broad class of problems:

(i) it generates a contraction semigroup on \( L^1, \)

(ii) it conserves the integral,

(iii) it does not possess a conventional nontrivial Liapunov functional,

(iv) it is not strongly monotone.

Equation (2) is related to the Gurtin–Pipkin equation (see also [SKT])

\[ u_t = [u(u + V)_x]_x = \frac{1}{2}(u^2)_x + (uV)_x \]  

(3)

that arises in a biological context. Stabilization for a class of equations modelled after (3) is established in [BH, BGHP] by different methods. That work provided some of the motivation for the present paper.

Finally we consider the general equation

\[ u_t = [a(x, u, \phi(u))_x + b(x, u)]_x \quad \text{for } 0 < x < 1 \text{ and } t > 0; \]

\[ a(x, u, \phi(u))_x + b(x, u) = 0 \quad \text{at } x = 0, 1, \text{ and } t > 0; \]

\[ u(x, 0) = u_0(x) \quad \text{in } [0, 1]. \]  

(4)
Under appropriate hypotheses on \( a \) and \( \phi \) that include as special cases the operators
\[
a(x, u, \phi(u)_x) = (|u|^{m-1} u)_x \quad \text{(porous medium)}
\]
\[
a(x, u, \phi(u)_x) = |u_x|^{m-1} u_x \quad \text{(m-Laplacian)},
\]
assuming regularity, we establish stabilization for bounded solutions. In addition to the Invariance Principle our method relies on the set of equilibria having a certain but fairly general structure which is present for (2) and (3) (see Types \( T_1 \) and \( T_2 \) in Section 3). In the Appendix we develop the necessary regularity properties of the solutions to (4) when \( a \) and \( b \) have a prescribed form general enough to include Eqs. (2) and (3) and a case where \( a \) is the sum of the porous medium and \( m \)-Laplacian operators.

1. DEFINITIONS

Let \( X \) be a Banach space with a partial order defined by a cone, \( K \), that is, a closed convex subset of \( X \) such that (i) \( \lambda x \in K \) whenever \( \lambda \geq 0 \) and \( x \in K \), and (ii) \( K \cap (-K) = \{0\} \). We shall use some of the notation and results found in Krasnoselskii [Kr] (see also Amann [Am]). The order is given by \( y \geq x \) if and only if \( y - x \in K \). A set, \( S \), is said to be order bounded if and only if there exist elements \( y, z \in X \) such that \( y \geq x \geq z \) for all \( x \in S \). Assume

(1) \( K \) is regular, that is, order bounded monotone sequences converge in \( X \).

(II) \( K \) is minihedral, that is, given \( x, y \in X \) there exists \( z \in X \) such that \( z \leq x, z \leq y \) and if \( w \leq x, w \leq y \) for some \( w \in X \) then \( w \leq z \). We write \( z = \inf(x, y) \).

(III) There exists a strictly monotone functional, \( J \), on \( X \). That is, \( J \) satisfies
\[
y \geq x \quad \text{and} \quad y \neq x \quad \text{implies} \quad J(y) > J(x).
\]

Remark. If \( X \) is separable then there exists a continuous linear functional, \( L \), such that \( L(x) > 0 \) for all \( x \in K \setminus \{0\} \), so (III) holds (see [Kr]).

Remark. If \( X \) is a space of real valued functions and \( K \) is the cone of functions which are nonnegative then, if we do not impose more than Lipschitz continuity, (II) is satisfied, the greatest lower bound of two functions being their pointwise minimum. If \( X = L^p \) for some \( p \geq 1 \), then \( K \) is regular.
Let $S(t)$ be a semidynamical system on $X$, that is, a one parameter family of maps from $X$ into itself parameterized by $t \in \mathbb{R}^+$ and satisfying the axioms

(a) $S(t): X \to X$ is continuous for each $t \geq 0$,
(b) $S(t + \tau)f = S(t)S(\tau)f$ for $t \geq 0, \tau \geq 0$, and $f \in X$,
(c) $S(\cdot)f: [0, \infty) \to X$ is continuous for each $f \in X$.

We assume in addition that $S(t)$ is order preserving, that is, it satisfies

(d) $f \leq g \Rightarrow S(t)f \leq S(t)g$ for $t \geq 0$.

Following Matano [M] and Amann [Am] we give the

**Definition 1.1.** An element $f \in X$ is called a supersolution if $S(t)f \leq f$ for all $t > 0$ and a subsolution if $S(t)f \geq f$ for all $t \geq 0$.

**Definition 1.2.** For $p \in X$ let $\Sigma_p = \{ \rho \in X: \rho \geq p \text{ and } \rho \text{ is a supersolution} \}$ and $\sigma_p = \{ \rho \in X: \rho \leq p \text{ and } \rho \text{ is a subsolution} \}$.

From this definition one can easily show

(c) $\inf(x, y) \in \Sigma_p$ if $x, y \in \Sigma_p$ and $\sup(x, y) = -\inf(-x, -y) \in \sigma_p$ if $x, y \in \sigma_p$.

In the following we say that $u$ is a minimal element of a set $S$ provided $u \in S$ and $u \leq v$ for all $v \in S$.

**Lemma 1.3.** If $\Sigma_p$ is not empty, then it contains a unique minimal element. If $\sigma_p$ is not empty, then it contains a unique maximal element.

**Proof.** Suppose that $\Sigma_p \neq \emptyset$. Let $J$ be as in (III) and for $u \in \Sigma_p$ define

$$m(u) = \sup\{J(u) - J(v): v \in \Sigma_p, v \leq u\}.$$ 

Note that $m(u) \geq 0$ by (III). Also for $v \in \Sigma_p$

$$J(u) - J(v) \leq J(u) - J(p)$$

and so $m(u) \leq J(u) - J(p) < \infty$. Observe that if $u$ is not minimal in $\Sigma_p$ then by (II) and (e) there exists $w \in \Sigma_p$ such that $\bar{v} = \inf(u, w) \in \Sigma_p$ and $\bar{v} \neq u$. By (III) $J(\bar{v}) < J(u)$ and therefore $m(u) > 0$.

Let $u_0 \in \Sigma_p$ be arbitrary and select a sequence $\{u_n\}_{n=0}^{\infty}$ recursively by choosing $u_{n+1}$ such that

(i) $u_{n+1} \leq u_n$,
(ii) $u_{n+1} \in \Sigma_p$,
(iii) $J(u_n) - J(u_{n+1}) \geq \frac{1}{2}m(u_n)$. 

Since \( J(u_n) - J(u_{n+1}) + J(u_{n+1}) - J(v) = J(u_n) - J(v) \leq m(u_n) \) for each \( v \in \Sigma_p \) with \( v \leq u_{n+1} \), we have that
\[
\frac{1}{2} m(u_n) + m(u_{n+1}) \leq m(u_n).
\]
It follows that \( 0 \leq m(u_{n+1}) \leq \frac{1}{2} m(u_n) \) and so \( m(u_n) \to 0 \) as \( n \to \infty \).

By (I), \( \lim_{n \to \infty} u_n \) exists; call it \( u_\infty \). Since
\[
p \leq u_n \quad \text{and} \quad S(t) u_n \leq u_n
\]
for each \( t \geq 0 \) and all \( n \geq 1 \), using the facts that \( K \) is closed and \( S(t) \) is continuous we find
\[
p \leq u_\infty \quad \text{and} \quad S(t) u_\infty \leq u_\infty.
\]
This means that \( u_\infty \in \Sigma_p \). If \( u_\infty \) is not minimal then \( m(u_\infty) > 0 \). Since \( u_\infty \leq u_n \) we have
\[
J(u_\infty) - J(v) \leq J(u_n) - J(v) \leq m(u_n)
\]
for all \( v \in \Sigma_p \) such that \( v \leq u_\infty \).

Thus \( 0 < m(u_\infty) \leq m(u_n) \) for all \( n \). This contradicts the fact that \( m(u_n) \to 0 \) as \( n \to \infty \) and so proves that \( \Sigma_p \) has a minimal element.

If \( x \) and \( y \) were distinct minimal elements of \( \Sigma_p \) then by (e) \( z = \inf\{x, y\} \in \Sigma_p \) would be strictly smaller (in the order on \( X \)) than either \( x \) or \( y \), contradicting minimality. This establishes that part of the lemma concerning \( \Sigma_p \). The proof of that part concerning \( \sigma_p \), which we omit, is similar.

**Definition 1.4.** The upper Liapunov operator, \( \bar{V} \), is given by
\[
dom \bar{V} = \{ p \in X : \Sigma_p \neq \emptyset \}
\]
\[
\bar{V}(p) = \text{minimal element of } \Sigma_p.
\]

The lower Liapunov operator, \( V \), is given by
\[
dom V = \{ p \in X : \sigma_p \neq \emptyset \}
\]
\[
V(p) = \text{maximal element of } \sigma_p.
\]

**Remarks.** (i) The technique used in the construction of \( \bar{V} \) and \( V \) is from the unpublished work [CFP].

(ii) Even in the simple case where \( S(t) \) is the semigroup determined by the heat equation with homogeneous Dirichlet boundary conditions,
acting in $X = L^2(\Omega)$ for some bounded domain $\Omega$, $\bar{V}$ is nontrivial. In that case if $p$ is a function with compact support in $\Omega$, it can be shown that $\bar{V}(p)$ coincides with the solution of the obstacle problem [KS].

2. The Invariance Principle

As may be anticipated from their names, these Liapunov operators have a certain monotonicity along trajectories. Our results will be stated for $\bar{V}$, however, analogous statements are true for $V$. In this section we shall denote $\bar{V}$ by $V$.

**Proposition 2.1.** Let $u \in \text{dom } V$, then for $t \geq 0$,

$$S(t) u \in \text{dom } V \quad \text{and} \quad V(S(t) u) \leq V(u).$$

*Proof.* Since $u \in \text{dom } V$ there is a supersolution $q$ with $u \leq q$. Since $S(t)$ is order preserving $S(t) u \leq S(t) q \leq q$. Thus we see that $\emptyset \neq \Sigma_u \subset \Sigma_{S(t)u}$ and so $S(t) u \in \text{dom } V$.

Now, $u \leq V(u)$, by definition.

By (d), for $t \geq 0$

$$S(t) u \leq S(t) V(u),$$

$$\leq V(u),$$

since $V(u)$ is a supersolution. By minimality, $V(S(t) u) \leq V(u)$.

**Definitions 2.2.**

(i) The orbit or trajectory of $u \in X$ is the set

$$\gamma(u) = \{ S(t) u : t \geq 0 \}.$$

(ii) $\gamma(u)$ is called Liapunov stable if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\| u - v \| < \delta$ implies $\| S(t) u - S(t) v \| < \varepsilon$ for all $t \geq 0$.

Note that contraction semigroups produce Liapunov stable orbits trivially.

(iii) The $\omega$-limit set of $u$ is

$$\omega(u) = \{ v \in X : v = \lim_{n \to \infty} S(t_n) u \text{ for some } t_n \to \infty \}.$$ 

This brings us to our main abstract result:

**Theorem 2.3 (The Invariance Principle).** Let $u \in \text{dom } V$. Suppose $\gamma(u)$ is relatively compact and bounded below. Suppose that the trajectory through
any point of $\omega(u)$ is Liapunov stable. Then $V$ takes the same value on $\omega(u)$ and that value is an equilibrium, i.e., it is fixed under $S(t)$ for all $t \geq 0$.

Proof. Step I: If $v \in \omega(u)$ then $\omega(v) = \omega(u)$.

Verification. Clearly, $\omega(v) \subset \omega(u)$. Let $w \in \omega(u)$, then for two sequences $s_n, t_n \to \infty$ as $n \to \infty$ we have $v = \lim S(s_n) u$ and $w = \lim S(t_n) u$. Choose a subsequence $\{t_{n_j}\}$ of $\{t_n\}$ such that $\tau_j = t_{n_j} - s_j \to \infty$ as $j \to \infty$. Note that

$$
\|S(\tau_j) v - w\| \leq \|S(\tau_j) v - S(\tau_j) S(s_j) u\| + \|S(\tau_j + s_j) u - w\|.
$$

The first term on the right approaches zero by Liapunov stability and the second does so because $\tau_j + s_j = t_{n_j}$. Thus, $w \in \omega(v)$ and since $w \in \omega(u)$ was arbitrary, we are done. Note that the compactness of $\overline{y(u)}$ implies $\omega(u) \neq \emptyset$.

Step II: $\omega(u) \subset \text{dom } V$ and $V$ takes the same value on $\omega(u)$.

Verification. Let $v \in \omega(u)$ and let $\{s_n\}$ be such that $s_n \to \infty$ and $S(s_n) u \to w$ as $n \to \infty$. As in the proof of Proposition 2.1 we have $S(s_n) u \leq q$ for any supersolution $q$ with $u \leq q$. Since $K$ is closed, letting $n \to \infty$ gives $v \leq q$. Thus, we have $\emptyset \neq \Sigma_u \subset \Sigma_v$ for all $v \in \omega(u)$ and, hence, $\omega(u) \subset \text{dom } V$.

Now suppose that $w$ is another element of $\omega(u)$ and let $\{t_n\}$ be chosen so that $t_n \to \infty$ monotonically and $S(t_n) v \to w$ as $n \to \infty$. By Proposition 2.1 $\{V(S(t_n) v)\}$ is nonincreasing. Also $V(S(t_n) v) \geq S(t_n) v \in \overline{y(u)}$ which is bounded below. Assumption (I) (Regularity) on $K$ implies that $\lim_{n \to \infty} V(S(t_n) v) = q_0$ exists. By the continuity of $S(t)$ for each $t \geq 0$, we see that $q_0$ is a supersolution. Also $V(S(t_n) v) \geq S(t_n) v$ implies that $q_0 \geq w$. The last two observations imply that $q_0 \geq V(w)$. We have

$$
V(v) = V(S(0) u) \geq V(S(t_n) v) \geq q_0 \geq V(w).
$$

Since $v$ and $w$ were arbitrary elements of $\omega(u)$, symmetry shows that $V$ takes on only one value on $\omega(u)$, namely $q_0$.

Step III: $q_0$ is an equilibrium.

Verification. From the above, for any $v \in \omega(u)$ we have $V(v) = q_0$. By definition of $V$, $v \leq q_0$ and by (d) $S(t) v \leq S(t) q_0$ for $t \geq 0$. But $\omega(u)$ is positively invariant so $q_0 = V(S(t) v)$ and since $q_0$ is a supersolution $V(S(t) q_0) = S(t) q_0 \leq q_0$. It is easy to see that $a, b \in \text{dom } V$ and $a \leq b$ implies that $V(a) \leq V(b)$ and so we have $V(S(t) v) \leq V(S(t) q_0)$. Putting these observations together yields

$$
q_0 = V(S(t) v) \leq V(S(t) q_0) = S(t) q_0 \leq q_0 \quad \text{for all } t \geq 0.
$$
This shows that $S(t) q_0 = q_0$ for all $t \geq 0$ and completes the proof of the theorem.

**Remark 2.4.** Under the hypotheses of Theorem 2.3, to establish the stabilization of $u$ it is now sufficient to show that $\omega(u)$ consists of equilibria (or just supersolutions) since these remain fixed under $V$. This argument actually shows that $\omega(u)$ is a single equilibrium provided (a) $\omega(u)$ contains a supersolution $\tilde{q}$ and (b) $\tilde{q} \leq v$ for all $v \in \omega(u)$.

### 3A. An Example with Strong Monotonicity Due to Hirsch

Consider

\[
\frac{\partial u}{\partial t} = Au + g(x, u) \quad \text{on } \Omega \times (0, \infty)
\]

\[
\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0, \infty)
\]

\[u(x, 0) = u_0(x) \quad \text{on } \Omega,
\]

$\Omega \subseteq \mathbb{R}^N$, a smooth bounded domain,

\[
A = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial}{\partial x_i}
\]

\[
\lambda_1 |\xi|^2 \geq \sum_{i,j=1}^{N} a_{ij} \xi_i \xi_j \geq \lambda_2 |\xi|^2, \quad \lambda_1, \lambda_2 > 0,
\]

where $a_{ij}(\cdot) \in C^1(\Omega)$, $b(\cdot) \in C^0(\Omega)$, $g$ is Hölder in $x$ and Lipschitz in $u$, and $\partial/\partial \nu$ is the derivative operator along an outward-pointing vector transverse to $\partial \Omega$. We assume that there are real constants $c_1, c_2$ such that $c_2 > c_1$ and

\[g(x, c_1) > 0 > g(x, c_2), \quad x \in \Omega \]

and also for all $y$ in $[c_1, c_2]$

\[
\frac{\partial g}{\partial y}(x, y) + \lambda_A \leq 0, \quad x \in \Omega,
\]

where $\lambda_A$ is the maximal eigenvalue of $A$, real by the Krein–Rutman theorem.
The Setting

(3.1) generates a semidynamical system on $X \equiv L^2(\Omega)$ (see [H]). Inequality (3.2) implies via the maximum principle that the order interval

$[c_1, c_2] = \{q(\cdot) \in L^2(\Omega): c_1 \leq q \leq c_2\}$, with the natural ordering,

is positively invariant and (3.3) implies that (3.1) generates a contraction semigroup on $[c_1, c_2]$. The strong maximum principle and regularity theory [H] imply that if $t > 0$, $u_0 \leq v_0$, and $u_0 \neq v_0$, then

$$S(t)u_0 - S(t)v_0 \in \text{Int} C^0_+ (\Omega),$$

where $C^0_+ (\Omega) = \{f: \bar{\Omega} \to \mathbb{R} \text{ is continuous and } f \geq 0 \text{ on } \bar{\Omega}\}$, with the uniform topology.

The property expressed in (3.4) is strong monotonicity. We write

$$S(t)u_0 \preceq S(t)v_0.$$

**Proposition 3.1.** Every solution of (3.1) with $u_0 \in [c_1, c_2]$ converges as $t \to +\infty$, uniformly on $\bar{\Omega}$ to an equilibrium, that is, a time independent solution of (3.1).

**Proof.** The regularity theory in [H] implies that the set

$$\{S(t)u_0: \tau > 0\}$$

is relatively compact in $C^0(\bar{\Omega})$ hence in $L^2(\Omega)$. Let $v \in \omega(u_0)$. By Theorem 2.3

$$\bar{V}(S(t)v) = \bar{q}, \quad V(S(t)v) = q,$$

(3.5)

where $\bar{q}$ and $q$ are equilibria. By strong monotonicity

$q \preceq S(t)v \preceq \bar{q}$

(3.6)

unless $v$ coincides with one of the equilibria (in which case we are done). Since $S(t)$ is a contraction on a Hilbert space, by a well-known fact, the set of equilibria is convex. Therefore

$$q_\lambda = \lambda q + (1 - \lambda) \bar{q} \quad \text{with } \lambda \in (0, 1)$$

is an equilibrium. By choosing $\lambda$ sufficiently small we can achieve for some fixed $t > 0$,

$$S(t)v \preceq q_\lambda \preceq \bar{q}$$

(3.7)

which contradicts the minimality of $\bar{q}$ (cf. (3.5)). Therefore $v$ is an
equilibrium and since $v$ was arbitrary in $\omega(u_0)$ this set is a singleton (see Remark 2.4).

Remarks 3.2. Proposition 3.1 was established with completely different methods in [Hi].

An abstract version of the proof of Proposition 3.1 establishes stabilization of relatively compact orbits of a strongly monotone contraction semigroup $S(t)$ if $X$ is a strictly convex Banach space. On the other hand strong monotonicity is important, as can be seen from the following counterexample. Consider $\mathbb{R}^3$ and take a straight line $L$ through the origin which we may identify with the $x_1$ axis. Let $P$ be the orthogonal projection on $L$ and introduce the cone

$$C = \{x: Px \geq 0, |Px| \geq |P^+x|\}.$$

Finally define the semigroup $S(t)$ to be rotation by $t$ radians about $L$. It is easily seen that $S(t)$ is only weakly monotone and contractive and that given $x$ in $\mathbb{R}^3$, $\bar{V}(x)$ and $V(x)$ are well defined. All our abstract hypotheses are verified (take $J(x) = Px$). On the other hand the only convergent orbits are the ones corresponding to the equilibria (points of $L$).

For information on convergence for contraction semigroups in general we refer to [Pa].

3B. **An Example on $L^1$ without Strong Monotonicity**

Consider

$$u_x = [(u_x)^m]_x + [u^2(u - 1) V_x]_x \quad \text{on } (0, 1) \times (0, \infty)$$

$$(u_x)^m + u^2(u - 1) V_x = 0 \quad \text{on } \{0, 1\} \times (0, \infty) \quad (3.8)$$

$$u(x, 0) = u_0(x) \geq 0, \quad u_0(\cdot) \in L^\infty(0, 1).$$

We take $m > 1$, $\alpha \geq m$ and $V$ a $C^2$ function with a graph as in Fig. 1. If $q$ is a nonnegative $C^1$ function satisfying

$$(q_x)^m + q^2(q - 1) V_x = 0 \quad \text{on } [0, 1]$$

then $S(t)q = q$ where $S(t)$ is the semigroup on $L^1(0, 1)$ associated with (3.8). Let $q_u$ be the minimal unbounded equilibrium.

The Setting

Let $X = L^1(0, 1)$ with the natural ordering. By the results in the Appendix (3.8) generates a semigroup $S(t)$ that is a contraction on $L^1(0, 1)$, order

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1 We are indebted to Professor M. Hirsch for this.
preserving, and conserves the integral. Moreover the abstract semigroup solution satisfies the equation in the sense of distributions and equilibria are $C^1$ time independent solutions of (3.8). Furthermore, for $a < 2m - 1$ the orbit

$$\{ S(t)u_0; t \geq \tau > 0 \}$$

is relatively compact in $C^1[0, 1]$.

**Theorem 3.3.**

(i) If $a < 2m - 1$, then $u(\cdot, t)$ converges in $C^1[0, 1]$ to an equilibrium as $t \to \infty$.

(ii) If $a > 2m - 1$, then $\int q_u < \infty$ (see Fig. 1) and any solution with $\int u_0 \geq \int q_u$ becomes unbounded in $L^\infty$ as $t \to \infty$.

**Note.** (a) The case $a = 2m - 1$ has been settled by C. Grant who showed global existence and boundedness.

(b) We have recently shown in [AB] that in case (ii), solutions blow up in finite time.

**Proof of Part (i).** Since we have a contraction semigroup, for stabilization it is sufficient to establish that the $\omega$-limit set consists of equilibria. For the convenience of the reader we first treat data for which an a priori $L^\infty$ bound can be established by means of comparison.

**Step 1:** $u_0 < q_u$. Let $\xi \in \omega(u_0)$. We will assume that $\xi$ is not an equilibrium and we will reach a contradiction. By Theorem 2.3 for $t > 0$

$$\bar{V}(S(t)\xi) = \bar{q}, \quad V(S(t)\xi) = q,$$

where $\bar{q}, q$ are fixed equilibria. Moreover $\bar{V}, V$ take these values on $\omega(u_0)$.

Even though it is not essential for our argument to separate cases we
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note that by the continuity of $\xi$, near any point where $\xi \neq 0$, the equation is nondegenerate and so using the distribution of the equilibria (Fig. 1) and the strong maximum principle we can conclude that necessarily $\tilde{q}$ and $q$ make a degenerate contact with $\xi$. We consider a typical case where $0 \leq \xi \leq 1$, $\xi$ contacts $\tilde{q}$ where $\tilde{q} = 1$ and contacts $q$ where $q = 0$ (Fig. 2). This case serves to demonstrate the general argument which will be given in Section 3C.

To reach a contradiction it will be sufficient to show that $\xi$ under the action of $S(t)$ will move below an equilibrium lower than $\tilde{q}$. For comparison purposes define a $C^1$ function

$$p(x) = \begin{cases} \text{transverse to the equilibria crossed} & \text{on } [0, x_1] \\ \tilde{q}(x) & \text{on } [x_1, 1]. \end{cases}$$

See Fig. 2 and refer to Section 3C for a more detailed description.

By construction we have

$$(p_x)^m + p^*(p - 1) V_x \geq 0.$$  \hspace{1cm} (3.9)

Let

$$w(x, t) = \int_0^x S(t) p, \quad w_0(x) = \int_0^x p$$

and note that $w$ satisfies

$$w_x = \left| w_{xx} \right|^{m-1} w_{xx} + (w_x)^2(w_x - 1) V_x$$

$$w(0, t) = 0, \quad w(1, t) = \int_0^1 S(t) p = \int_0^1 p$$  \hspace{1cm} (3.10)

(by conservation).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Figure 2}
\end{figure}
By (3.9)

\[ |w_{0xx}|^{m-1} w_{0xx} + (w_{0x})^*(w_{0x} - 1) V_x \geq 0. \]  

(3.11)

Therefore by adapting a classical result on subsolutions found in [Sa] \( w(x, t) \) is monotonically increasing in \( t \) for fixed \( x \). Since \( w(x, t) \) is bounded it converges to some function \( w(x) \) as \( t \to \infty \). It is easy to see then that \( S(t) p \) converges as \( t \to \infty \) (in \( C^1(\Omega) \)) to an equilibrium \( q \). Since

\[ \int p < \int \bar{q} \]

this equilibrium has to lie below \( \bar{q} \) (see Fig. 1). On the other hand,

\[ \xi \leq p \Rightarrow S(t) \xi \leq S(t) p \]

and so the desired contradiction is reached.

**Step II: Relaxing \( u_0 < q_u \).** Since \( u_0 \) is in \( L^\infty \) and \( \alpha < 2m - 1 \) by Lemma A2, \( |u(\cdot, t)|_{L^\infty} < C \). Let \( f(u) = u^\alpha (u - 1) \) for \( |u| < C + 1 \) and extend this function in a positive smooth and bounded fashion, \( \tilde{f} \), on \( \mathbb{R}^+ \). Note that the equilibria of the new system do not blow up for finite \( x \) and so the hypothesis of Step I is verified. Therefore, by the argument in Step I, \( u(x, t) \) stabilizes to some equilibrium \( q \),

\[ (q_x)^m + \tilde{f}(q) V_x = 0. \]

Hence \( q \leq C \) and so \( q \) is an equilibrium solution to (3.8).

**Proof of Part (ii).** Note that near the blowup point \( x = 1 \), \( q_u \) satisfies essentially an equation of the form

\[ q''_u = q_u^{(\alpha + 1)/m}. \]

Therefore

\[ q_u(x) \approx (1 - x)^{\lambda/(1 - \lambda)}, \quad \lambda = \frac{\alpha + 1}{m} \]

and so if \( \alpha < 2m - 1 \) the "mass" of \( q_u \) is infinite. A consequence of this is that in this range there are bounded equilibria of arbitrarily large mass. On the other hand if \( \alpha > 2m - 1 \), \( q_u \) has finite "mass." If \( |u(\cdot, t)|_{L^\infty} \) is uniformly bounded, by the proof of (i) above, the solution has to converge to a bounded equilibrium of (3.8), an impossibility by conservation since all the bounded equilibria have mass less than that of \( q_u \) due to the ordering.
Note. From Fig. 1 one can see that the integral does not uniquely determine the equilibrium.

3C. THE GENERAL THEOREM FOR A CLASS OF EQUATIONS IN DIVERGENCE FORM

Consider

$$u_t = [a(x, u, \phi(u)_x) + b(x, u)]_x, \quad 0 < x < 1, \quad t > 0$$

$$a(x, u, \phi(u)_x) + b(x, u) = 0 \quad \text{at} \quad x = 0, 1, \quad t > 0 \quad (3.12)$$

$$u(x, 0) = u_0(x), \quad 0 \leq x \leq 1.$$ 

We assume

(H1) $a$ is continuous and $\phi$ is $C^1$.

(H2) $a(x, u, \cdot)$ and $\phi$ are strictly increasing.

(H3) $a(x, u, 0) = 0 = \phi(0)$ and $a(x, u, v) \to \pm \infty$ as $v \to \pm \infty$.

Remarks 3.4. (i) We could consider the more general situation where

$$a(x, u, v) = g(x, u) A(x, u, v) \quad \text{and} \quad b(x, u) = g(x, u) B(x, u) \quad \text{with} \quad A \text{ satisfying (H1)-(H3)} \quad \text{and} \quad g \text{ continuous nonnegative and zero exactly on a set of the form } \{(x, u): x \in \{x_i\}_{i=1}^n \cup \{(x, u): u = h_i(x), 1 \leq i \leq m\} \}, \text{ where the functions } \{h_i\}_{i=1}^m \text{ are Lipschitz continuous. We would simply work on each fixed interval } [x_j, x_{j+1}], 0 \leq j \leq n, \text{ where } x_0 = 0 \text{ and } x_{n+1} = 1. \text{ The curves } u = h_i(x), 0 \leq i \leq m, \text{ are just additional equilibria and do not create any difficulties.}$$

(ii) Several important operators are included in (3.12) for instance:

Taking $a(x, u, v) = v$ and $\phi(u) = |u|^{n-1}u, n > 1$, gives the porous medium operator.

Taking $a(x, u, v) = |v|^{m-1}v$ and $\phi(u) = u, m > 1$, gives the $m$-Laplacian.

Taking $a(x, u, v) = \mu |v|^{m-1}v + vn |u|^{n-1}v$ and $\phi(u) = u$ gives the sum of the previous two. This is provided that weak solutions are sufficiently smooth that $(|u|^{n-1}u)_x = n |u|^{n-1}u_x$.

In the following we describe our method for showing that $\omega$-limit sets consist of equilibria. Our aim is not to delve into questions of regularity, existence of the semigroup, etc.; these questions are answered in the Appen-
dix for some model cases. Therefore, we assume that (3.12) generates an order-preserving semigroup, $S(t)$, on $L^1(0, 1)$ which conserves the integral, and that $L^\infty$-bounded orbits are relatively compact in $C([0, 1])$. We also assume that the semigroup solution with $L^\infty$ initial data satisfies the equation in the sense of distributions and has appropriate regularity properties and in particular the semigroup equilibria are weak time independent solutions of (3.12)

**Lemma 3.5.** Suppose that $p$ is continuous and piecewise continuously differentiable and satisfies

$$a(x, p(x), \phi(p(x))) + b(x, p(x)) \geq 0 \quad (3.13)$$

and

$$p(x) \leq q(x)$$

for all $x \in [0, 1]$, where $q$ is an equilibrium and derivatives are interpreted from the left and right.

Then $p$ stabilizes, that is, $\omega(p) = \{ \bar{q} \}$ where $\bar{q}$ is an equilibrium.

**Proof.** Define

$$v_0(x) = \int_0^x p(y) \, dy$$

and

$$v(x, t) = \int_0^x (S(t)p)(y) \, dy.$$ 

Then $v$ satisfies

$$v_t = a(x, v, \phi(v_x)) + b(x, v_x) \quad 0 < x < 1, \quad t > 0$$

$$v(0, t) = 0, \quad v(1, t) = \bar{p} = \int_0^1 p(y) \, dy, \quad t > 0 \quad (3.14)$$

$$v(x, 0) = v_0(x).$$

Note that conservation of the integral was used for the boundary condition at $x = 1$. Now

$$a(x, v_0x, \phi(v_0x)) + b(x, v_0x) = a(x, p, \phi(p_x)) + b(x, p) \geq 0 \quad \text{on} \quad [0, 1].$$

By adapting a well-known result in [Sa] (see the Remark 3.7 below) we deduce that $v(x, \cdot)$ is nondecreasing for each $x \in [0, 1]$. 
Since \( v(x, t) \leq \int_0^3 q(y) \, dy \) we conclude that \( \lim_{t \to \infty} v(x, t) \) exists; call it \( v_\infty(x) \). Since orbits for \( S(t) \) are compact in \( C([0, 1]) \) we have \( \omega(p) \neq \emptyset \). If \( \tilde{q} \in \omega(p) \) then for some unbounded increasing sequence \( \{ t_n \} \),

\[
v_\infty(x) = \lim_{n \to \infty} \int_0^x (S(t_n) p)(y) \, dy = \int_0^x \tilde{q}(y) \, dy.
\]

This implies that \( v_\infty \) is \( C^1 \) and \( \bar{q} = v'_\infty \). Thus we see that \( \bar{q} \) is unique and \( \omega(p) = \{ \bar{q} \} \). Since the \( \omega \)-limit set is invariant, \( \bar{q} \) is an equilibrium.

**Remark 3.6.** A similar result holds when both inequalities in the statement of the lemma are reversed.

**Remark 3.7.** Following the proof of Theorem 2.5.1 in [Sa] one has the result: If \( w \) satisfies

\[
w_t = w_{xx} + b(x, w_x) \\
w(0, t) = 0, \quad w(1, t) = \bar{w} \\
w(x, 0) = w_0(x) \quad \text{where} \quad w_{0xx} + b(x, w_{0x}) \geq 0
\]

then for each \( x \in [0, 1] \), \( w_t(x, t) \geq 0 \).

The idea of proof is to take the derivative with respect to \( t \) in the equation and use the Maximum Principle. For degenerate equations some additional care is needed. One approach is to regularize the equation, as is done in the Appendix, by perturbing the operator into one which is strictly parabolic, prove the desired result for the perturbed equation, and then deduce the result for the given equation.

Now we give conditions on the distribution of equilibria for (3.12) which allow the use of the previous lemma to deduce stabilization of all trajectories.

**Definitions.** We say that the equilibrium equation

\[
a(x, u, \phi(u)_x) + b(x, u) = 0 \quad 0 < x < 1
\]

is of

**Type T_1.** If no two equilibria agree at two points of \([0, 1]\) and of

**Type T_2.** If whenever \( q \leq Q \) are two equilibria which agree at two points, \( \alpha < \beta \) say, of \([0, 1]\) then there is an equilibrium \( \tilde{q} \) with \( q \leq \tilde{q} \leq Q \) and satisfying: If \( \tilde{q} \) is a distinct equilibrium with either \( \tilde{q} \leq q \leq Q \) or \( q \leq \tilde{q} \leq \tilde{q} \) on \([\alpha, \beta]\) then \( \tilde{q} = \tilde{q} < Q \) or, respectively, \( \tilde{q} = \tilde{q} > q \) on an open subinterval of \([\alpha, \beta]\).
Remark 3.8. To visualize "Type $T_2$" equations we may say that if two solutions agree at two points then we allow at most three initially distinct solutions to emanate from one of these points. Thus we do not allow the situation depicted in Fig. 3 but we do allow that in Fig. 4. For instance in the case that (3.15) is
\[ u_x - 3(1/2 - x) u^{1/3}, \]
then $Q(x) = \left[ \frac{1}{4} - (\frac{1}{2} - x)^2 \right]^{3/2} = -q(x)$ and $\bar{q}(x) \equiv 0$ fit the definition.

**Theorem 3.9.** Under the assumptions of this section if (3.15) is of type $T_1$ or type $T_2$ the solution to (3.12) with $u_0 \in L^\infty(0, 1)$ stabilizes, provided $u_0$ is bounded below by a subsolution and above by a supersolution.

**Proof.** Let $\zeta \in \omega(u_0)$, $q = \mathcal{V}(\zeta)$, and $Q = \mathcal{V}(\bar{\zeta})$. Suppose that $\zeta$ is not an equilibrium, then $q \leq \zeta \leq Q$ and $q \neq Q$. By taking a subinterval if necessary, we may assume $q < Q$ on $(0, 1)$.

*Type $T_1$: Case 1.* $q(1) < \zeta(1)$. 

![Figure 3](image3.png)

![Figure 4](image4.png)
For $\varepsilon > 0$ small and fixed define

$$p(x) = \begin{cases} q(x) & \text{on } [0, 1 - \varepsilon] \\ \phi^{-1}(M(x - 1 + \varepsilon) + \phi(q(1 - \varepsilon))) & \text{on } [1 - \varepsilon, 1] \end{cases},$$

where $M = \max \{\phi(r(x)), q\} < r \leq Q$ and $r$ is an equilibrium and $\varepsilon$ is so small that $p < \xi$ on $[1 - \varepsilon, 1]$ (see Fig. 5).

The point is that $p$ is constructed to be a continuous, piecewise $C^1$ function which is transverse to the vector field defined by (3.15), that is, $p$ satisfies (3.13). To see this fix $x \in (1 - \varepsilon, 1)$ and consider the point $(x_1, p(x_1))$. There is some equilibrium $r$ passing through this point and satisfying $q \leq r \leq Q$. At $x$, we have $d(p)_x = M > 0$, and so

$$a(x, p, \phi(p)_x) + b(x, p) \geq a(x, p, \phi(r)_x) + b(x, p) = a(x, r, \phi(r)_x) + b(x, r) = 0.$$

at $x_1$. Since $x_1 \in (1 - \varepsilon, 1)$ is arbitrary (3.13) holds. From Lemma 3.5, $p$ stabilizes to an equilibrium $\tilde{q}$. By comparison we have

$$q \leq \tilde{q} \leq \xi \quad \text{for all } \xi \in \omega(\tilde{q}).$$

Also $\tilde{q} \neq q$ since $\int_0^1 \tilde{q} = \int_0^1 p > \int_0^1 q$ by conservation and construction. This contradicts the fact that $\tilde{q} \leq V(\tilde{\xi}) = V(\omega(\tilde{q})) = V(\omega(u_o)) = q$.

The cases $Q(1) > \xi(1)$, $q(0) < \xi(0)$, and $Q(0) > \xi(0)$ are similar and are omitted.

Type $T_2$: Case 1. Assume that there is a point $x_0 \in (0, 1)$ with

$$(S(t) \xi)(x_0) = q(x_0) \quad \text{for all } t \geq 0.$$
Fix $\varepsilon > 0$ small and let

$$
p_1(x) = \begin{cases} 
Q(x) & \text{on } [0, x_0 - \varepsilon] \\
\phi^{-1} \left( \frac{[\phi(q(x_0 + \varepsilon)) - \phi(Q(x_0 - \varepsilon))]}{2\varepsilon} \right) (x - x_0 + \varepsilon) + \phi(Q(x_0 - \varepsilon)) & \text{on } [x_0 - \varepsilon, x_0 + \varepsilon] \\
q(x) & \text{on } [x_0 + \varepsilon, 1] \\
\phi^{-1} \left( \frac{[\phi(Q(x_0 + \varepsilon)) - \phi(q(x_0 - \varepsilon))]}{2\varepsilon} \right) (x - x_0 + \varepsilon) + \phi(q(x_0 - \varepsilon)) & \text{on } [0, x_0 - \varepsilon] \\
Q(x) & \text{on } [x_0 - \varepsilon, x_0 + \varepsilon] \\
q(x) & \text{on } [x_0 + \varepsilon, 1] 
\end{cases}
$$

(see Fig. 6).

The requirement on $\varepsilon$ is that $p_1$ and $p_2$ are transverse to the vector field determined by (3.15) in the sense that on $[0, 1]$

$$a(x, p_1, \phi(p_1)_x) + b(x, p_1) \leq 0 \leq a(x, p_2, \phi(p_2)_x) + b(x, p_2),$$

where left and right sided derivatives are taken at $x_0 \pm \varepsilon$. We also require $p_1 \geq \xi$ on $[0, x_0]$ and $p_2 \geq \xi$ on $[x_0, 1]$, which holds for $\varepsilon$ small.

By Lemma 3.5 we know that $\omega(p_1) = \{q_1\}$ and $\omega(p_2) = \{q_2\}$ where $q_1$ and $q_2$ are equilibria.
Not only this but from the proof of Lemma 3.5 we actually have that

\[
\int_{0}^{x_0} (S(t) p_1)(y) \, dy \quad \text{is nonincreasing in } t
\]

and

\[
\int_{x_0}^{1} (S(t) p_2)(y) \, dy \quad \text{is nonincreasing in } t.
\]

Since

\[
\int_{0}^{x_0} p_1(y) \, dy < \int_{0}^{x_0} Q(y) \, dy
\]

and

\[
\int_{x_0}^{1} p_2(y) \, dy < \int_{x_0}^{1} Q(y) \, dy
\]

and

\[p_1, p_2 \leq Q \quad \text{on } [0, 1],\]

we have \(q_1, q_2 \leq Q\) on \([0, 1]\) with \(q_1 \neq Q\) on \([0, x_0]\) and \(q_2 \neq Q\) on \([x_0, 1]\).

Because \((S(t) \xi)(x_0) = q(x_0)\) is fixed for all \(t \geq 0\) we find that on \([0, x_0]\), \(S(t) \xi\) coincides with \(S(t) \xi_1\) where

\[
\xi_1 = \begin{cases} 
\xi & \text{on } [0, x_0] \\
q & \text{on } [x_0, 1].
\end{cases}
\]

Similarly, on \([x_0, 1]\), \(S(t) \xi\) coincides with \(S(t) \xi_2\) where

\[
\xi_2 = \begin{cases} 
q & \text{on } [0, x_0] \\
\xi & \text{on } [x_0, 1].
\end{cases}
\]

The foregoing uses some mild regularity of solutions which was assumed to hold from the outset. Let \(\zeta \in \omega(\xi)\).

Since \(p_1 \geq \xi_1\) and \(p_2 \geq \xi_2\) we find that \(q_1 \geq \xi_1\) on \([0, x_0]\) and \(q_2 \geq \xi_2\) on \([x_0, 1]\). Now suppose \(q_1(x_0) \leq q_2(x_0)\), then

\[
\bar{q}(x) = \begin{cases} 
\max\{q_1(x), q_2(x)\} & \text{on } [0, x_0] \\
q_2(x) & \text{on } [x_0, 1].
\end{cases}
\]
is an equilibrium which satisfies

\[ \xi \preceq \tilde{q} \preceq Q \quad \text{and} \quad \tilde{q} \not\equiv Q. \]

But this contradicts the fact that \( \xi \in \omega(\xi) = \omega(u_0) \) and so \( \bar{V}(\xi) = Q \) is the minimal equilibrium greater than \( \xi \). If \( q_1(x_0) \geq q_2(x_0) \) taking

\[ q_i(x) = \max\{q_1(x), q_2(x)\} \]

on \([0, x_0]\) provides a contradiction. The result is therefore established in this case. A similar argument can be given for the case where \( (S(t), \xi)(x_0) = Q(x_0) \) for all \( t \geq 0 \), for some \( x_0 \in (0, 1) \).

**Case 2.** Let \( \tilde{q} \) be as given in the definition of "Type T2." Since \( \xi \) is not an equilibrium, there exists \( \tilde{x} \in [0, 1] \) such that \( q(\tilde{x}) < \xi(\tilde{x}) < Q(\tilde{x}) \). Since \( q \leq \tilde{q} \leq Q \) it follows that either \( \tilde{q}(\tilde{x}) > q(\tilde{x}) \) or \( \tilde{q}(\tilde{x}) < Q(\tilde{x}) \). We shall assume that the second alternative holds; the analysis for the other alternative is similar and will be omitted.

Let \( \bar{u} = \max\{\xi, \tilde{q}\} \), then \( \bar{u} \) is continuous, \( \bar{u} \leq \tilde{u} \leq Q \), and \( \tilde{q} \not\equiv Q \) on \([0, 1]\). Let \( \tilde{w} \in \omega(\tilde{u}) \), then \( \tilde{w} \leq Q \) and \( \tilde{w} \neq Q \) because \( \int_0^1 \tilde{u} < \int_0^1 Q \). Furthermore \( \xi \leq \tilde{u} \leq Q \) implies that \( \bar{V}(\tilde{w}) = Q \) so \( \tilde{w} \) is not an equilibrium, nor can \( \omega(\tilde{w}) \) contain any equilibrium.

Let \( q_1 = \bar{V}(\tilde{w}) \) and let \( q_2 \) be another equilibrium such that \( q_1 \leq q_2 \leq Q \) with \( q_1 \neq q_2 \neq Q \). Since \( q_2 = \tilde{q} \) on an open subinterval of \([0, 1]\), by definition of \( T_2 \), and since \( \tilde{q} \leq q_1 \leq q_2 \), there must be an open subinterval, \( I \), of \([0, 1]\) on which \( q_2 = q_1 \) (see Fig. 7).
Take $I$ to be maximal with this property. Since $q_2 \neq q_1$ on $[0, 1]$ either the left or the right endpoint of $I$ lies in $(0, 1)$. Assume the latter and call this endpoint $x_0$. By the definition of $T_2$ it is possible to ensure that $Q(x_0) > q_2(x_0) = q_1(x_0)$. By Case 1, since $\omega(\tilde{w})$ contains no equilibrium, $(S(t) \tilde{w})(x_0) \neq q_1(x_0)$ for some $t > 0$ so there is no loss of generality in assuming that $\tilde{w}(x_0) > q_1(x_0) = q_2(x_0)$. For $\varepsilon > 0$ and small define

$$p(x) = \begin{cases} 
q_1(x) & \text{on } [0, x_0] \cup [x_0 + 2\varepsilon, 1] \\
q_2(x) & \text{on } [x_0, x_0 + \varepsilon] \\
\phi^{-1}\left(\frac{[\phi(q_1(x_0 + 2\varepsilon)) - \phi(q_2(x_0 + \varepsilon))]}{\varepsilon}(x - x_0 - \varepsilon) + \phi(q_2(x_0 + \varepsilon))\right) & \text{on } [x_0 + \varepsilon, x_0 + 2\varepsilon].
\end{cases}$$

By continuity, $\varepsilon$ may be chosen so small that on $[0, 1]$, $p \leq \tilde{w}$ and $a(x, p, \phi(p)_x) + b(x, p) \leq 0$ (see Fig. 7).

By Lemma 3.5 we know that $p$ stabilizes, that is, $\omega(p) = \{q_3\}$ where $q_3$ is an equilibrium. Since $q_1 \leq p \leq \tilde{w}$ we have $q_1 \leq q_3 \leq \int \omega(\tilde{w}) = q_1$. But then $\int q_1 < \int p = \int q_3$ provides a contradiction. This completes the proof of the theorem.

**Appendix**

We give a brief introduction to the regularity theory omitting most proofs. We consider the model equation

$$u_t = \{\mu(u_x)^m + v(u^n)_x + b(x, u)\}_x \quad \text{in } (0, 0, \infty)$$

$$u_x = \mu(u_x)^m + v(u^n)_x + b(x, u) \quad \text{on } \{0, 1\} \times (0, \infty) \quad \text{(A1)}$$

$$u(x, 0) = u_0(x) \in L^\infty(0, 1).$$

For brevity we write $v^p$ for $|v|^{p-1}v$.

Assume that $\mu, v \geq 0$ with $\mu^2 + v^2 \neq 0$ and that $m, n \geq 1$ are real numbers. Assume that $b$ is locally Lipschitz continuous in $(x, u)$ and satisfies the growth condition

$$|b(x, u)| \leq c_1 + c_2 |u|^q$$

where

$$q < 2m \quad \text{if } \mu > 0$$

and

$$q < n + 1 \quad \text{if } \mu = 0.$$

We will denote $(0, 1)$ by $\Omega$ and $\Omega \times [0, t]$ by $Q_t$. 
Definition of Solution

$u : [0, \infty ] \to L^1(\Omega)$ is a weak solution of (A1) if

(i) $u \in C([0, t]; L^1(\Omega)) \cap L^\infty(\Omega_t)$ for all $t \in (0, \infty ),$

(ii) $\int_\Omega \int |u_x|^{p+1} < \infty$ if $\mu > 0$, and $\int_\Omega \int |(u^n)_x|^2 < \infty$ if $\mu = 0$,

(iii) $\int_\Omega u(t) \psi(t) = \int_\Omega u_0 \psi_0 + \int_\Omega \int \{ u\psi_t - [\mu(u^n)_x + \nu(u^n)^2 + b(x, u^n)] \psi_x \}$ for all $\psi \in C^1(\Omega_t), t > 0$.

We will be working with the regularized version of (A1),

$$u^\varepsilon(x, 0) = u_0(x),$$

$u^\varepsilon(x, 0) = u_0(x)$, with the zero flux condition on the boundary.

Here $g_\varepsilon(z) = z(z^2 + \varepsilon)^{(m-1)/2}$, $\phi_\varepsilon(z) = z^n + \varepsilon z$, where $\varepsilon > 0$.

**Lemma A1.** Let $u^\varepsilon$ be a classical solution to $(A1)_\varepsilon$. Then

$$\mu \int_0^t \int_\Omega [(u^\varepsilon)_x^2 + \varepsilon]^{(m+1)/2} \leq C(t)$$

and

$$\nu \int_0^t \int_\Omega [\phi_\varepsilon(u^\varepsilon)_x]^2 \leq C(t),$$

where $C$ is independent of $\varepsilon$.

**Proof.** In the case $\mu > 0$, multiply by $u^\varepsilon$, integrate by parts, and use the fact that $u^\varepsilon$ is bounded on $\Omega \times [0, t]$. In the case that $\mu = 0$, multiply by $(u^\varepsilon)_x^2 + \varepsilon u^\varepsilon$ and integrate by parts.

**Lemma A2.** Let $u^\varepsilon$ be a solution to $(A1)_\varepsilon$ with $q < 2m$ if $\mu > 0$ ($q < n + 1$ if $\mu = 0$). Then

$$|u^\varepsilon(x, t)| \leq M < \infty,$$

where $M$ is independent of $\varepsilon$.

The proof of Lemma A2 is a suitable variation of Moser's iteration technique in the vein of [A1], and it will appear elsewhere. Notice that (A2) cannot be obtained in general by a comparison argument as is evident from the example in Section 3B (take $u_0$ not to lie below $q_u$). The same example shows that (A2) fails in general for $q > 2m$ ($q > n + 1$ if $\mu = 0$). The equality cases remain open.
LEMMA A3. Let \( u^\varepsilon \) be a solution to \((A1)_\varepsilon \).

(i) If \( \mu = 0, \nu > 0 \), then for every \( \tau > 0 \) and \( T > \tau \) there exists a continuous function \( w_\varepsilon \) such that \( w_\varepsilon (0) = 0 \) and

\[
|u^\varepsilon (x_1, t_1) - u^\varepsilon (x_2, t_2)| \leq w_\varepsilon (|x_1 - x_2| + |t_1 - t_2|^{1/2})
\]

for all \( (x_1, t_1), (x_2, t_2) \in \bar{\Omega} \times [\tau, T] \). The function \( w_\varepsilon \) does not depend on \( \varepsilon \) and \( T \).

(ii) If \( \mu > 0 \) and \( \nu = 0 \), then for every \( \tau > 0 \) and \( T > \tau \) there exists a continuous function \( w_\varepsilon \) such that \( w_\varepsilon (0) = 0 \) and

\[
|u^\varepsilon (x_1, t_1) - u^\varepsilon (x_2, t_2)| \leq w_\varepsilon (|x_1 - x_2| + |t_1 - t_2|^{1/2})
\]

for all \( (x_1, t_1), (x_2, t_2) \in \bar{\Omega} \times [\tau, T] \). The function \( w_\varepsilon \) does not depend on \( \varepsilon \) and \( T \).

Remark. (i) above follows from the a priori bound \((A2)\) and a result of DiBenedetto in \([Di]\); (ii) follows from \((A2)\) and results of Weigner in \([W]\).

The a priori estimates above suffice for associating a contraction semigroup to \((A1)\) with the desired compactness properties for the case when either \( \mu \) or \( \nu \) is zero. This can be done by defining

\[
u(t) = S(t) u_0 = \lim_{\varepsilon \to 0} S_\varepsilon (t) u_0 \quad \text{in } L^1,
\]

where \( S_\varepsilon (t) u_0 = u^\varepsilon (x, t) \), the solution to \((A1)_\varepsilon \). It follows that for \( u_0, v_0 \in L^\infty (\Omega) \)

(a) \( \|S(t) u_0 - S(t) v_0\|_{L^1(\Omega)} \leq \|u_0 - v_0\|_{L^1(\Omega)} \) (contraction),
(b) \( u_0 \geq v_0 \Rightarrow S(t) u_0 \geq S(t) v_0 \) (comparison),
(c) \( \int \Omega S(t) u_0 = \int \Omega u_0 \) (conservation).

The estimates above together with appropriate adjustments of the arguments in \([BH]\), \([VH]\) allow us to establish that \( S(t) u_0 \) is the unique weak solution to \((A1)\).

We summarize these facts in

**THEOREM A4** \((\mu > 0, \nu = 0 \text{ or } \mu = 0, \nu > 0)\). Let \( b(x, u) \) be locally Lipschitz in \( x, u \) and satisfy the growth condition \( |b(x, u)| \leq C_1 + C_2 |u|^q \) where \( q < 2m \) if \( \mu > 0, \nu = 0 \) and \( q < n + 1 \) if \( \mu = 0, \nu > 0 \). Then \((A1)\) has a unique weak solution existing for all \( t > 0 \). Hence, if we define \( S(t) u_0 = u(x, t; u_0) \), the solution of \((A1)\) with data \( u_0 \in L^\infty (\Omega) \), then \( S(t) \) can
be extended to a contraction semigroup on $L^1(\Omega)$ which preserves pointwise order and conserves the average. Finally, for $u_0 \in L^\infty(\Omega)$ the orbit

$$Q_\tau = \{ S(t)u_0 : t \geq \tau > 0 \}$$

is relatively compact in $C(\Omega)$ for $\mu = 0$, $v > 0$ and relatively compact in $C^1(\Omega)$ for $\mu > 0$, $v = 0$.

The case $\mu > 0$, $v > 0$ is different. The source of difficulty is the lack of a sufficiently strong estimate on $u^\varepsilon$. Notice that in the case $v = 0$ this has been obtained by differentiating the equation with respect to $x$ [AE, DiF, W].

We have been able to establish that the abstract semigroup solution (given by (A3)) solves (A1) weakly under the additional hypotheses

(i) $u_0 \geq 0$
(ii) $n \leq 2$.

Now we choose a different regularization. We replace $u_0(x)$ by $u_0^\varepsilon(x) = u_0(x) + \varepsilon$ for $\varepsilon > 0$ and let $g = g_0$ and $\phi = \phi_0$. The feature of this regularization we exploit is the monotonicity of the family $\{ u^\varepsilon \}$ in $\varepsilon$. We remark that Lemmas A1–A3 are still valid in this new set of circumstances. To avoid congestion we occasionally write $u_\varepsilon$ for $u^\varepsilon$.

**Lemma A5.** Let $\mu > 0$, $v > 0$ and $b(x, u)$ be $C^2$ and let $n \leq 2$. Then for any subcylinder $Q \subset Q_t$ we have the estimate

$$|u^\varepsilon| < C,$$

(A4)

$$C = C(Q).$$

The proof of Lemma A5 is based on Berstein's device and the argument is a suitable adaptation of that in [Ar].

**Lemma A6.** Under the hypotheses of Lemma A5 we have the estimate

$$\int_Q \int |(u_t^{(\alpha + 1)/2})|^2 \, dx \, dt < C(Q)$$

for

$$\alpha \geq \max \{ 1, 3 + 2(n - 2)(m + 1) \}.$$}

The proof of Lemma A6 is a suitable adaptation of the arguments in [Be 2].
THEOREM A7 \((\mu > 0, \nu > 0, n \leq 2, u_0 \geq 0)\). Let \(b(x, u)\) be \(C^2\) in both arguments and satisfy the growth conditions

\[ |b(x, u)| \leq C_1 + C_2 |u|^q, \quad q < 2m. \]

For \(u_0 \in L^\infty(\Omega)\) let \(S_\varepsilon(t) u_0\) stand for the solution of the regularization described above \((u_0^\varepsilon = u_0 + \varepsilon)\). Then define

\[ u(t) = S(t) u_0 = \lim_{\varepsilon \to 0} S_\varepsilon(t) u_0^\varepsilon \quad \text{(in } L^1). \]

\(S(t)\) can be extended to a contraction semigroup on \(L^1(\Omega)\) that preserves pointwise order and conserves the average. Moreover \(S(t) u_0\) is a weak solution to (A1).

Proof. The statement that requires proof is that \(S(t) u_0\) is a weak solution to (A1). Consider the weak formulation for the regularization

\[ \int_{\Omega} \mu'(t) \psi(t) = \int_{\Omega} u_0^\varepsilon \psi_0 + \int_{\Omega} \{ \mu(u_0^\varepsilon) + v \phi(u_0^\varepsilon)_x + b(x, u_0^\varepsilon) \} \psi_x \].

Step 1.

\[ \int (g(u_0^\varepsilon) - g(v_x))(z^2(u_0^\varepsilon - v))_x \geq -c \int (u_0^\varepsilon - v)^2, \quad (A5) \]

where \(v\) is a function such that \(u_0^\varepsilon - v \in C^1(\overline{\Omega})\) and \(z\) is a test function with compact support in \(\Omega\).

Verification of (A5).

\[ \int (g(u_0^\varepsilon) - g(v_x))(z^2(u_0^\varepsilon - v))_x \]

\[ = \int z^2(g(u_0^\varepsilon) - g(v_x))(u_0^\varepsilon - v_x) \]

\[ + 2 \int z_x(g(u_0^\varepsilon) - g(v_x))(u_0^\varepsilon - v). \]

Using Young’s inequality in the second integral we find that this is bounded below by

\[ (1 - \delta \sup g' \int z^2(g(u_0^\varepsilon) - g(v_x))(u_0^\varepsilon - v_x) - \frac{C}{\delta} \int (u_0^\varepsilon - v)^2. \]

The supremum is taken over a set which contains the ranges of all \(u_0^\varepsilon\), compact by (A4) and choice of \(z\). Choosing \(\delta = 1/\sup g'\) gives (A5).
From Lemma A1 we know that \( \{ g(u^\varepsilon) \} \) is bounded in \( L^{(m+1)/m}(Q) \) and \( \{ \phi(u^\varepsilon) \} \) is bounded in \( L^2(Q) \). Passing to the limit in the weak formulation of the equation we obtain

\[
\int_Q u(t) \psi(t) = \int_Q u_0 \psi_0 + \iint \{ u \psi_x - [\mu \chi + v \phi(u)x + b(x, u)] \psi_x \},
\]

where \( \chi \) is the weak limit of \( \{ g(u^\varepsilon) \} \) in \( L^{(m+1)/m} \).

**Step 2.**

\( \chi = g(u_\ast) \).

**Verification of (A6).** By (A5) we have

\[
\int \{ \mu (g(u^\varepsilon) - g(u_\ast)) + v(\phi(u^\varepsilon) - \phi(v)) \} (z^2(u^\varepsilon - v))_x
\]

\[
\geq -c \int (u^\varepsilon - v)^2 + \int v(\phi(u^\varepsilon) - \phi(v))_x (z^2(u^\varepsilon - v))_x.
\]

Using an argument involving weak lower semicontinuity and the localization available with \( z \), we pass to the limit along a subsequence to obtain

\[
\lim_{\varepsilon \to 0} \text{L.H.S.} \geq -c \int (u - v)^2 + \int v(\phi(u) - \phi(v))_x (z^2(u - v))_x, \quad (A7)
\]

where L.H.S. stands for left hand side of the inequality above.

The most crucial part in computing the limit of L.H.S. is to show that

\[
\int \{ \mu(g(u^\varepsilon)_x + v \phi(u^\varepsilon)) (z^2u^\varepsilon)_x \} \to \int \{ \mu \chi + v \phi(u)_x (z^2u)_x \} \quad \text{as} \quad \varepsilon \to 0.
\]

By taking \( \alpha = 3 \) in Lemma A6 we obtain

\[
\int_Q \int |(\mu g(u^\varepsilon)_x + v \phi(u^\varepsilon))_x|^2 u^\varepsilon < C(Q). \quad (A8)
\]

Writing \( f_\varepsilon = (\mu g(u^\varepsilon)_x + v \phi(u^\varepsilon))_x \), we have

\[
\iint f_\varepsilon u^\varepsilon z^2 = \iint f_\varepsilon (u^\varepsilon - u) z^2 + \iint f_\varepsilon u z^2
\]

\[
= \iint f_\varepsilon u^\varepsilon (1 - u/u^\varepsilon) z^2 + \iint f_\varepsilon u z^2.
\]
Using the fact that $uz$ is absolutely continuous the second term gives

\[
\iint f_\varepsilon uz^2 = \iint \left( \frac{\partial}{\partial x} \int_0^{x_\varepsilon} f_\varepsilon z \right) uz
\]

\[
= -\iint \left( \int_0^{x_\varepsilon} f_\varepsilon z \right) \frac{\partial}{\partial x} (uz)
\]

\[
= -\iint \left[ \mu g(u_\varepsilon^\varepsilon) + v\phi(u^\varepsilon)_x \right] z \frac{\partial}{\partial x} (uz)
\]

\[
+ \iint \left( \int_0^{x_\varepsilon} \left[ \mu g(u_\varepsilon^\varepsilon) + v\phi(u^\varepsilon)_x \right] z_x \right) \frac{\partial}{\partial x} (uz).
\]

By passing to the limit and integrating by parts we find that

\[
-\iint f_\varepsilon uz^2 \to \iint (\mu \chi + v\phi(u)_x)(z^2 u)_x \quad \text{as} \quad \varepsilon \to 0.
\]

On the other hand the first term goes to zero since

\[
\iint f_\varepsilon u^\varepsilon (1 - u/u^\varepsilon) \leq \left( \iint (f_\varepsilon u^\varepsilon)^2 \right)^{1/2} \left( \iint (1 - u/u^\varepsilon)^2 \right)^{1/2},
\]

$u^\varepsilon \to u$ monotonically and we have (A8). Therefore, (A7) gives, after cancelling terms,

\[
\iint \mu(\chi - g(v_x))(z^2(u - v))_x \geq -c \iint (u - v)^2.
\]

Finally we let $v = u - \sigma\xi$ and letting $\sigma \to 0$ we obtain as in [DH]

\[
\iint (\chi - g(u_x))(z^2\xi)_x \geq 0
\]

and therefore

\[
\chi = g(u_x)
\]

ACKNOWLEDGMENTS

We thank L. Peletier for making available to us the unpublished results [CFP], R. Rostamian for a number of specific suggestions in the second application, and P. Sacks for a useful discussion on the regularity part of the paper. We also thank J. Ralston and M. Hirsch for helpful communications.
Note added in proof. After the completion of the present paper it was pointed out in [AH] that (3.3) may not suffice for the semigroup to be a contraction. However, (3.3) suffices to ensure that orbits are Liapunov stable, as shown in [AHM].

REFERENCES


STABILIZATION OF SOLUTIONS


