Stability and attractivity of periodic solutions of parabolic systems with time delays

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Abstract

This paper is concerned with the existence, stability, and global attractivity of time-periodic solutions for a class of coupled parabolic equations in a bounded domain. The problem under consideration includes coupled system of parabolic and ordinary differential equations, and time delays may appear in the nonlinear reaction functions. Our approach to the problem is by the method of upper and lower solutions and its associated monotone iterations. The existence of time-periodic solutions is for a class of locally Lipschitz continuous reaction functions without any quasimonotone requirement using Schauder fixed point theorem, while the stability and attractivity analysis is for quasimonotone nondecreasing and mixed quasimonotone reaction functions using the monotone iterative scheme. The results for the general system are applied to the standard parabolic equations without time delay and to the corresponding ordinary differential system. Applications are also given to three Lotka–Volterra reaction diffusion model problems, and in each problem a sufficient condition on the reaction rates is obtained to ensure the stability and global attractivity of positive periodic solutions.

Keywords: Periodic solution; Parabolic boundary problem; Upper and lower solutions; Quasimonotone function; Asymptotic stability; Global attractivity; Time delays; Lotka–Volterra diffusion models

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1. Introduction

Periodic behavior of solutions of parabolic boundary problems arises from many fields of applied sciences and various methods have been developed for the study of the existence, stability, and attractivity of periodic solutions. Most of the discussions in the earlier literature are devoted either to scalar parabolic equations or to coupled system of equations in specific model problems such as population growth problems in ecology (cf. [1–16,27–34]). In recent years, attention has been given to coupled systems of parabolic equations where time delays may be taken into consideration in the nonlinear reaction functions (cf. [12,18,19,35]). In this paper, we consider a coupled system of parabolic boundary problem in the form

\[
\begin{align*}
\frac{\partial u_i}{\partial t} - L_i u_i &= f_i(t, x, u, u_\tau) \quad (t > 0, \ x \in \Omega), \\
B_i u_i &= h_i(t, x) \quad (t > 0, \ x \in \partial \Omega), \ i = 1, \ldots, N, \tag{1.1}
\end{align*}
\]

with the periodic condition

\[
\begin{align*}
u_i(t, x) &= u_i(t + T, x) \quad (\tau_i \leq t \leq 0, \ x \in \Omega), \tag{1.2a}
\end{align*}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with boundary \(\partial \Omega\), \(u \equiv (u_1(t, x), \ldots, u_N(t, x))\), \(u_\tau \equiv (u_1(t - \tau_1, x), \ldots, u_N(t - \tau_N, x))\) for some time delays \(\tau_1, \ldots, \tau_N\), and for each \(i = 1, \ldots, N\),

\[
\begin{align*}
L_i u_i &= \sum_{j,k=1}^{n} a_{ik}^{(j)}(t, x) \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_{ik}^{(j)} \frac{\partial u_i}{\partial x_j}, \\
B_i u_i &= \alpha_i \frac{\partial u_i}{\partial \nu} + \beta_i(t, x) u_i,
\end{align*}
\]

with \(\partial/\partial \nu\) denoting the outward normal derivative on \(\partial \Omega\). It is assumed that the boundary \(\partial \Omega\) is of class \(C^{1+\alpha}\), and for each \(i, \tau_i > 0, L_i\) is a uniform elliptic operator, and \(B_i\) is either of Dirichlet type (\(\alpha_i = 0, \beta_i = 1\)) or of Neumann–Robin type (\(\alpha_i = 1, \beta_i \geq 0\)), and is allowed to be different type for different \(i\). It is also assumed that the functions \(a_{ik}^{(j)}(t, x), b_{ik}^{(j)}(t, x)\) and \(f_i(t, x, \cdot)\) are H"older continuous in \([0, \infty) \times \partial \Omega\), \(\beta_i(t, x)\) and \(h_i(t, x)\) are H"older continuous in \([0, \infty) \times \partial \Omega\) and \(h_i\) is a \(C^{2+\alpha}\) function if \(\alpha_i = 0, \beta_i = 1\). We allow that \(L_i = 0\) (and without the corresponding boundary condition) for some or all \(i\). This implies that problem (1.1) may consist of a coupled system of parabolic and ordinary differential equations which appear quite often in specific reaction diffusion type of problems. To emphasize the dependence of \(L_i\) and \(B_i\) on \(t\) we sometimes write \(L_i = L_i(t)\) and \(B_i = B_i(t)\).

To investigate the stability and attractivity of \(T\)-periodic solutions of (1.1), (1.2a) we also consider the system (1.1) under the initial condition

\[
\begin{align*}
u_i(t, x) &= \eta_i(t, x) \quad (\tau_i \leq t \leq 0, \ x \in \Omega), \ i = 1, \ldots, N, \tag{1.2b}
\end{align*}
\]

where \(\eta_i(t, x)\) is given. We assume that \(\eta_i \in C^{\alpha,2+\alpha}(D^{(i)}_0)\) if \(\alpha_i = 0\) and \(\beta_i = 1\), \(\eta_i \in C^{\alpha/2,\alpha}(D^{(i)}_0)\) if \(\alpha_i = 1\) and \(\beta_i \geq 0\), and \(\eta_i\) satisfies the compatibility condition \(\beta_i(0, x) \eta_i(0, x) = h_i(0, x)\) for \(x \in \partial \Omega\) when \(\alpha_0 = 0\), where \(D^{(i)}_0 = [-\tau_i, 0] \times \Omega, i =

The set of functions \( \eta = (\eta_1, \ldots, \eta_N) \) satisfying the above standard conditions is denoted by \( \mathcal{X}(D_0) \).

When \( f_i \equiv f_i(t, x, u) \) is independent of \( u \), system (1.1) is reduced to the standard parabolic boundary problem (without time delays)

\[
\frac{\partial u_i}{\partial t} - L_i u_i = f_i(t, x, u) \quad (t > 0, \ x \in \Omega),
\]

\[
B_i u_i = h_i(t, x) \quad (t > 0, \ x \in \partial \Omega), \ i = 1, \ldots, N. \tag{1.3}
\]

The periodic condition (1.2a) and the initial condition (1.2b) are reduced, respectively, to

\[
u_i(0, x) = u_i(T, x) \quad (x \in \Omega), \ i = 1, \ldots, N, \tag{1.4}_a
\]

and

\[
u_i(0, x) = \eta_i(x) \quad (x \in \Omega), \ i = 1, \ldots, N. \tag{1.4}_b
\]

On the other hand, if \( L_i = 0 \) for all \( i \) then problem (1.1), (1.2a) is reduced to the ordinary differential system

\[
\frac{du_i}{dt} = f_i(t, u, u_\tau) \quad (t > 0),
\]

\[
u_i(t) = \nu_i(t + T) \quad (-\tau_i \leq t \leq 0). \tag{1.5}
\]

We shall see that all the conclusions about the existence, local stability, and global attractivity of \( T \)-periodic solutions of the general system (1.1), (1.2a) are directly applicable to the above two special systems.

Literature dealing with the standard initial-boundary problem (1.3), (1.4b) is extensive, and many stability and global attractivity of the steady-state solution of the corresponding elliptic boundary value problem

\[
-L_i u_i = f_i(x, u) \quad \text{in} \ \Omega,
\]

\[
B_i u_i = h_i(x) \quad \text{on} \ \partial \Omega, \ i = 1, \ldots, N, \tag{1.6}
\]

has been obtained (cf. [17] and references therein). However the results for the above problem are mostly for the autonomous system in the sense that the coefficients of \( L_i, B_i \) and the functions \( f_i, h_i \) are all independent of \( t \). On the other hand, the existence problem for the general periodic boundary problem (1.1), (1.2a) has been investigated in [12,18,19,35] using the method of upper and lower solutions. The monotone iterative scheme associated with this method leads to various computational algorithms for numerical solutions of the periodic boundary problem (cf. [23]). A major requirement of the monotone iteration in these works is that the reaction function \( f \equiv (f_1, \ldots, f_N) \) in the system be quasimonotone nondecreasing. In this paper, we show the existence of a solution for problem (1.1), (1.2a) by the method of upper and lower solutions without any quasimonotone requirement on the reaction function \( f \). Furthermore, we investigate the stability and attractivity problem of periodic solutions of (1.1), (1.2a) when \( f \) is either quasimonotone nondecreasing or mixed quasimonotone. Specifically, we show the existence of a \( T \)-periodic solution of (1.1), (1.2a) for any Lipschitz continuous reaction function \( f \), the local stability of maximal and minimal \( T \)-periodic solutions for quasimonotone nondecreasing reaction functions, and the global attractivity of a \( T \)-periodic solution for mixed quasimonotone reaction functions. Since
every solution \( u(x) \equiv (u_1(x), \ldots, u_N(x)) \) of (1.6) can be regarded as a \( T \)-periodic solution of (1.1), (1.2) for any period \( T \) when \( L_i, B_i, f_i \), and \( h_i \) are all independent of \( t \) these results may be considered as an extension of the autonomous system to the corresponding nonautonomous system.

The plan of the paper is as follows. In Section 2 we show the existence of \( T \)-periodic solutions of (1.1), (1.2) for a general class of nonquasimonotone functions using the method of upper and lower solutions and Schauder fixed point theorem. Based on the monotone convergence property of the maximal and minimal sequences developed in [19] we show in Section 3 the local stability of the maximal and minimal \( T \)-periodic solutions for quasi-monotone nondecreasing functions as well as the global attractivity of the sector between upper and lower solutions. The stability and attractivity of \( T \)-periodic solutions is in the sense of (3.11) and (3.13) in Theorem 3.1. A sufficient condition is given to ensure the uniqueness and the global attractivity of a \( T \)-periodic solution. Section 4 is devoted to the attractivity of the \( T \)-periodic quasisolutions, including the global attraction of a unique \( T \)-periodic solution, for the system (1.1), (1.2) with mixed quasimonotone reaction functions. Finally in Section 5 we give some applications of the results in Sections 3 and 4 to three Lotka–Volterra type of competition diffusion model problems in population dynamics. Here some sufficient conditions on the various reaction rates in the reaction mechanism are obtained to guarantee the existence and attractivity of positive \( T \)-periodic solutions, including the global attraction of a unique positive solution. All the results in the above sections are applicable to the standard periodic system (1.3), (1.4) without time delays, and to the ordinary differential system (1.5) with or without time delays.

2. Existence of \( T \)-periodic solutions

Let \( D = [0, \infty) \times \Omega \), \( \bar{D} = [0, \infty) \times \bar{\Omega} \), \( \Gamma = [0, \infty) \times \partial \Omega \), and for each \( i = 1, \ldots, N \), we set
\[
D_0^{(i)} = [-\tau_i, 0] \times \Omega, \quad Q^{(i)} = [-\tau_i, \infty] \times \bar{\Omega},
\]
\[
D_0 = D_0^{(1)} \times \cdots \times D_0^{(N)}, \quad Q = Q^{(1)} \times \cdots \times Q^{(N)}.
\]
Denote by \( C^{m+\alpha}(D^*) \) (respectively, \( C^{\alpha/2,\alpha}(D^*) \)) the set of functions in \( C^m(D^*) \) that are \( \alpha \)-Holder continuous (respectively, \( \alpha/2 \)-Holder continuous in \( t \) and \( \alpha \)-Holder continuous in \( x \)) with exponent \( \alpha \in (0, 1) \), where \( D^* \) is any one of the above domains. The set of \( N \)-vector functions of the above spaces are denoted by \( C^{m+\alpha}(D^*) \) and \( C^{\alpha/2,\alpha}(D^*) \), respectively. Similar notations are defined for other function spaces. Throughout this paper we assume, in addition to the general smoothness assumptions on \( \partial \Omega \) and the given functions in the introduction, the following basic hypothesis:

\((H_1)\) For each \( i = 1, \ldots, N \), the coefficients \( a_{j,k}^{(i)}(t,x), b_j^{(i)}(t,x) \) of \( L_i \) and the function \( f_i(t,x,\cdot) \) are in \( C^\alpha(D) \), \( \beta_j(t,x) \) and \( h_i(t,x) \) are in \( C^\alpha(\Gamma) \), and all of these functions are \( T \)-periodic in \( t \).
Theorem 2.1. Let \( T \)-periodic solution.

\[
K_i(u - u_i| + |v - v'|) \quad (i = 1, \ldots, N)
\]

for \((u, v), (u', v') \in S \times S_t\), (2.1)

where \( K_i \) is a positive constant, \( |w| = |w_1| + \cdots + |w_N| \) for any \( w = (w_1, \ldots, w_N) \in \mathbb{R}^N \), and the subsets \( S, S_t \) are given by (2.3) below. These subsets are the sectors between a pair of coupled upper and lower solutions which are defined as follows.

Definition 2.1. A pair of functions \( \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_N), \tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_N) \) in \( C^{1,2}(D) \cap C^{\alpha/2, \alpha}(Q) \) are called coupled upper and lower solutions of (1.1), (1.2) if \( \tilde{u} \geq \tilde{v} \) and if for each \( i = 1, \ldots, N \) \( u_i \equiv \tilde{u}_i \) and \( v_i \equiv \tilde{v}_i \), \( u_i \geq \tilde{u}_i \) \((t, x) \in D\), \( B_i u_i \leq h_i(t, x) \geq B_i \tilde{u}_i \) \((t, x) \in \Gamma\), \( \tilde{u}_i(t, x) \geq \tilde{u}_i(t + T, x), \tilde{u}_i(t, x) \leq \tilde{u}_i(t + T, x) \) \((t, x) \in D_0^{(i)}\),

\[
\partial \tilde{u}_i / \partial t - L_i \tilde{u}_i \leq f_i(t, x, u, v) \quad (i = 1, \ldots, N),
\]

(2.2)

where

\[
S = \{ u \in C(\bar{D}); \tilde{u} \leq u \leq \tilde{u} \text{ on } \bar{D}\},
\]

\[
S_t = \{ v \in C(Q); \tilde{v} \leq v \leq \tilde{v} \text{ on } \bar{D}\}.
\]

In the above definition, upper and lower solutions are not required to be \( T \)-periodic in \( t \). To show the existence of a \( T \)-periodic solution for problem (1.1) we may assume without loss of generality that \( h_i = 0 \) for all \( i \). The following theorem gives the existence of a \( T \)-periodic solution.

Theorem 2.1. Let \( \tilde{u}, \tilde{v} \) be a pair of coupled upper and lower solutions of (1.1), (1.2) and let hypothesis \( (H_1) \) and condition (2.1) hold. Then problem (1.1), (1.2) has at least one \( T \)-periodic solution \( u \in \bar{S} \).

Proof. For each \( i = 1, \ldots, N \), we define operators \( \mathcal{L}_i : D_i \rightarrow \mathcal{R}_i \) and \( \mathcal{L} : D \rightarrow \mathcal{R} \) by

\[
\mathcal{L}_i u_i = \partial u_i / \partial t - L_i u_i + K_i u_i \quad (i = 1, \ldots, N),
\]

\[
\mathcal{L} u = (\mathcal{L}_1 u_1, \ldots, \mathcal{L}_N u_N),
\]

(2.4)

where \( K_i \) is the Lipschitz constant in (2.1) and

\[
D_i = \{ w_i \in C^{1+\alpha/2, 2+\alpha}(D); B_i w_i = 0 \text{ on } \Gamma, w_i(t, x) = w_i(t + T, x) \in D_0^{(i)}\},
\]

\[
\mathcal{R}_i = \{ w_i \in C^{\alpha/2, \alpha}(D); w_i(t, x) = w(t + T, x) \in \mathcal{R}_0\},
\]

\[
D = D_1 \times \cdots \times D_N, \quad \mathcal{R} = \mathcal{R}_1 \times \cdots \times \mathcal{R}_N.
\]

Define also
\( (F_i(u, u'))(t, x) = K_iu_i(t, x) + f_i(t, x, u(t, x), u_r(t, x)) \),
\[ F(u, u') = (F_1(u, u'), \ldots, F_N(u, u')). \quad (2.6) \]

Then system (1.1), (1.2) may be written in the form
\[ \mathcal{L}u = F(u, u') \quad (u \in \mathcal{D}). \quad (2.7) \]

Given any \( w \in \mathcal{S} \) and any \( i = 1, \ldots, N \), we let \( q_i(t, x) = (F_i(w, w'))(t, x) \), and consider the linear scalar periodic boundary problem
\[ \mathcal{L}_i u_i = q_i \quad \text{in } D, \quad B_i u_i = 0 \quad \text{on } \Gamma, \quad u_i(t, x) = u_i(t + T, x) \quad \text{in } D_0^{(i)}, \]
\[ i = 1, \ldots, N. \quad (2.8) \]

Since by hypothesis \((H_1)\) and condition (2.1), \( q_i \in \mathcal{R}_i \), we see that problem (2.8) has a unique solution \( u_i^* \in \mathcal{D} \). In fact, the inverse \( L_i^{-1} : \mathcal{R}_i \to \mathcal{D}_i \) exists and is a positive compact operator on \( \mathcal{R}_i \) (cf. [9, p. 36]). This implies that the equation
\[ \mathcal{L}u = F(w, w') \quad (2.9) \]
has a unique solution \( u = L^{-1}[F(w, w')] \) and \( L^{-1} : \mathcal{R} \to \mathcal{D} \) is a (continuous) compact operator on \( \mathcal{R} \). Let \( \mathcal{X} \) be the closed bounded convex subset given by
\[ \mathcal{X} = \{ u \in \mathcal{R} ; \ \hat{u} \leq u \leq \breve{u} \}. \quad (2.10) \]

By the compact property of \( L^{-1} \) and the hypothesis on \( f_i \) the operator \( L^{-1}F \) is compact on \( \mathcal{X} \). We show that \( L^{-1}F \) maps \( \mathcal{X} \) onto itself.

Let \( w \in \mathcal{X} \) be given, and let \( w^{(i)} \) denote the vector \( w \) whose \( i \)th component \( w_i \) is replaced by \( \hat{u}_i \). By (2.4), (2.2) and (2.1),
\[ \mathcal{L}_i \hat{u}_i = \partial \hat{u}_i / \partial t - L_i \hat{u}_i + K_i \hat{u}_i \geq f_i(t, x, w^{(i)}, w) + K_i \hat{u}_i \]
\[ \geq K_i w_i + f_i(t, x, w, w) = q_i(t, x), \quad i = 1, \ldots, N. \]

Since \( B_i \hat{u}_i \geq 0 \) on \( \Gamma \) and \( \hat{u}_i(t, x) \geq \hat{u}_i(t + T, x) \) in \( D_0^{(i)} \), \( \hat{u}_i \) is an upper solution of (2.8).

A similar argument shows that \( 
\bar{u}_i \) is a lower solution of (2.8), \( i = 1, \ldots, N \). By Theorem 2.1 of [19], problem (2.8) has at least one solution \( u_i \) such that \( \hat{u}_i \leq u_i \leq \bar{u}_i \) for each \( i \). The uniqueness of the solution \( u_i^* \) ensures that \( u_i = u_i^* \).

This shows that \( \hat{u} \leq u^* \leq \breve{u} \) and therefore \( L^{-1}F \) maps \( \mathcal{X} \) onto itself. It follows from the Schauder fixed point theorem that Eq. (2.9) has at least one solution \( u \in \mathcal{X} \). \[ \square \]

When \( f_i \equiv f_i(t, x, u) \) is independent of \( u \), for every \( i \) the definition of upper and lower solutions is the same as that in Definition 2.1 except without the variable \( v \) in (2.2). The Lipschitz condition (2.1) is reduced to
\[ |f_i(\cdot, u) - f_i(\cdot, u')| \leq K_i |u - u'| \quad \text{for } u, u' \in \mathcal{S}. \]

As a consequence of Theorem 2.1 we have the following existence result for the periodic boundary problem (1.3), (1.4) without time delays.

**Corollary 2.1.** Let \( \hat{u}, \breve{u} \) be coupled upper and lower solutions of (1.3), (1.4), (with \( h_i \equiv 0 \)), and let hypothesis \((H_1)\) and condition (2.11) hold. Then problem (1.3), (1.4) has at least one \( T \)-periodic solution \( u \in \mathcal{X} \).
3. Attractivity and stability of periodic solutions

To investigate the attractivity and stability of periodic solutions we need to impose some quasimonotone conditions on the reaction function

\[ f(t, x, u, v) = \left( f_1(t, x, u, v), \ldots, f_N(t, x, u, v) \right) \]

Recall that an \( N \)-vector function \( f(\cdot, u, v) \) is said to have a mixed quasimonotone property in \( S \times S \) if for each \( i = 1, \ldots, N \), there exist nonnegative integers \( a_i, b_i, c_i \) and \( d_i \) with

\[ a_i + b_i = N - 1, \quad c_i + d_i = N \]

such that \( f_i(\cdot, u, [u]_{a_i}, [u]_{b_i}, [v]_{c_i}, [v]_{d_i}) \) is nondecreasing in \([u]_{a_i}, [v]_{c_i}\) and nonincreasing in \([u]_{b_i}, [v]_{d_i}\) for all \((u, v) \in S \times S \), where \( u \) and \( v \) are written in the respective split forms

\[ u = (u_i, [u]_{a_i}, [u]_{b_i}), \quad v = ([v]_{c_i}, [v]_{d_i}) \]

and \([u]_{c_i}\) (respectively, \([v]_{d_i}\)) denotes \( c_i \)-components of \( u \) (respectively, \( v \)) (cf. [17,20]). In particular, if \( b_i = d_i = 0 \) (so that \( a_i = N - 1, b_i = N \)) for all \( i \) then \( f(\cdot, u, v) \) is said to be quasimonotone nondecreasing in \( S \times S \). It is obvious that if \( f(\cdot, u, v) \) is a \( C^1 \)-function of \((u, v)\) then the quasimonotone nondecreasing property of \( f(\cdot, u, v) \) becomes

\[ \frac{\partial f_i}{\partial u_j} f(\cdot, u, v) \geq 0 \quad \text{for } j \neq i \quad \text{and} \quad \frac{\partial f_i}{\partial v_j} f(\cdot, u, v) \geq 0 \quad \text{for all } j, \]

\((u, v) \in S \times S \).

(3.2)

For quasimonotone nondecreasing functions, the inequalities of an upper solution \( \bar{u} \) in (2.2) are equivalent to

\[ \frac{\partial \bar{u}_i}{\partial t} - L_i \bar{u}_i \geq f_i(t, x, \bar{u}, \bar{u}) \quad (t, x) \in D, \]

\[ B_i \bar{u}_i \geq h_i(t, x) \quad (t, x) \in \Gamma', \]

\[ \bar{u}_i(t, x) \geq \bar{u}_i(t + T, x) \quad (t, x) \in D_0^{(i)}, \quad i = 1, \ldots, N, \]

(3.3)

and those of a lower solution \( \underline{u} \) become the inequalities in (3.3) but in reversed order. In this situation, upper and lower solutions are not coupled, and every \( T \)-periodic solution of (1.1), (1.2) is an upper solution as well as a lower solution. We refer to \( \bar{u}, \underline{u} \) as ordered upper and lower solutions if \( \bar{u} \geq \underline{u} \). Throughout this section we make the following hypothesis:

\[(H_2) \text{ There exist a pair of ordered upper and lower solutions } \bar{u}, \underline{u}, \text{ and } f(\cdot, u, v) \text{ is a quasimonotone nondecreasing } C^1 \text{-function in } S \times S.\]

Using either \( u^{(0)} = \bar{u} \) or \( u^{(0)} = \underline{u} \) as an initial iteration we construct a sequence \( \{u^{(m)}\} \equiv \{u_1^{(m)}, \ldots, u_N^{(m)}\} \) from the linear (uncoupled) initial boundary problem

\[ \mathcal{L} u_i^{(m)} = f_i(t, x, u^{(m-1)}, u^{(m-1)}), \quad (t, x) \in D, \]

\[ B_i u_i^{(m)} = h_i(t, x) \quad (t, x) \in \Gamma', \]

\[ u_i^{(m)}(t, x) = u_i^{(m-1)}(t + T, x) \quad (t, x) \in D_0^{(i)}, \quad i = 1, \ldots, N, \]

(3.4)
where \( m = 1, 2, \ldots \), and \( L_i \) and \( F_i \) are given by (2.4) and (2.6), respectively. It is clear that this sequence is well defined and can be computed by solving a linear scalar parabolic initial boundary problem. Denote the sequence by \( \{ \tilde{u}^{(m)} \} \) if \( u^{(0)} = \tilde{u} \) and by \( \{ u^{(m)} \} \) if \( u^{(0)} = \hat{u} \), and refer to them as maximal and minimal sequences, respectively. The following theorem from [19] gives the monotone convergence of these sequences.

**Theorem A.** Let hypotheses \((H_1), (H_2)\) be satisfied. Then the sequence \( \{ \tilde{u}^{(m)} \} \) converges monotonically to a maximal \( T \)-periodic solution \( \tilde{u} \) and \( \{ u^{(m)} \} \) converges monotonically to a minimal \( T \)-periodic solution \( u \), and they satisfy

\[
\tilde{u} \leq u^{(m)} \leq u^{(m+1)} \leq \tilde{u} \leq u \quad \text{on } Q.
\]

If, in addition, \( \tilde{u}(0,x) = u(0,x) \) then \( \tilde{u} = u \) \( (\equiv u^* \) and \( u^* \) is the unique \( T \)-periodic solution of \((1.1), (1.2) \) \( b \) for a certain class of initial functions \( \eta \equiv (\eta_1, \ldots, \eta_N) \). The definition of upper and lower solutions for this initial boundary problem, denoted again by \( \tilde{u} \) and \( u \), are the same as that for \((1.1), (1.2) \) \( a \) except with the initial condition replaced by

\[
\hat{u}_i(t,x) \leq \eta_i(t,x) \leq \tilde{u}(t,x) \quad \text{for } (t,x) \in D_0^{(i)}, \ i = 1, \ldots, N.
\]

It is clear from this definition that every pair of ordered upper and lower solutions of \((1.1), (1.2) \) \( a \) are also ordered upper and lower solutions of \((1.1), (1.2) \) \( b \) whenever \( \eta \in S_0 \), where

\[
S_0 = \{ \eta \in \mathcal{X}(D_0) : \tilde{u} \leq \eta \leq \hat{u} \text{ in } D_0 \}.
\]

For the initial boundary problem \((1.1), (1.2) \) \( b \) we have the following existence-comparison theorem from [20,21].

**Theorem B.** Let hypothesis \((H_2)\) be satisfied, and let \( \{ \hat{u}_I^{(m)} \}, \{ \tilde{u}_I^{(m)} \} \) be the maximal and minimal sequences obtained from (3.4) with \( \hat{u}_I^{(0)} = \hat{u}, \tilde{u}_I^{(0)} = \tilde{u} \) and with the initial condition replaced by

\[
u_i^{(m)}(t,x) = \eta_i(t,x) \quad (t,x) \in D, \ i = 1, \ldots, N,
\]

where \( m = 1, 2, \ldots \) and \( \eta \equiv (\eta_1, \ldots, \eta_N) \in S_0 \). Then \( \{ \hat{u}_I^{(m)} \} \) and \( \{ \tilde{u}_I^{(m)} \} \) converge monotonically to a unique solution \( u \equiv u(t,x) \) of \((1.1), (1.2) \) \( b \) and satisfy the relation

\[
\hat{u} \leq u_I^{(m)} \leq u_I^{(m+1)} \leq \hat{u} \quad \text{on } Q, \ m = 1, 2, \ldots
\]

Based on the monotone convergence property of the maximal and minimal sequences in the above theorems we investigate the local stability of the maximal and minimal \( T \)-periodic solutions \( \hat{u} \) and \( \tilde{u} \) in Theorem A. To emphasize the dependence of the solution \( u(t,x) \) on the initial function \( \eta(t,x) \) in Theorem B we write \( u \equiv u(t,x; \eta) \). The following lemma gives some additional properties of the maximal and minimal sequences.
Lemma 3.1. For any \( m \) and \( m' \), the pair \( \bar{u}^{(m)}, \underline{u}^{(m')} \) are ordered upper and lower solutions of (1.1), (1.2) \( b \). They are also ordered upper and lower solutions of (1.1), (1.2) \( b \) provided that \( \bar{u}^{(m)} \leq \eta \leq \underline{u}^{(m)} \) in \( D_0 \).

Proof. Consider the maximal sequence \{\( \bar{u}^{(m)} \)\}. By (3.4) (with \( \bar{u}^{(m)} = \bar{u}^{(1)} \)), (2.4) and (2.6),
\[
\bar{u}_t^{(m)} - L_1 \bar{u}_{x}^{(m)} = \left[K_1(\bar{u}^{(m-1)} - \bar{u}^{(m)}) + f_i(t, x, \bar{u}^{(m-1)}, \bar{u}^{(m)})\right].
\]
In view of (2.1), (3.5) and the quasimonotone nondecreasing property of \( f(t, u, \sigma_u) \) we have
\[
\bar{u}_t^{(m)} - L_1 \bar{u}_{x}^{(m)} \geq f_i(t, x, \bar{u}^{(m)}, \bar{u}^{(m)}) \quad \text{in} \quad D.
\]
Since (3.4) and (3.5),
\[
\begin{align*}
B_i \bar{u}_t^{(m)} &= h_i(t, x) \quad \text{on} \quad \Gamma, \\
\bar{u}_i^{(m)}(t, x) &= \bar{u}_i^{(m-1)}(t, x) \quad \text{in} \quad D_0^{(i)},
\end{align*}
\]
we see that \( \bar{u}^{(m)} = (\bar{u}_1^{(m)}, \ldots, \bar{u}_N^{(m)}) \) is an upper solution of (1.1), (1.2) \( b \). A similar argument shows that \( \underline{u}^{(m)} \) is a lower solution. The ordering relation \( \bar{u}^{(m)} \geq \underline{u}^{(m')} \) follows from (3.5). This proves the lemma for the periodic boundary problem (1.1), (1.2) \( b \). The proof for the initial boundary problem (1.1), (1.2) \( a \) is similar. \( \square \)

Our next lemma plays a major role in the analysis of the stability and attractivity of \( T \)-periodic solutions.

Lemma 3.2. Let \( u(t, x; \eta) \) be the solution of (1.1), (1.2) \( b \) with any \( \eta \in S_0 \). Then
\[
\bar{u}^{(m)}(t, x) \leq u(t + mT, x; \eta) \leq \bar{u}^{(m)}(t, x) \quad \text{on} \quad \bar{D} \quad \text{for every} \quad m.
\]

Proof. Let \( u_0(t, x) = u(t + mT, x; \eta) \). By Theorem B and \( \eta \in S_0 \), the solution \( u_m(t, x) \) are in \( S \) for every \( m \) and, in particular, \( \bar{u}(t + T, x) \leq u_1(t, x) \leq \underline{u}(t + T, x) \) on \( Q \). Consider the system (1.1) with the initial condition \( \eta(t, x) = u_1(t, x) \) in \( D_0 \). Since by the initial condition in (3.4) for \( m = 1 \), \( u^{(1)}(t, x) = u^{(0)}(t + T, x) = \bar{u}(t, x) \) and \( \underline{u}^{(1)}(t, x) = \underline{u}^{(0)}(t + T, x) = \bar{u}(t, x) \), we see that \( \bar{u}^{(1)}(t, x) \leq u_1(t, x) \leq \underline{u}^{(1)}(t, x) \) in \( D_0 \). By Lemma 3.1, \( \bar{u}^{(1)}(t, x) \) and \( \underline{u}^{(1)}(t, x) \) are ordered upper and lower solutions of (1.1), (1.2) \( b \) when \( \eta = u_1(t, x) \). However, since by \( (H_1) \), \( u_1(t, x) \) is the unique solution of (1.1), (1.2) \( b \), we conclude from Theorem B that \( \bar{u}^{(1)}(t, x) \leq u_1(t, x) \leq \underline{u}^{(1)}(t, x) \) on \( \bar{D} \). Assume, by induction, that \( \bar{u}^{(m-1)} \leq u_m(t, x) \leq \bar{u}^{(m-1)} \) on \( \bar{D} \) for some \( m > 1 \). Then \( \bar{u}^{(m-1)}(t + T, x) \leq u_m(t, x) \leq \bar{u}^{(m-1)}(t + T, x) \) because \( u_m(t, x) = u_{m-1}(t + T, x) \). Now consider the system (1.1), (1.2) \( b \) with \( \eta(t, x) = u_m(t, x) \) in \( D_0 \). Since by the initial condition in (3.4), \( \bar{u}^{(m)}(t, x) = u^{(m-1)}(t + T, x) \) and \( \underline{u}^{(m)}(t, x) = u^{(m-1)}(t + T, x) \) in \( D_0 \), we see that \( \bar{u}^{(m)}(t, x) \leq u_m(t, x) \leq \bar{u}^{(m)}(t, x) \) in \( D_0 \). It follows again from Lemma 3.1 that \( \bar{u}^{(m)}(t, x) \) and \( \underline{u}^{(m)}(t, x) \) are ordered upper and lower solutions of (1.1), (1.2) \( b \) when \( \eta = u_m(t, x) \). But \( u_m(t, x) \) is its unique solution, an application of Theorem B leads to \( \bar{u}^{(m)}(t, x) \leq u_m(t, x) \leq \bar{u}^{(m)}(t, x) \) on \( \bar{D} \). The conclusion in (3.10) follows from the principle of induction. \( \square \)
Based on Lemma 3.2 we have the following convergence result for the solution \( u(t, x; \eta) \) of the initial boundary problem (1.1), (1.2)\(_b\) in relation to the maximal and minimal \( T \)-periodic solutions of (1.1), (1.2)\(_b\).

**Theorem 3.1.** Let hypotheses \((H_1), (H_2)\) be satisfied, and let \( \tilde{u}(t, x), u(t, x) \) be the respective maximal and minimal \( T \)-periodic solutions of (1.1), (1.2)\(_b\). Denote by \( u(t, x; \eta) \) the solution of (1.1), (1.2)\(_b\). Then

\[
\lim_{m \to \infty} u(t + mT, x; \eta) = \begin{cases} u(t, x) & \text{if } \hat{u} \leq \eta \leq u \text{ in } D_0, \\ \tilde{u}(t, x) & \text{if } \hat{u} \leq \eta \leq \tilde{u} \text{ in } D_0. \end{cases}
\]  

(3.11)

Moreover, for arbitrary \( \eta \in S_0 \),

\[
\bar{u}(t, x) \leq u(t + mT, x; \eta) \leq \tilde{u}(t, x) \quad \text{on } \tilde{D} \text{ as } m \to \infty,
\]

and if \( u(t, x) = \bar{u}(t, x) \) (\( \equiv u^*(t, x) \)) then

\[
\lim_{m \to \infty} u(t + mT, x; \eta) = u^*(t, x) \quad \text{if } (t, x) \in \tilde{D}.
\]  

(3.13)

**Proof.** Consider the case \( \hat{u} \leq \eta \leq u \) in \( D_0 \). By considering the solution \( u(t, x) \) as an upper solution of (1.1), (1.2)\(_b\), Theorem B implies that \( \bar{u}(t, x) \leq u(t, x; \eta) \leq \tilde{u}(t, x) \) on \( \tilde{D} \) and, in particular,

\[
u(t + mT, x; \eta) \leq u(t + mT, x) = u(t, x) \quad \text{on } \tilde{D}
\]

for every \( m \). On the other hand, since by (3.10) in Lemma 3.2 \( u(t + mT, x; \eta) \geq u^{(m)}(t, x) \) we conclude that

\[
u^{(m)}(t, x) \leq u(t + mT, x; \eta) \leq u(t, x) \quad \text{on } \tilde{D}, \; m = 1, 2, \ldots.
\]

The relation (3.11) for the case \( \hat{u} \leq \eta \leq \tilde{u} \) follows from Theorem A by letting \( m \to \infty \). The proof for the case \( \tilde{u} \leq \eta \leq \hat{u} \) is similar. It is obvious that relation (3.12) follows from Theorem A and (3.10), and in the case of \( \tilde{u} = u = u^* \), relation (3.13) is a consequence of (3.12). This proves the theorem. \( \square \)

Theorem 3.1 implies that the sector \([u, \tilde{u}]\) between the maximal and minimal \( T \)-periodic solutions \( u \) and \( \tilde{u} \) is a global attractor of the system relative to the region \( S \), and both \( u \) and \( \tilde{u} \) are one-sided asymptotically stable. Moreover, if \( \tilde{u} = u = u^* \) then \( u^* \) is a global attractor relative to \( S \). Here and in the following discussions the global attractivity and asymptotic stability are in the sense of (3.12) and (3.11), respectively. In the following theorem we give a sufficient condition for the global attractivity of the sector \([u, \tilde{u}]\) for every allowable initial function \( \eta \).

**Theorem 3.2.** Let hypotheses \((H_1), (H_2)\) be satisfied, and let \( u(t, x; \eta) \) be the solution of (1.1), (1.2)\(_b\) corresponding to an arbitrary \( \eta \in X(D_0) \). If there exists \( t^* > 0 \) such that for every \( i = 1, \ldots, N \),

\[
\hat{u}_i(t, x) \leq u_i(t, x; \eta) \leq \tilde{u}_i(t, x) \quad \text{when } t^* - \tau_i \leq t \leq t^*, \; x \in \Omega,
\]

(3.14)

then relation (3.12) holds for all \( t \geq t^*, \; x \in \tilde{\Omega} \). If, in addition, \( \tilde{u} = u = u^* \) then

\[
\lim_{m \to \infty} u(t + mT, x; \eta) = u^*(t, x) \quad \text{for } t \geq t^*, \; x \in \tilde{\Omega}.
\]  

(3.15)
Proof. Let \( s = t - t^* \), \( w_i(s, x) = u_i(s + t^*, x) \), \( i = 1, \ldots, N \), where we have dropped the dependence of \( w_i \) and \( u_i \) on \( \eta \). Then \( w_i(s, x) \) satisfies the equations of the initial boundary problem

\[
\begin{aligned}
\frac{\partial w_i}{\partial s} - L_i w_i &= f_i(s + t^*, x, w(s, x), w_T(s, x)) \quad (s > 0, \ x \in \Omega), \\
B_i w_i &= h_i(s + t^*, x) \quad (s > 0, \ x \in \partial \Omega), \\
w_i(s, x) &= \xi_i(s, x) \quad (-\tau_i \leq s \leq 0, \ x \in \Omega),
\end{aligned}
\]

(3.16)

where \( \xi_i(s, x) = u_i(s + t^*, x) \). It is clear that the pair \( \hat{w}(s, x) = \hat{u}(s + t^*, x) \) and \( \hat{w}(s, x) = \hat{u}(s + t^*, x) \) are ordered upper and lower solutions of (3.16) if

\[
\hat{w}_i(s, x) \leq \xi_i(s, x) \leq \hat{w}_i(s, x) \quad \text{for} \ -\tau_i \leq s \leq 0, \ x \in \Omega.
\]

But this relation follows from (3.14) with \( t = s + t^* \). By an application of Theorem 3.1 to (3.16) we obtain

\[
\hat{w}(s, x) \leq w(s + mT, x) \leq \hat{w}(s, x) \quad \text{on} \ \hat{D} \ \text{as} \ m \to \infty,
\]

where \( \hat{w}(s, x) = \hat{u}(s + t^*, x) = \hat{u}(t, x) \) and \( \hat{w}(s, x) = \hat{u}(s + t^*, x) = \hat{u}(t, x) \). This proves relation (3.12) for \( t \geq t^* \). The relation (3.15) is a consequence of (3.12) if \( \hat{u} = \hat{u} \equiv u^* \). 

For the periodic boundary problem (1.3), (1.4) without time delays the condition in \((H_2)\) with respect to \( u_\tau \) is trivially satisfied. The requirements of upper and lower solutions remain the same except with \( f_i \equiv f_i(t, x, u) \) and with \( \tau_i = 0 \) in the initial condition. Because of its usefulness in applications we state the following results from Theorems 3.1 and 3.2 as a theorem.

Theorem 3.3. Let hypotheses \((H_1), (H_2)\) be satisfied for \( f \equiv f(t, x, u) \) which is independent of \( u_\tau \). Then all the conclusions in Theorems 3.1 and 3.2 hold true for the system (1.3), (1.4).

It is easy to see from the arguments in the proofs of Theorems 3.1 and 3.2 that all the conclusions in these theorems hold true if \( L_i = 0 \) (and without the corresponding boundary condition) for some or all \( i \) (see also the proofs of Theorems A and B in [19,20]). In particular, if \( L_i = 0 \) for all \( i \) then problem (1.1), (1.2) is reduced to the ordinary differential system (1.5). In this situation, upper and lower solutions \( \hat{u}(t), \hat{u}_\tau(t) \) are required to satisfy the relation

\[
\begin{aligned}
d\hat{u}_i/dt - f_i(t, \hat{u}, \hat{u}_\tau) &\geq 0 \geq d\hat{u}_i/dt - f_i(t, \hat{u}, \hat{u}_\tau) \quad (t > 0) \\
\hat{u}_i(t) \geq \hat{u}_i(t + T), &\hat{u}_i(t) \leq \hat{u}_i(t + T) \quad (-\tau_i \leq t \leq 0),
\end{aligned}
\]

(3.17)

Since hypothesis \((H_1)\) is satisfied if \( f_i(t, \cdot) \) is \( T \)-periodic we have the following results for the ordinary differential system (1.5).

Theorem 3.4. Let \( \hat{u}(t), \hat{u}_\tau(t) \) satisfy \( \hat{u}(t) \geq \hat{u}_\tau(t) \) and relation (3.17), and let \( f_i \equiv f_i(t, u, u_T) \) be \( T \)-periodic in \( t \) and quasimonotone nondecreasing in \((u, u_T) \in \mathcal{S} \times \mathcal{S}_T\). Then all the conclusions in Theorems 3.1 and 3.2 hold true for the ordinary differential system (1.5).
Remarks. (a) Since for each $i$ and each $m$ the components of the maximal and minimal sequences $\bar{u}(m)$, $\hat{u}(m)$ can be obtained by solving a linear scalar initial boundary problem the iterative scheme (3.4) is well suitable for the computation of numerical solutions of the periodic system (1.1), (1.2) similar to that in [23] for scalar boundary problem.

(b) In the definition of upper and lower solutions, $\bar{u}$ and $\hat{u}$ are not required to be $T$-periodic in $t$. It can be shown that if $\bar{u}$ (respectively, $\hat{u}$) is $T$-periodic then $\bar{u}(m)$ (respectively, $\hat{u}(m)$) is also $T$-periodic for every $m$. It can also be shown by the maximum principle that strict inequalities for the sequences $\bar{u}(m)$ (respectively, $\hat{u}(m)$) in (3.5) hold true until $\bar{u}(m)$ (respectively, $\hat{u}(m)$) becomes a true solution of (1.1), (1.2) for some $m$.

(c) For scalar periodic boundary problem (1.1), (1.2) if $f_1 \equiv f(t, x, u)$ is independent of $u_\tau$ then the quasimonotone nondecreasing property of $f$ in $(H_2)$ is trivially satisfied, and in this case hypothesis $(H_2)$ is fulfilled if there exist a pair of ordered upper and lower solutions.

(d) The convergence property (3.11) implies that no $T$-periodic solution between $\bar{u}$ and $\bar{u}$ or between $\hat{u}$ and $\hat{u}$ can exist. This confirms the fact that $\bar{u}$ and $\hat{u}$ are the respective maximal and minimal $T$-periodic solutions in $\mathcal{S}$.

(e) It is clear that for the problem (1.1), (1.2) with the homogeneous Neumann boundary condition

$$B_i u_i = \partial u_i/\partial v = 0 \quad (t, x) \in \Gamma, \quad i = 1, \ldots, N$$

(3.18)

(that is, $\beta_i = h_i = 0$ and $\alpha_i = 1$), every pair of upper and lower solutions of (1.5) are also upper and lower solutions of (1.1), (1.2) if $f \equiv f(t, u, u_\tau)$ is independent of $x$. This is true even if the coefficients $a^{(i)}_{j,k}, b^{(i)}_{j,k}$ of $L_i$ are spatially dependent. Moreover, a solution of (1.5) is also a solution of (1.1), (1.2) if $f \equiv f(t, u, u_\tau)$ depends explicitly on $x$.

4. Mixed quasimonotone functions

In this section we investigate the asymptotic behavior of the solution of the initial boundary problem (1.1), (1.2) in relation to the $T$-periodic solution or quasisolutions of (1.1), (1.2) if $f \equiv f(t, u, u_\tau)$ possesses a mixed quasimonotone property in $\mathcal{S} \times \mathcal{S}_\tau$. It is easy to see from Definition 2.1 that for mixed quasimonotone functions the requirement of upper and lower solutions in (2.2) is equivalent to

$$\partial \bar{u}_i/\partial t - L_i \bar{u}_i \geq f_i(t, x, \bar{u}_i, [\bar{u}]_{\beta_i}, [\bar{u}]_{\alpha_i}, [\bar{u}]_{\tau_i}, [\bar{u}]_{\tau_i}) \quad (t, x) \in D),$$

$$\partial \hat{u}_i/\partial t - L_i \hat{u}_i \leq f_i(t, x, \hat{u}_i, [\hat{u}]_{\beta_i}, [\hat{u}]_{\alpha_i}, [\hat{u}]_{\tau_i}, [\hat{u}]_{\tau_i}) \quad (t, x) \in D),$$

$$B_i \bar{u}_i \geq h_i(t, x) \geq B_i \hat{u}_i \quad (t, x) \in \Gamma, \quad i = 1, \ldots, N,$$

$$\bar{u}_i(t, x) \geq \bar{u}_i(t + T, x), \quad \hat{u}_i(t, x) \leq \hat{u}_i(t + T, x) \quad (t, x) \in D_0^{(i)}.$$

(4.1)

Instead of hypothesis $(H_2)$ we impose the following conditions on $f(\cdot, u, u_\tau)$:
(H3) There exist a pair of coupled upper and lower solutions \( \hat{u}, \tilde{u} \) and \( f(\cdot, u, \tau) \) is a mixed quasimonotone \( C^1 \)-function in \( S \times S \).

Using \( \hat{u}^{(0)} = \hat{u} \) and \( \tilde{u}^{(0)} = \tilde{u} \) as a coupled initial iterations we construct two sequences \( \{\hat{u}^{(m)}\} = \{\hat{u}_1^{(m)}, \ldots, \hat{u}_N^{(m)}\}, \{\tilde{u}^{(m)}\} = \{\tilde{u}_1^{(m)}, \ldots, \tilde{u}_N^{(m)}\} \) from the linear iteration process

\[
\mathcal{L}_i \hat{u}_i^{(m)} = K_i \hat{u}_i^{(m-1)} + f_i(t, x, \hat{u}_i^{(m-1)}, [\hat{u}_i^{(m-1)}]_{l_i}, [\tilde{u}_i^{(m-1)}]_{r_i}, [\tilde{u}_i^{(m-1)}]_{l_i}) ,
\]

\[
\mathcal{L}_i \tilde{u}_i^{(m)} = K_i \tilde{u}_i^{(m-1)} + f_i(t, x, \tilde{u}_i^{(m-1)}, [\hat{u}_i^{(m-1)}]_{l_i}, [\tilde{u}_i^{(m-1)}]_{l_i}, [\tilde{u}_i^{(m-1)}]_{l_i}) ,
\]

\[
B_i \hat{u}_i^{(m)} = B_i \tilde{u}_i^{(m)} = h_i(t, x),
\]

\[
\hat{u}_i^{(m)}(t, x) = \hat{u}_i^{(m-1)}(t + T, x), \quad \tilde{u}_i^{(m)}(t, x) = \tilde{u}_i^{(m-1)}(t + T, x)
\]

(\( i = 1, \ldots, N \)),

where \( \mathcal{L}_i \) is given by (2.4). It is obvious that these two sequences are well defined and can be computed by solving a scalar initial boundary problem for each \( i \) and \( m \). The following lemma gives the monotone property of these sequences.

**Lemma 4.1.** Let hypotheses (H1), (H3) be satisfied. Then the sequences \( \{\hat{u}^{(m)}\}, \{\tilde{u}^{(m)}\} \) governed by (4.2) possess the monotone property

\[
\hat{u} \leq \tilde{u}^{(m)} \leq \tilde{u}^{(m+1)} \leq \hat{u}^{(m)} \leq \tilde{u} \quad \text{on} \quad Q.
\]

Moreover, for every \( m = 1, 2, \ldots, \tilde{u}^{(m)} \) and \( \hat{u}^{(m)} \) are coupled upper and lower solutions of (1.1), (1.2a).

**Proof.** Let \( w_i^{(0)} = \hat{u}_i^{(1)} - \tilde{u}_i^{(0)} = \tilde{u}_i^{(1)} - \hat{u}_i, \quad i = 1, \ldots, N \). By (4.1) and (4.2),

\[
\mathcal{L}_i w_i^{(0)} = K_i \hat{u}_i^{(0)} + f_i(\cdot, \hat{u}_i^{(0)}, [\hat{u}_i^{(0)}]_{l_i}, [\tilde{u}_i^{(0)}]_{l_i}, [\tilde{u}_i^{(0)}]_{l_i}) - \mathcal{L}_i \tilde{u}_i - f_i(\cdot, \hat{u}_i^{(0)}, [\hat{u}_i^{(0)}]_{l_i}, [\tilde{u}_i^{(0)}]_{l_i}, [\tilde{u}_i^{(0)}]_{l_i}) \geq 0 \quad \text{in} \quad D,
\]

\[
B_i w_i^{(0)} = h_i(t, x) - B_i \tilde{u}_i \geq 0 \quad \text{on} \quad \Gamma, \quad w_i^{(0)}(t, x) = \hat{u}_i^{(0)}(t + T, x) - \tilde{u}_i(t, x) \geq 0 \quad \text{in} \quad D_0^{(1)}(i = 1, \ldots, N).
\]

The positivity lemma for parabolic boundary problems implies that \( w_i^{(0)} \geq 0 \) on \( D \) (cf. [17]). This leads to \( \tilde{u}^{(1)} \geq \hat{u}^{(0)} \) on \( Q \). A similar argument using the property of a lower solution gives \( \hat{u}^{(1)} \leq \tilde{u}^{(0)} \) on \( Q \). Moreover, by (4.2), (2.1) and the mixed quasimonotone property of \( f(\cdot, u, \tau) \), \( w_i^{(1)} = \hat{u}_i^{(1)} - \tilde{u}_i^{(1)} \) satisfies

\[
\mathcal{L}_i w_i^{(1)} = K_i (\hat{u}_i^{(0)} - \tilde{u}_i^{(0)}) + f_i(\cdot, \hat{u}_i^{(0)}, [\hat{u}_i^{(0)}]_{l_i}, [\tilde{u}_i^{(0)}]_{l_i}, [\tilde{u}_i^{(0)}]_{l_i}) - f_i(\cdot, \tilde{u}_i^{(0)}, [\hat{u}_i^{(0)}]_{l_i}, [\tilde{u}_i^{(0)}]_{l_i}, [\tilde{u}_i^{(0)}]_{l_i}) \geq 0 \quad \text{in} \quad D,
\]

\[
B_i w_i^{(1)} = h_i(t, x) - h_i(t, x) = 0 \quad \text{on} \quad \Gamma, \quad w_i^{(1)}(t, x) = \hat{u}_i^{(0)}(t + T, x) - \tilde{u}_i^{(0)}(t + T, x) \geq 0 \quad \text{in} \quad D_0^{(1)}.
\]
This yields \( u_i^{(1)} \geq 0 \) on \( D, \) \( i = 1, \ldots, N. \) The above conclusions show that \( u^{(0)} = \tilde{u}^{(0)} \leq \bar{u}^{(1)} \leq u^{(0)} \) on \( Q. \) The monotone property (4.3) follows by an induction argument. Since by (4.2), (4.3) and the mixed quasimonotone property of \( f(\cdot, u, v), \)

\[
\frac{\partial \tilde{u}_i^{(m)}}{\partial t} - L_i \tilde{u}_i^{(m)} = K_i (\tilde{u}_i^{(m-1)} - \tilde{u}_i^{(m)}) + f_i (\cdot, \tilde{u}_i^{(m-1)}, [\tilde{u}_i^{(m-1)}], [u_i^{(m-1)}], [\tilde{u}_i^{(m-1)}], [u_i^{(m-1)}]) \]

\[
\geq f_i (\cdot, \bar{u}_i^{(m)}, [\bar{u}_i^{(m)}], [u_i^{(m)}], [\bar{u}_i^{(m)}], [u_i^{(m)}]) \text{ in } D,
\]

\[
B_i \tilde{u}_i^{(m)} = h_i (t, x) \text{ on } \Gamma,
\]

\[
\tilde{u}_i^{(m)} = \bar{u}_i^{(m)} (t + T, x) \geq \bar{u}_i^{(m)} (t + T, x) \text{ in } D_0 (i = 1, \ldots, N),
\]

and similar inequalities for \( \bar{u}_i^{(m)} \) (but in reversed order), we conclude that \( \bar{u}_i^{(m)} \) and \( u_i^{(m)} \) are coupled upper and lower solutions of (1.1), (1.2). This proves the lemma.

It is seen from the monotone property (4.3) that the pointwise limits

\[
\lim_{m \to \infty} \tilde{u}_i^{(m)} (t, x) = \tilde{u}_i (t, x), \quad \lim_{m \to \infty} u_i^{(m)} (t, x) = u_i (t, x)
\]

exist and satisfy a similar relation in the form of (3.5). By letting \( m \to \infty \) in (4.2), a regularity argument as that in [17,20] shows that \( \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_N) \) and \( u = (u_1, \ldots, u_N) \) satisfy the equations

\[
\frac{\partial \tilde{u}_i}{\partial t} - L_i \tilde{u}_i = f_i (t, x, \tilde{u}_i, [\tilde{u}_i], [u_i], [\tilde{u}_i], [u_i]) \quad \text{in } D,
\]

\[
\frac{\partial u_i}{\partial t} - L_i u_i = f_i (t, x, u_i, [u_i], [\tilde{u}_i], [u_i], [\tilde{u}_i]) \quad \text{in } D,
\]

\[
B_i \tilde{u}_i = B_i u_i \text{ on } \Gamma,
\]

\[
\tilde{u}_i (t + T, x) = u_i (t + T, x) \quad \text{in } D_0 (i = 1, \ldots, N).
\]

(4.4)

In the following theorem we show that \( \tilde{u} \) and \( u \), called quasisolutions of (1.1), (1.2), are \( T \)-periodic in \( D \), and every solution of (1.1), (1.2) in \( \mathcal{S} \) must be in between \( u \) and \( \tilde{u} \).

**Theorem 4.1.** Let hypotheses \((H_1), (H_2)\) be satisfied. Then

(i) the sequences \( \tilde{u}_i^{(m)}, u_i^{(m)} \) obtained from (4.2) with \( \tilde{u}_i^{(0)} = \tilde{u}_i \) and \( u_i^{(0)} = u_i \) converge monotonically to some \( T \)-periodic quasisolutions \( \tilde{u}, u \) that satisfy (4.4),

(ii) every \( T \)-periodic solution \( u^* \) of (1.1), (1.2) in \( \mathcal{S} \) must satisfy the relation \( \tilde{u} \leq u^* \leq u \), and there exists at least one such a solution, and

(iii) \( u^* \) is the unique \( T \)-periodic solution in \( \mathcal{S} \) if \( \tilde{u} = u \) (or \( \tilde{u} = u^* \)).

**Proof.** (i) For part (i) of the theorem it suffices to show that \( \tilde{u}_i (t, x) \) and \( u_i (t, x) \) are \( T \)-periodic in \( t \). Let

\[
\tilde{u}_i (t, x) = \tilde{u}_i (t, x) - \bar{u}_i (t + T, x), \quad u_i (t, x) = u_i (t + T, x) - \bar{u}_i (t, x),
\]

\( i = 1, \ldots, N. \)
By (1.1), hypothesis \((H_1)\), and the \(T\)-periodic property of \(\bar{u}_i\) and \(u_i\) in \(D_0^{(i)}\), we have
\[
B_i \bar{w}_i = h_i - h_i = 0, \quad B_i \bar{z}_i = h_i - h_i = 0 \quad \text{on } \Gamma,
\]
\[
\bar{w}_i(t,x) = 0, \quad \bar{z}_i(t,x) = 0, \quad \text{in } D_0^{(i)}.
\]
Moreover, by the \(T\)-periodic property of \(L_i(t), f_i(t,\cdot)\) and the mean-value theorem, we also have
\[
\partial \bar{w}_i/\partial t - L_i \bar{w}_i = f_i(t,x,\bar{u}_i(t,x),[\bar{u}(t,x)]_{a_i}, [\bar{u}_i(t,x)]_{b_i}, [\bar{u}_r(t,x)]_{c_i}, [\bar{u}_r(t+x)]_{d_i})
\]
\[
- f_i(t,x,\bar{u}_i(t+T,x),[\bar{u}(t+T,x)]_{a_i}, [\bar{u}(t+T,x)]_{b_i}, [\bar{u}_r(t+T,x)]_{c_i}, [\bar{u}_r(t+T+T,x)]_{d_i})
\]
\[
= \left( \frac{\partial f_i}{\partial u_i}(\xi,\theta) \right) \bar{w}_i + \sum_{j=1}^{a_i} \left( \frac{\partial f_i}{\partial u_j}(\xi,\theta) \right) \bar{w}_j - \sum_{j=1}^{b_i} \left( \frac{\partial f_i}{\partial u_j}(\xi,\theta) \right) \bar{z}_j
\]
\[
+ \sum_{j=1}^{c_i} \left( \frac{\partial f_i}{\partial (u_r)_j}(\xi,\theta) \right) \bar{w}_r - \sum_{j=1}^{d_i} \left( \frac{\partial f_i}{\partial (u_r)_j}(\xi,\theta) \right) \bar{z}_r
\]
\[
= \alpha^{(i)}(\bar{w}_j) + \sum_{j=1}^{a_i} \alpha^{(i)}(\bar{w}_j) + \sum_{j=1}^{b_i} \beta^{(i)}(\bar{z}_j) + \sum_{j=1}^{c_i} \gamma^{(i)}(\bar{w}_r) + \sum_{j=1}^{d_i} \delta^{(i)}(\bar{z}_r)
\]
\[
(i = 1, \ldots, N),
\]
where \((\xi,\theta)\) (which may depend on \(i\)) is an intermediate value between \((\bar{u}(t,x), u(t,x))\) and \((\bar{u}(t+T,x), u(t+T,x))\), and
\[
\alpha^{(i)} = \frac{\partial f_i}{\partial u_i}(\xi,\theta), \quad \alpha^{(i)} = \frac{\partial f_i}{\partial u_j}(\xi,\theta) \quad (j = 1, \ldots, a_i),
\]
\[
\beta^{(i)} = \frac{\partial f_i}{\partial u_j}(\xi,\theta) \quad (j = 1, \ldots, b_i),
\]
\[
\gamma^{(i)} = \frac{\partial f_i}{\partial (u_r)_j}(\xi,\theta) \quad (j = 1, \ldots, c_i),
\]
\[
\delta^{(i)} = \frac{\partial f_i}{\partial (u_r)_j}(\xi,\theta) \quad (j = 1, \ldots, d_i).
\]
Similarly
\[
\partial \bar{z}_i/\partial t - L_i \bar{z}_i = \alpha^{(i)}(\bar{z}_j) + \sum_{j=1}^{a_i} \alpha^{(i)}(\bar{z}_j) + \sum_{j=1}^{b_i} \beta^{(i)}(\bar{z}_j) + \sum_{j=1}^{c_i} \gamma^{(i)}(\bar{w}_r) + \sum_{j=1}^{d_i} \delta^{(i)}(\bar{z}_r)
\]
\[
(i = 1, \ldots, N),
\]
where \(\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}\) and \(\delta^{(i)}\) are given in the form of (4.7) with possibly a different intermediate value \((\xi,\theta)\). Since \((\xi,\theta)\) is also in between \((\bar{u}(t,x), u(t,x))\) and \((\bar{u}(t+T,x), u(t+T,x))\), the mixed quasimonotone property of \(f(\cdot, u, u_r)\) ensures that
the coefficients \( a_j^{(i)} (j \neq i) \), \( \beta_j^{(i)} \), \( \gamma_j^{(i)} \) and \( \delta_j^{(i)} \) in (4.7) are all nonnegative. The same is true for the coefficients \( \alpha_j^{(i)} \), \( \beta_j^{(i)} \), \( \gamma_j^{(i)} \) and \( \delta_j^{(i)} \). It follows from the positivity lemma for the 2N system (4.5), (4.6) and (4.8) that \((\tilde{u}_i, \tilde{z}_i) \geq (0, 0), i = 1, \ldots, N \) (cf. [21, Lemma 3.1]). It is easy to verify that the relations in (4.5), (4.6) and (4.8) remain unchanged when \((\tilde{u}_i, \tilde{z}_i) \) is replaced by \((-\tilde{u}_i, -\tilde{z}_i)\). This implies that \((\tilde{u}_i, \tilde{z}_i) \leq (0, 0) \) which leads to \((\tilde{u}_i, \tilde{z}_i) = (0, 0), i = 1, \ldots, N \). Hence \( \tilde{u}(t, x) = \tilde{u}(t + T, x) \) and \( u(t, x) = u(t + T, x) \).

(ii) To show the relation \( \underline{u} \leq u^* \leq \overline{u} \) for any \( T \)-periodic solution \( u^* \) in \( S \) we consider \( \tilde{u}_i^{(m)} = \tilde{u}_i^{(m)} - u_i^* \) and \( \tilde{w}_i^{(m)} = u_i^* - u_i^{(m)} \), \( i = 1, \ldots, N \), where \( u^* = (u_1^*, \ldots, u_N^*) \). By (4.2) and (1.1) we have

\[
L_i \tilde{u}_i^{(m)} = K_i \left( u_i^{(m-1)} - u_i^* \right) + f_i (\cdot, \alpha_j^{(i)}, \beta_{j_k}^{(i)}, \gamma_{j_k}^{(i)}, \delta_{j_k}^{(i)}, u_i^{(m-1)}_{j_k}) - f_i (\cdot, u_i^*, u_i^{(m-1)}_{j_k}) \ni \left( u_i^{(m-1)}_{j_k} \right)_{j_k} \right) \ni \left( u_i^{(m-1)}_{j_k} \right)_{j_k}
\]

\[
= (K_i + \alpha_j^{(i)}) \tilde{u}_i^{(m-1)} - \sum_{j=1}^{a_i} \alpha_j^{(i)} \tilde{u}_j^{(m-1)} + \sum_{j=1}^{b_i} \beta_j^{(i)} \tilde{w}_j^{(m-1)}
\]

\[
+ \sum_{j=1}^{c_i} \gamma_j^{(i)} \tilde{u}_j^{(m-1)} + \sum_{j=1}^{d_i} \delta_j^{(i)} \tilde{w}_j^{(m-1)} \quad (i = 1, \ldots, N)
\]

and similarly

\[
L_i \tilde{w}_i^{(m)} = (K_i + \alpha_i^{(m)}) \tilde{w}_i^{(m-1)} + \sum_{j=1}^{a_i} \alpha_j^{(m-1)} \tilde{u}_j^{(m-1)} + \sum_{j=1}^{b_i} \beta_j^{(m-1)} \tilde{w}_j^{(m-1)}
\]

\[
+ \sum_{j=1}^{c_i} \gamma_j^{(m-1)} \tilde{u}_j^{(m-1)} + \sum_{j=1}^{d_i} \delta_j^{(m-1)} \tilde{w}_j^{(m-1)} \quad (i = 1, \ldots, N)
\]

where \( a_j^{(i)}, \alpha_i^{(i)}, \gamma_j^{(i)}, \delta_j^{(i)}, \) etc. are given by (4.7) with possibly some different intermediate values in \( S \times S \). Moreover, the boundary and initial conditions in (4.2) and (1.1) yield

\[
B_i \tilde{u}_i^{(m)} = B_i \tilde{w}_i^{(m)} = h_i - h_i = 0 \quad \text{for} \quad m = 1, 2, \ldots, \\
B_i \tilde{u}_i^{(0)} = B_i \tilde{w}_i^{(0)} = h_i - h_i = 0 \quad \text{in} \quad \Gamma,
\]

and

\[
\tilde{u}_i^{(m)} (t, x) = \tilde{u}_i^{(m-1)} (t + T, x) - u^* (t + T, x),
\]

\[
\tilde{w}_i^{(m)} (t, x) = u_i^* (t + T, x) - u_i^{(m-1)} (t + T, x) \quad \text{in} \quad D_0^{(i)}, \quad m = 1, 2, \ldots.
\]

It is obvious from (4.12) (with \( m = 1 \)) that \( \tilde{u}_i^{(1)} (t, x) = \tilde{u}_i (t + T, x) - u^* (t + T, x) \geq 0 \) and \( \tilde{w}_i^{(1)} (t, x) = u_i^* (t + T, x) - \tilde{u}_i (t + T, x) \geq 0 \) in \( D_0^{(i)} \). Since by the mixed quasimonotone property of \( \mathfrak{f} (\cdot, u, u_r) \), the coefficients \( \alpha_j^{(i)}, \alpha_i^{(i)}, \) etc. in (4.9) and (4.10) are all nonnegative we see from (4.9)–(4.11) (with \( m = 1 \)) and the positivity lemma in [21] that \( \tilde{u}_i^{(1)}, \tilde{w}_i^{(1)} \geq 0 \).
we imbed the system (1.1), (1.2) a into an extended system (1.2) b are extended, respectively, to (4.14), (4.15) a we prepare the following lemma.

It is easy to see from the mixed quasimonotone property of \( f(\cdot, \mathbf{u}, \mathbf{u}_r) \) that \( F(\cdot, \mathbf{w}, \mathbf{w}_r) \) is quasimonotone nondecreasing in \( S^* \). To apply Theorem 3.1 to the extended system (4.14), (4.15) a we prepare the following lemma.
Lemma 4.2. The pair \( \tilde{w} = (\tilde{u}, \tilde{v}) \) and \( \hat{w} = (\hat{u}, \hat{v}) \), where \( \tilde{v} = M - \tilde{u} \) and \( \hat{v} = M - \hat{u} \), are ordered upper and lower solutions of (4.14), (4.15), if and only if \( \tilde{u} \) and \( \hat{u} \) are coupled upper and lower solutions of (1.1), (1.2).

Proof. Suppose \( \tilde{u}, \hat{u} \) are coupled upper and lower solutions of (1.1), (1.2). By (4.1) and (4.13) the components \( (\tilde{u}_i, \tilde{v}_i) \) of \( \tilde{w} \equiv (\tilde{u}, \tilde{v}) \equiv (\tilde{u}, M - \tilde{u}) \) satisfy the relation
\[
\frac{\partial \tilde{u}_i}{\partial t} - L_i \tilde{u}_i \geq f_i (\cdot, \tilde{u}_i, [\tilde{u} - \tilde{v}]_i, [\hat{u} - \tilde{v}]_i, [M - \tilde{v}]_i) = F_i (\cdot, \tilde{v}, \hat{u}, \hat{v}, \tilde{v}_i),
\]
\[
\frac{\partial \tilde{v}_i}{\partial t} - L_i \tilde{v}_i \geq -f_i (\cdot, M_i - \tilde{v}_i, [\hat{u} - \tilde{v}]_i, [\hat{u} - \tilde{v}]_i, [M - \tilde{v}]_i) = G_i (\cdot, \hat{v}, \tilde{v}, \tilde{v}_i) \quad ((t, x) \in D).
\]
\[
B_i \tilde{u}_i \geq h_i (t, x), \quad B_i \tilde{v}_i \geq M_i \beta_i (t, x) - h_i (t, x) = h^*_i (t, x) \quad ((t, x) \in \Gamma),
\]
\[
\tilde{u}_i (t,x) \geq \tilde{v}_i (t+T,x), \quad \tilde{v}_i (t,x) \geq M_i - \tilde{u}_i (t+T,x) = \tilde{v}_i (t+T,x) \quad (t,x) \in D^{(i)}_0, \quad i = 1, \ldots, N.
\] (4.18)

In view of the quasimonotone nondecreasing property of \( F(\cdot, w, x) \) the above relation implies that \( \tilde{w} \equiv (\tilde{u}, \tilde{v}) \) is an upper solution of (4.14), (4.15). A similar argument shows that \( \hat{w} = (\hat{u}, \hat{v}) \) is a lower solution of (4.14), (4.15). The ordering relation \( \hat{w} \supseteq \tilde{w} \) follows from \( \tilde{u} \geq \hat{u} \). This proves the “if part.” The proof for the converse is similar and is omitted. \( \square \)

Using \( \tilde{w} = (\tilde{u}, M - \tilde{u}) \) or \( \hat{w} = (\hat{u}, M - \hat{u}) \) as an initial iteration we construct a sequence \( \tilde{w}^{(m)} = [u_1^{(m)}, \ldots, u_N^{(m)}, v_1^{(m)}, \ldots, v_N^{(m)}] \) from the linear (uncoupled) iteration process
\[
L_i u_i^{(m)} = K_i u_i^{(m-1)} + F_i (t, x, u^{(m-1)}, v^{(m-1)}, u_t^{(m-1)}, v_t^{(m-1)}),
\]
\[
L_i v_i^{(m)} = K_i v_i^{(m-1)} + G_i (t, x, u^{(m-1)}, v^{(m-1)}, u_t^{(m-1)}, v_t^{(m-1)}) \quad ((t,x) \in D),
\]
\[
B_i u_i^{(m)} = h_i (t, x), \quad B_i v_i^{(m)} = h^*_i (t, x) \quad ((t,x) \in \Gamma),
\]
\[
u_i^{(m)} (t,x) = \nu_i^{(m-1)} (t+T,x), \quad v_i^{(m)} (t,x) = v_i^{(m-1)} (t+T,x) \quad (t,x) \in D^{(i)}_0, \quad i = 1, \ldots, N.
\] (4.19)

Since for each \( m \) and each \( i \) the above system is a linear scalar initial boundary problem the sequence \( [u_i^{(m)}, v_i^{(m)}] \) is well defined. Denote the sequence by \( \{\tilde{u}^{(m)}\} = [\tilde{u}_1^{(m)}, \tilde{u}_2^{(m)}, \ldots, \tilde{u}_N^{(m)}, \tilde{v}_1^{(m)}, \ldots, \tilde{v}_N^{(m)}] \) if \( w^{(0)} = \tilde{w} \) and by \( [\hat{u}^{(m)}] = [u_1^{(m)}, \ldots, u_N^{(m)}], \hat{v}^{(m)} \) if \( w^{(0)} = \hat{w} \). The following lemma gives an equivalence relation between the two iteration process (4.2) and (4.19).

Lemma 4.3. The sequences \( \{\tilde{w}^{(m)}\} = [\tilde{u}^{(m)}, \tilde{v}^{(m)}], \{\hat{w}^{(m)}\} = [\hat{u}^{(m)}, \hat{v}^{(m)}] \) governed by (4.19) coincide with the sequences \( \{\tilde{u}^{(m)}, M - \tilde{u}^{(m)}\} \) and \( \{\hat{u}^{(m)}, M - \hat{u}^{(m)}\} \), respectively, where \( \{\tilde{u}^{(m)}\} \) and \( \{\hat{u}^{(m)}\} \) are the sequences governed by (4.2).

Proof. It is easily seen from the definition of \( F_i (\cdot, u, v, u_t, v_t) \) and \( G_i (\cdot, u, v, u_t, v_t) \) in (4.13) that the sequences \( \{\tilde{u}^{(m)}\}, \{\hat{u}^{(m)}\}, i = 1, \ldots, N, \) governed by (4.2) are in the same form as that in (4.19) with \( v_i = m_i - \tilde{u}_i \) when \( \tilde{u}^{(m)} \) and \( \hat{u}^{(m)} \) in \( [\cdot, \cdot, \cdot, \cdot] \) at the right-hand side of (4.2) are replaced by \( M - \tilde{u}^{(m)} \) and \( M - \hat{u}^{(m)} \), respectively. The uniqueness of
the sequences in these iteration processes ensures that \((\bar{u}(m), \bar{v}(m)) = (\bar{u}(m), M - \bar{u}(m))\) and \((u(m), v(m)) = (u(m), M - u(m))\) for every \(m\).

As a consequence of Lemmas 4.2 and 4.3 we have the following result.

**Theorem 4.2.** Let hypotheses \((H_1), (H_3)\) be satisfied, and let \(\bar{u}(t, x), u(t, x)\) be the \(T\)-periodic quasisolutions of (1.1), (1.2)_b that satisfy (4.4). Denote by \(u(t, x; \eta)\) the solution of (1.1), (1.2)_b with an arbitrary \(\eta \in S_0\). Then

\[
\bar{u}(t, x) \leq u(t + mT, x; \eta) \leq \bar{u}(t, x) \quad \text{as} \quad m \to \infty \quad ((t, x) \in \bar{D}).
\]

If \(\bar{u}(t, x) = u(t, x) (\equiv u^*(t, x))\), then

\[
\lim_{m \to \infty} u(t + mT, x; \eta) = u^*(t, x) \quad ((t, x) \in \bar{D}).
\]

**Proof.** Consider the sequences \(\{\bar{w}(m)\}, \{\bar{w}(m)\}\) governed by (4.19) with \(\bar{w}(0) = (\bar{u}, M - \bar{u})\) and \(\bar{w}(0) = (\bar{u}, M - \bar{u})\). Since the function \(F(, w, w)\) in (4.16) is quasimonotone nondecreasing in \(S^* \times S^*_+\), and by Lemma 4.1, \(\bar{w}(0)\) and \(\bar{w}(0)\) are ordered upper and lower solutions of (4.14), (4.15)_a, Theorem A ensures that \(\{\bar{w}(m)\}\) and \(\{\bar{w}(m)\}\) converge monotonically from above and below, respectively, to the maximal and minimal \(T\)-periodic solutions \(\bar{w}\) and \(\bar{w}\). Since the relation \(\bar{\eta}_i = M_i - \bar{u}_i\) implies \(M_i - \bar{u}_i \leq \bar{\eta}_i \leq M_i - \bar{u}_i\) for any \(\bar{\eta} \equiv (\eta_1, \ldots, \eta_N) \in S_0\) the initial function \(\bar{\eta}, \bar{\eta}^*\) in (4.15)_b satisfies the requirement \(\bar{w} \leq (\bar{\eta}, \bar{\eta}) \leq \bar{w} \in D_0\). By an application of Lemma 3.2 to the extended problem (4.14), (4.15)_b we conclude that the solution \(w = (u, \bar{v})\) of this initial boundary problem possesses the property

\[
\bar{w}(m)(t, x) \leq w(t + mT, x; \eta, \bar{\eta}^*) \leq \bar{w}(m)(t, x) \quad \text{on} \quad \bar{D}, \quad m = 1, 2, \ldots.
\]

In view of Lemma 4.3, the above relation is equivalent to

\[
\bar{u}(m)(t, x), M - \bar{u}(m)(t, x) \leq (u(t + mT, x; \eta), \bar{v}(t + mT, x; \bar{\eta}^*)) \leq (\bar{u}(m)(t, x), M - u(m)(t, x)),
\]

and, in particular,

\[
u(m)(t, x) \leq u(t + mT, x; \eta) \leq \bar{u}(m)(t, x) \quad \text{on} \quad \bar{D}, \quad m = 1, 2, \ldots.
\]

The relation (4.20) follows from (4.22) by letting \(m \to \infty\) and an application of Theorem 4.1. It is obvious from (4.20) that (4.21) holds if \(\bar{u} = u \equiv u^*\).

Theorem 4.2 implies that the sector between the two \(T\)-periodic quasisolutions \(\bar{u}\) and \(u\) is an attractor (in the sense of (4.20)) of all the solutions of (1.1), (1.2)_b with \(\eta \in S_0\), and if these quasisolutions coincide then their common value \(u^*\) is a true \(T\)-periodic solution and is a global attractor relative to \(S_0\). As in the case of quasimonotone nondecreasing functions, we have the following global attraction property of \(u^*\) for all initial functions \(\eta\).

**Theorem 4.3.** Let hypotheses \((H_1), (H_3)\) be satisfied, and let \(u(t, x; \eta)\) be the solution of (1.1), (1.2)_b with an arbitrary \(\eta \in X(D_0)\). If condition (3.14) holds for some \(t^* > 0\) then \(u(t + mT, x; \eta)\) possesses the property (4.20). Moreover, (4.21) holds if \(\bar{u} = u \equiv u^*\).
The proof is similar to that for Theorem 3.2 and is omitted.

For the periodic boundary problem (1.3), (1.4)_a without time-delays the requirements of coupled upper and lower solutions remain the same as that in (4.1) except that the differential inequalities for \((\bar{u}_i, \hat{u}_i)\) are replaced by

\[
\frac{\partial \bar{u}_i}{\partial t} - L_i \bar{u}_i \geq f_i(t, x, \bar{u}_i, [\bar{u}]_{\tilde{a} l}, [\bar{u}]_{\tilde{a} h}),
\]

\[
\frac{\partial \hat{u}_i}{\partial t} - L_i \hat{u}_i \leq f_i(t, x, \hat{u}_i, [\hat{u}]_{\tilde{a} l}, [\hat{u}]_{\tilde{a} h}).
\]

By considering problem (1.3) as a special case of problem (1.1) we have the following conclusions.

**Theorem 4.4.** Let hypotheses \((H_1), (H_3)\) be satisfied, where \(\bar{u}\) and \(\hat{u}\) satisfy (4.23), and let \(u(t, x; \eta)\) be the solution of (1.3), (1.4)_b with an arbitrary \(\eta \in X(D_0)\). If condition (3.14) holds then \(u(t + mT, x; \eta)\) possesses the property (4.20). Moreover, \(u(t + mT, x; \eta) \rightarrow u^*(t, x)\) on \(D\) as \(m \rightarrow \infty\) if \(\bar{u} = \hat{u} \equiv u^*\).

If the coefficients of \(L_i, B_i\) and the functions \(f_i, h_i\) are independent of \(t\) then every solution \(u_i(x)\) of the elliptic system (1.6) is a \(T\)-periodic solution of (1.1), (1.2)_a for any period \(T\). Moreover, every pair of coupled or ordered upper and lower solutions \(\bar{u}_i, \hat{u}_i\) of (1.6) (in the usual sense for elliptic systems) are also coupled or ordered upper and lower solutions of (1.1), (1.2)_a (cf. [17]). Using \(\tilde{u}^{(0)} = \bar{u}_i(x)\) and \(\tilde{u}^{(0)} = \hat{u}_i(x)\) as the initial iterations in the iteration process (4.2) with \(J_i = (-L_i + K_i)\) and without the initial condition, the corresponding sequences \([\tilde{u}^{(m)}], [\tilde{u}^{(m)}]\) are time-independent and converge monotonically to some quasisolutions \(\tilde{u}_i(x), \tilde{u}_i(x)\) of (1.6) in the sense that \(\tilde{u}_i \equiv (\bar{u}_1, \ldots, \bar{u}_N)\) and \(\tilde{u}_i \equiv (\bar{u}_1, \ldots, \tilde{u}_N)\) satisfy the equation

\[
-L_i \bar{u}_i = f_i(x, \bar{u}_i, [\bar{u}]_{\tilde{a} l}, [\bar{u}]_{\tilde{a} l}, [\bar{u}]_{\tilde{a} l}, [\bar{u}]_{\tilde{a} l}) (x \in \Omega),
\]

\[
-B_i \bar{u}_i = B_i \bar{u}_i = h_i(x) (x \in \partial \Omega), \quad i = 1, \ldots, N. \quad (4.24)
\]

In particular, if \(f(t, u, \eta)\) is quasimonotone nondecreasing in \(S \times S\) (that is, \(b_i = d_i = 0\) for all \(i\)) then \(\tilde{u}_i\) and \(\tilde{u}_i\) are the respective maximal and minimal solutions of (1.6) in \(S\) (cf. [17,21]). This observation leads to the following conclusion.

**Corollary 4.1.** Let \(L_i, B_i, f_i\) and \(h_i\) be all independent of \(t\), and let \(\tilde{u}_i(x), \tilde{u}_i(x)\) be a pair of coupled upper and lower solutions of the elliptic system (1.6). If hypothesis \((H_3)\) holds with \(\tilde{u} = \tilde{u}_i(x)\) and \(\tilde{u} = \tilde{u}_i(x)\) then all the conclusions in Theorems 4.1–4.4 hold true with respect to the \(T\)-periodic quasisolutions \(\tilde{u}(t, x) \equiv \tilde{u}_i(x), \tilde{u}(t, x) \equiv \tilde{u}_i(x)\) for any period \(T > 0\). Similarly, if hypothesis \((H_2)\) holds with \(\tilde{u} = \tilde{u}_i(x)\) and \(\tilde{u} = \tilde{u}_i(x)\) then all the conclusions in Theorems 3.1–3.3 hold true with respect to the maximal and minimal solutions \(\tilde{u}(t, x) \equiv \tilde{u}_i(x), \tilde{u}(t, x) \equiv \tilde{u}_i(x)\) for any period \(T > 0\).

**5. Applications to Lotka–Volterra systems**

In this section we give some applications of the theorems in Sections 3 and 4 to three Lotka–Volterra reaction–diffusion model problems that arise from ecology. These model
problems are special cases of (1.1) with some specific reaction functions which are either quasimonotone nondecreasing or mixed quasimonotone. Hence our main concern in these applications is the construction of suitable upper and lower solutions so that the existence of a $T$-periodic solution and its global attractivity can be determined from the corresponding theorem for the general system. Of particular concern is the construction of positive upper and lower solutions which ensures the coexistence and permanence of the ecological system as well as the global stability of a positive $T$-periodic solution. The global stability result implies that the trivial solution and the various semitrivial $T$-periodic solutions are necessarily unstable.

5.1. A logistic model

Consider the logistic diffusion model problem

$$
\frac{\partial u}{\partial t} - Lu = au(1 - bu) \quad \text{in } D, \quad Bu = 0 \quad \text{on } \Gamma,
$$

(5.1)

under either the periodic condition (1.4)$_a$ or the initial condition (1.4)$_b$ (with $N = 1$), where $L$ and $B$ are in the same form as $L_i$ and $B_i$, respectively, and $a \equiv a(t, x)$ and $b \equiv b(t, x)$ are smooth positive $T$-periodic functions on $\bar{D}$. It is obvious that problem (5.1), (1.4)$_a$ has always the trivial solution $u = 0$. To show the existence and stability of a positive $T$-periodic solution we consider the periodic eigenvalue problem

$$
\frac{\partial \phi}{\partial t} - L\phi - a\phi = \lambda \phi \quad \text{in } D, \quad B\phi = 0 \quad \text{on } \Gamma, \quad \phi(0, x) = \phi(t, x) \quad \text{in } \Omega.
$$

(5.2)

It is known that for any $T$-periodic function $a \equiv a(t, x)$ the principle eigenvalue of (5.2), denoted by $\lambda(a)$, is real and its corresponding eigenfunction $\phi \equiv \phi(t, x)$ may be chosen positive in $(0, \infty) \times \Omega$ (cf. [9]). In the special case of constant $a$ and the coefficients of $L$ and $B$ are independent of $t$ we have $\lambda(a) = \lambda_0 - a$ and $\phi \equiv \phi(x)$ is independent of $t$, where $\lambda_0$ and $\phi(x)$ are the smallest eigenvalue and its corresponding eigenfunction of the classical eigenvalue problem

$$
L\phi + \lambda\phi = 0 \quad \text{in } \Omega, \quad B\phi = 0 \quad \text{on } \partial\Omega.
$$

(5.3)

We normalize $\phi(t, x)$ so that $\max[\phi(t, x)] = 1$ on $[0, T] \times \bar{\Omega}$.

It is easily seen that for any constant $\rho \geq 1/b(t, x)$ the pair $\tilde{u} = \rho$ and $\hat{u} = 0$ are ordered upper and lower solutions of (5.1), (1.4)$_a$. By Theorem A, problem (5.1), (1.4)$_a$ has a maximal $T$-periodic solution $\bar{u}(t, x)$ and a minimal $T$-periodic solution $\underline{u}(t, x)$ such that $0 \leq u \leq \rho$. It is known that problem (5.1), (1.4)$_a$ has only the trivial solution $u = 0$ if $\lambda(a) \geq 0$, and it has a unique positive $T$-periodic solution $u^*(t, x) \leq 1/b$ if $\lambda(a) < 0$, where $b = \min[b(t, x)]$ on $[0, T] \times \bar{\Omega}$ (cf. [9]). Hence if $\lambda(a) \geq 0$ then $\tilde{u} = \rho = 0$, and by Theorem 3.1 and the arbitrariness of $\rho$ the solution $u(t, x; \eta)$ of (5.1), (1.4)$_b$ corresponding to any nonnegative initial function $\eta = \eta(x)$ possesses the convergence property $u(t + mT, x; \eta) \to 0$ as $m \to \infty$. On the other hand, if $\lambda(a) < 0$ then for a sufficiently small $\delta > 0$, $\bar{u} = \delta\phi(t, x)$ is a positive lower solution. Since $\bar{u} = \rho$ remains to be an upper solution we see from Theorem A that positive maximal and minimal $T$-periodic solutions $\bar{u}(t, x)$ and $\underline{u}(t, x)$ exist and satisfy the relation $\delta \phi \leq u \leq \rho$. The uniqueness property
of \( u^* \) implies that \( \bar{u} = u = u^* \). It follows from Theorem 3.1 that for any initial function \( \eta \geq \delta \phi \) the corresponding solution \( u(t + mT, x; \eta) \) of (5.1), (1.4)_b converges to \( u^*(t, x) \) as \( m \to \infty \). Since for any nontrivial nonnegative \( \eta(x) \) the maximal principle implies that \( u(t, x; \eta) \) is positive in \((0, \infty) \times \Omega\), the arbitrariness of \( \delta \) ensures that there exist \( t_0 > 0 \) and \( \delta > 0 \) such that \( u(t, x; \eta) \geq \delta \phi(x) \) in \([t_0, \infty) \times \Omega\). This shows that \( u(t, x; \eta) \) satisfies relation (3.14) for some \( t^* > 0 \). By Theorem 3.2, \( u(t + mT, x; \eta) \to u^*(t, x) \) as \( m \to \infty \).

The above discussion leads to the following result for the logistic diffusion model.

**Theorem 5.1.** Let \( u^*(t, x) \) be the unique positive solution of (5.1), (1.4)_a (with \( N = 1 \)) when \( \lambda(a) < 0 \), and let \( u(t, x; \eta) \) be the solution of (5.1), (1.4)_b corresponding to an arbitrary nontrivial nonnegative \( \eta(x) \). Then for any \( (t, x) \in (0, \infty) \times \Omega \),

\[
\lim_{m \to \infty} u(t + mT, x; \eta) = \begin{cases} 0 & \text{if } \lambda(a) \geq 0, \\
u^*(t, x) & \text{if } \lambda(a) < 0. \end{cases}
\]  

(5.4)

### 5.2. An extended two-competition model

According to the Holling–Tanner interaction hypothesis of two competing species, an extended Lotka–Volterra competition diffusion model is given by

\[
\begin{align*}
\partial u/\partial t - L_1 u &= u(a_1 - b_1 u - c_1 v)/(1 + \sigma_1 u), \\
\partial v/\partial t - L_2 v &= v(a_2 - b_2 u)/(1 + \sigma_2 u) - c_2 v) \\
B_1 u &= 0, & B_2 v &= 0 \\
(\Omega \times \Gamma).
\end{align*}
\]  

(5.5)

where \( a_i \equiv a_i(t, x), b_i \equiv b_i(t, x) \) and \( c_i \equiv c_i(t, x), i = 1, 2, \) are smooth positive \( T \)-periodic functions on \( D \), and \( \sigma_i \) is a nonnegative constant (cf. \([1, 8, 9, 15, 18, 29]\)). We allow either \( \sigma_i = 0 \) or \( \tau_i = 0 \) for some or both \( i \), and in the case of \( \sigma_i = \tau_i = 0 \) for both \( i \) the system is reduced to the standard competition model (e.g. see [17] and references therein). A major concern of the above diffusion model is whether and when the two-competing species can coexist and what is the long time behavior of the coexistent state. For the periodic system without time-delays (that is, \( \tau_1 = \tau_2 = 0 \)) this system has been treated in [8, 27] for Neumann boundary condition and in [18] for the general boundary condition (1.2)_a. Here we investigate the coexistence problem as well as the stability or instability of \( T \)-periodic solutions.

Let \( \lambda_i(a_i) \) and \( \phi_i(t, x), i = 1, 2, \) be the principal eigenvalue and its corresponding positive eigenfunction of (5.2) with \( L = L_i, B = B_i \) and \( a = a_i \). It is easy to see from the discussion of the logistic equation that problem (5.5) has only the trivial solution \((0, 0)\) if \( \lambda_i(a_i) \geq 0 \) for \( i = 1, 2, \) and it has also the semitrivial \( T \)-periodic solution \((U, 0)\) (respectively, \((0, V)\)) if \( \lambda_i(a_1) < 0 \) and \( \lambda_2(a_2) > 0 \) (respectively, \( \lambda_1(a_1) > 0 \) and \( \lambda_2(a_2) < 0 \)), where \( U \) and \( V \) are the positive \( T \)-periodic solutions of the respective problem

\[
\begin{align*}
U_t - L_1 U &= U(a_1 - b_1 U) & \text{in } D, & B_1 U &= 0 & \text{on } \Gamma, \\
V_t - L_2 V &= V(a_2 - c_2 V) & \text{in } D, & B_2 V &= 0 & \text{on } \Gamma.
\end{align*}
\]  

(5.6)

Hence for the existence of a positive \( T \)-periodic solution of (5.5), (1.2)_a a necessary (but not sufficient) condition is that \( \lambda_i(a_i) < 0, i = 1, 2 \). To show the existence problem we make a
transformation by letting \( w = M - v \) for a sufficiently large constant \( M > 0 \). Then problem (5.5) is transformed into the form of (1.1) with \( u \equiv (u_1, u_2) \equiv (u, v), (h_1, h_2) = (0, M\beta_2) \) and \((f_1, f_2)\) given by

\[
\begin{align*}
f_1(\cdot, u, w, u_{\tau_1}, w_{\tau_2}) &= u\left[a_1 - b_1u - c_1(M - w_{\tau_2})/(1 + \sigma_1u_{\tau_1})\right], \\
f_2(\cdot, u, w, u_{\tau_1}, w_{\tau_2}) &= -(M - w)\left[a_2 - b_2u_{\tau_1}/(1 + \sigma_2u_{\tau_1}) - c_2(M - w)\right].
\end{align*}
\]

(5.7)

It is easy to see that the above function \((f_1, f_2)\) is quasimonotone nondecreasing in \( S \times \mathbb{R}_+ \), where \( S = \mathbb{R}_+ \times [0, M] \). Hence for the existence of a positive \( T\)-periodic solution it suffices to find a pair of ordered upper and lower solutions of the transformed problem. We seek such a pair in the form

\[
\begin{align*}
\bar{u} &= (\bar{u}, \bar{w}) = (\rho_1, M - \delta_2\phi_2), \\
\hat{u} &= (\hat{u}, \hat{w}) = (\delta_1\phi_1, M - \rho_2)
\end{align*}
\]

for some positive constants \( \rho_i \) and \( \delta_i \) \((i = 1, 2)\), where \( \rho_2 < M \) and \( \delta_i \) is sufficiently small.

It is easy to verify that \( M - \bar{w} = \delta_2\phi_2 \) and \( M - \hat{w} = \rho_2 \) that \((\bar{u}, \bar{w})\) and \((\hat{u}, \hat{w})\) satisfy all the requirements of upper and lower solutions if

\[
\begin{align*}
0 &> \rho_1\left[a_1 - b_1\rho_1 - c_1(\delta_2\phi_2)_{\tau_1}/(1 + \sigma_1\rho_1)\right], \\
-\delta_2[\partial\phi_2/\partial t - L_2\phi_2] &\geq -(\delta_2\phi_2)\left[a_2 - b_2\rho_1/(1 + \sigma_2\rho_1) - c_2(\delta_2\phi_2)\right], \\
\delta_1[\partial\phi_1/\partial t - L_1\phi_1] &\leq (\delta_1\phi_1)\left[a_1 - b_1(\delta_1\phi_1) - c_1\rho_2/(1 + \sigma_1(\delta_1\phi_1)\tau_1)\right], \\
0 &\leq -\rho_2\left[a_2 - b_2(\delta_1\phi_1)_{\tau_1}/(1 + \sigma_2(\delta_1\phi_1)\tau_1) - c_2\rho_2\right].
\end{align*}
\]

In view of (5.2) the above inequalities are satisfied by some sufficiently small \( \delta_1, \delta_2 \) if

\[
\begin{align*}
0 > a_1 - b_1\rho_1, \quad -\lambda_2(a_2) > b_2\rho_1/(1 + \sigma_2\rho_1), \\
\lambda_1(a_1) < -c_1\rho_2, \quad 0 < c_2\rho_2 - a_2.
\end{align*}
\]

(5.8)

Set

\[
M_1 = \max_D\left[a_1(t, x)/b_1(t, x)\right], \quad M_2 = \max_D\left[a_2(t, x)/c_2(t, x)\right]
\]

(5.9)

Then the requirements in (5.8) are fulfilled by some \( \rho_i > M_i, \) \(i = 1, 2,\) if

\[
-\lambda_1(a_1) > c_1M_2 \quad \text{and} \quad -\lambda_2(a_2) > b_2M_1/(1 + \sigma_2M_1).
\]

(5.10)

This shows that under the condition (5.10) the transformed problem of (5.5), (1.2)_h with \( N = 2 \) has a maximal \( T\)-periodic solution \((\bar{u}, \bar{w})\) and a minimal \( T\)-periodic solution \((\hat{u}, \hat{w})\) such that

\[
(\delta_1\phi_1, M - \rho_2) \leq (u, w) \leq (\bar{u}, \bar{w}) \leq (\rho_1, M - \delta_2\phi_2).
\]

Moreover, by Theorem 3.1 the solution \( u \equiv (u, w) \) of the initial boundary problem (5.5), (1.2)_h possesses the convergence property (3.11) and (3.12) with respect to the initial function \((\eta_1^1, \eta_2^1) \equiv (\eta_1, M - \eta_2)\). Now by the transformation \( v = M - w \) the pair \((\bar{u}, \bar{v})\) and \((\hat{u}, \hat{v})\), where \( v = M - w, \hat{v} = M - \bar{w} \), are positive \( T\)-periodic solutions of the periodic boundary problem (5.5), (1.2)_h and satisfy the relation

\[
\delta_1\phi_1 \leq u \leq \rho_1, \quad \delta_2\phi_2 \leq v \leq \rho_2 \quad \text{on} \quad \bar{D}.
\]

(5.11)

Furthermore, for any \((\eta_1, \eta_2)\) satisfying \( \delta_1\phi_1 \leq \eta_1 \leq \rho_1, \delta_2\phi \leq \eta_2 \leq \rho_2 \) in \( D^1_0 \), \(i = 1, 2\), the solution of the initial boundary problem (5.5), (1.2)_h is given by \((u, v) \equiv (u, M - w)\)
and satisfies the relation $\delta_1 \phi_1 \leq u \leq \rho_1$, $\delta_2 \phi_2 \leq v \leq \rho_2$ on $\bar{D}$. For arbitrary nonnegative $(\eta_1, \eta_2)$ with $\eta_1(0, x) \neq 0$, a comparison between $(u, v)$ with the solution $(U, V)$ of (5.6) (with the same $(\eta_1, \eta_2)$) shows that $(u, v) \leq (U, V)$ on $[t_0, \infty) \times \Omega$ for some $t_0 \geq 0$. Since $(U, V) < (\rho_1, \rho_2)$ and $(\delta_1, \delta_2) > (0, 0)$ can be chosen arbitrarily small we see that there exist $t^* > 0$ and $(\delta_1, \delta_2) > (0, 0)$ such that

\[
\delta_1 \phi_1 \leq u \leq \rho_1 \quad \text{when } t^* - \tau_1 \leq t \leq t^*,
\]

\[
\delta_2 \phi_2 \leq v \leq \rho_2 \quad \text{when } t^* - \tau_2 \leq t \leq t^* \quad (x \in \Omega).
\]

By Theorem 3.2 the solution $(u, v)$ possesses the convergence property

\[
\lim_{m \to \infty} (u(t + mT, x; \eta_1), v(t + mT, x; \eta_2)) = \begin{cases} (\bar{u}, \bar{v}) & \text{if } 0 \leq \eta_1 \leq \bar{u}, \eta_2 \geq \bar{v}, \\ (\bar{u}, \bar{v}) & \text{if } \eta_1 \geq \bar{u}, 0 \leq \eta_2 \leq \bar{v}. \end{cases}
\] (5.12)

To summarize the above conclusions we have the following theorem.

**Theorem 5.2.** Let $(u(t, x; \eta_1), v(t, x; \eta_2))$ be the solution of (5.5), (1.2)$_b$ (with $N = 2$) for an arbitrary nonnegative $(\eta_1, \eta_2)$ with $\eta_i(0, x) \neq 0$, $i = 1, 2$, and let condition (5.10) be satisfied. Then

(i) problem (5.5), (1.2)$_b$ has positive $T$-periodic solutions $(\bar{u}, \bar{v}), (\bar{u}, \bar{v})$ such that $u \leq \bar{u}$ and $v \leq \bar{v}$ on $\bar{D}$,

(ii) the solution $(u(t, x; \eta_1), v(t, x; \eta_2))$ of (5.5), (1.2)$_b$ possesses the convergence property (5.12), and

(iii) if $(\bar{u}, \bar{v}) = (u^*, v^*)$ then

\[
\lim_{m \to \infty} (u(t + mT, x; \eta_1), v(t + mT, x; \eta_2)) = (u^*(t, x), v^*(t, x))
\]

$(t > 0, x \in \bar{\Omega})$. (5.13)

5.3. An $N$-species competition system

In the Lotka–Volterra system with $N$-competing species if the effect of dispersion and time delays are both taken into consideration, then the population densities of the competing species $u_1, \ldots, u_N$ are governed by the equations

\[
\partial u_i / \partial t - L_i u_i = a_i u_i \left( 1 - u_i - \sum_{j \neq i}^N b_{ij} u_j - \sum_{j=1}^N c_{ij}(u_j) \tau_j \right) \quad (t, x) \in D),
\]

\[
B_i u_i = 0 \quad (t, x) \in \Gamma), \quad i = 1, \ldots, N,
\] (5.14)

where $(u_j)_{\tau_j} = u_j(t - \tau_j, x)$ with $\tau_j > 0$, $a_i$, $b_{ij}$ and $c_{ij}$ are smooth $T$-periodic functions, and $a_i > 0$, $b_{ij} \geq 0$ and $c_{ij} \geq 0$ on $\bar{D}$ for all $i, j = 1, \ldots, N$. The above system is a special case of (1.1) with $u = (u_1, \ldots, u_N)$ and

\[
f_i(\cdot, u, u_\tau) = a_i u_i \left( 1 - u_i - \sum_{j \neq i}^N b_{ij} u_j - \sum_{j=1}^N c_{ij}(u_j) \tau_j \right).
\] (5.15)
It is clear from the nonnegative property of the various reaction rates that the vector function \((f_1(\cdot, u, u), \ldots, f_N(\cdot, u, u))\) from (5.15) is mixed quasimonotone in \(S \times S_\tau\), where \(S = S_\tau = (\mathbb{R}^N)^+ \times (\mathbb{R}^N)^+\). In fact, this function is quasimonotone nonincreasing in the sense that \(\partial f_i/\partial u_j \leq 0\) for \(j \neq i\) and \(\partial f_i/\partial (u_j)\tau_j \leq 0\) for all \(j, i = 1, \ldots, N\).

The asymptotic behavior of solutions of the Lotka–Volterra system (5.14), (1.2)_b, including the time-delayed ordinary differential system

\[
du_i/dt = a_iu_i \left(1 - u_i - \sum_{j \neq i} b_{ij}u_j - \sum_{j=1}^{N} c_{ij}(u_j)\tau_j\right) \quad (t > 0),
\]

\[
u_i(t) = \eta_i(t) \quad (-\tau_i \leq t \leq 0), \quad i = 1, \ldots, N,
\]

has been investigated by many researchers in the field of population dynamics. Most of the discussions in the literature are devoted either to systems without time delays, or to the autonomous system where the coefficients of \(L_i, B_i\) and the reaction rates \(a_i, b_{ij}\) and \(c_{ij}\) are all independent of \(t\) (cf. [11,22,24–26]). In view of the results obtained for the general system (1.1) we investigate the asymptotic behavior of the solution of (5.14), (1.2)_b in relation to positive \(T\)-periodic solutions of (5.14), (1.2)_b by a suitable construction of upper and lower solutions. Our main purpose for the system is to obtain a sufficient condition that ensures the existence and global attraction of positive \(T\)-periodic solutions. From an ecological point of view this result implies that the various competing species coexist and ultimately enter a sector between a pair of positive \(T\)-periodic quasisolutions and possibly converge to a unique positive \(T\)-periodic solution.

It is obvious that system (5.14), (1.2)_b has always the trivial solution \((0, \ldots, 0)\), and if \(\lambda_i(a_i) \geq 0\) for some \(i\) then there exists at least on nonnegative solution whose \(i\)th component is zero. In the case of \(\lambda_i(a_i) < 0\) for some \(i\), the system has at least on nonnegative semitrivial \(T\)-periodic solution whose \(i\)th component is positive. This implies that problem (5.14), (1.2)_b has various forms of nonnegative semitrivial solutions if \(\lambda_i(a_i) < 0\) for some or all \(i\), and a necessary condition for the existence of a positive \(T\)-periodic solution is that \(\lambda_i(a_i) < 0\) for all \(i\). To show the existence of positive \(T\)-periodic solutions we observe from the quasimonotone nonincreasing property of the reaction function that the requirements of upper and lower solutions in (4.1) become

\[
\partial \tilde{u}_i/\partial t - L_i\tilde{u}_i \geq a_i\tilde{u}_i \left(1 - \tilde{u}_i - \sum_{j \neq i} b_{ij}\tilde{u}_j - \sum_{j=1}^{N} c_{ij}(\tilde{u}_j)\tau_j\right),
\]

\[
\partial \hat{u}_i/\partial t - L_i\hat{u}_i \leq a_i\hat{u}_i \left(1 - \hat{u}_i - \sum_{j \neq i} b_{ij}\hat{u}_j - \sum_{j=1}^{N} c_{ij}(\hat{u}_j)\tau_j\right) \quad ((t, x) \in D),
\]

\[
B_i\tilde{u}_i \geq 0 \geq B_i\hat{u}_i \quad ((t, x) \in \Gamma),
\]

\[
\tilde{u}_i(t, x) \geq \tilde{u}_i(t + T, x) \quad \hat{u}_i(t, x) \leq \hat{u}_i(t + T, x) \quad ((t, x) \in D_0^{(i)}),
\]

\[i = 1, \ldots, N.
\]

We seek such a pair in the form

\[
\hat{u} = (1, \ldots, 1) \equiv E, \quad \tilde{u} = (\delta_1\phi_1, \ldots, \delta_N\phi_N)
\]

(5.17)
for some small positive constants \( \delta_i, i = 1, \ldots, N \), where \( \phi_i = \phi_i(t, x) \) is the (normalized) positive eigenfunction of (5.2) corresponding to \( \lambda_i(a_i) \) (with \( L = L_i, B = B_i \) and \( a = a_i \)). It is clear that this pair satisfy the boundary and initial inequalities as well as the first differential inequality for \( \tilde{u}_i \) in (5.17). The differential inequality for \( \tilde{u}_i \) is also satisfied if

\[
\delta_i \left( \frac{\partial \phi_i}{\partial t} - L_i \phi_i \right) \leq a_i (\delta_i \phi_i) \left( 1 - \delta_i \lambda_i \phi_i - \sum_{j \neq i} b_{ij} u_j - \sum_{j = 1}^N c_{ij} \phi_j \right).
\]

In view of (5.2) the above requirement is fulfilled by a sufficiently small \( \delta_i > 0 \) if

\[
-\lambda_i(a_i) > a_i \left( \sum_{j \neq i} b_{ij} + \sum_{j = 1}^N c_{ij} \right) \quad \text{on } \bar{D}, \ i = 1, \ldots, N.
\] (5.19)

Under the above condition the pair \( \tilde{u}, \hat{u} \) given by (5.18) are coupled upper and lower solutions of (5.14), (1.2)\(a\). They are also coupled upper and lower solutions of (5.14), (1.2)\(b\) if \( \delta_i \phi_i \leq \eta_i \leq 1 \) in \( D_{0}^{(i)} \). By Theorem 4.1, problem (5.14), (1.2)\(a\) has a pair of positive \( T\)-periodic quasisolutions \( \hat{u} \equiv (\hat{u}_1, \ldots, \hat{u}_N) \) and \( u \equiv (u_1, \ldots, u_N) \) that satisfy the relation

\[
\delta_i \phi_i \leq u_i \leq \bar{u}_i \leq 1 \quad \text{on } \bar{D}, \ i = 1, \ldots, N,
\] (5.20)

and the equations

\[
\begin{align*}
\partial \hat{u}_i / \partial t - L_i \hat{u}_i &= a_i \hat{u}_i \left( 1 - \hat{u}_i - \sum_{j \neq i} b_{ij} u_j - \sum_{j = 1}^N c_{ij} \phi_j \right), \\
\partial u_i / \partial t - L_i u_i &= a_i u_i \left( 1 - u_i - \sum_{j \neq i} b_{ij} u_j - \sum_{j = 1}^N c_{ij} \phi_j \right) \quad \text{in } D, \\
B_i \hat{u}_i = B_i u_i &= 0 \quad \text{on } \Gamma, \\
\hat{u}_i(t, x) &= \bar{u}_i(t + T, x), \quad u_j(t, x) = u_j(t + T, x) \quad \text{in } D_{0}^{(i)}.
\end{align*}
\] (5.21)

Moreover, for any initial function \( \eta \equiv (\eta_1, \ldots, \eta_N) \) with \( \delta_i \phi_i \leq \eta_i \leq 1 \) in \( D_{0}^{(i)} \) the corresponding solution \( u(t, x; \eta) \) of (5.14), (1.2)\(b\) satisfies (4.20), and if \( \tilde{u} = \hat{u} (\equiv u^*) \) then \( u(t + mT, x; \eta) \) converges to \( u^*(t, x) \) as \( m \to \infty \). For arbitrary nonnegative \( \eta \equiv (\eta_1, \ldots, \eta_N) \) with \( \eta_i(0, x) \neq 0 \), a similar argument as that for the system (5.5), (1.2)\(b\) and a comparison between (5.14) and (5.1) show that there exists \( t^* > 0 \) such that the components of the corresponding solution \( u(t, x; \eta) \) satisfy the relation

\[
\delta_i \phi_i \leq u_i(t, x; \eta) \leq 1 \quad \text{when } t^* - \tau_i \leq t \leq t^*+, \ x \in \Omega, \ i = 1, \ldots, N.
\]

As a consequence of Theorems 4.1–4.3 we have the following conclusions.

**Theorem 5.3.** Let \( \lambda_i(a_i) < 0 \) and satisfied condition (5.19). Then

(i) the system (5.14), (1.2)\(a\) has a pair of positive \( T\)-periodic quasisolutions \( \hat{u} \equiv (\hat{u}_1, \ldots, \hat{u}_N), \ u \equiv (u_1, \ldots, u_N) \) that satisfy relation (5.21),

(ii) for any nonnegative \( \eta \equiv (\eta_1, \ldots, \eta_N) \) with \( \eta_i(0, x) \neq 0 \) the corresponding solution \( u(t, x; \eta) \) of (5.14), (1.2)\(b\) possesses the property (4.20), and

(iii) if \( \tilde{u} = u (\equiv u^*) \) then \( u(t + mT, x; \eta) \) converges to \( u^*(t, x) \) as \( m \to \infty \).
For the ordinary differential system (5.16), upper and lower solutions are required to satisfy (5.17) with $\bar{L}_i = 0$ and without the boundary condition. It is obvious that in this situation the pair $\bar{u} = E$ and $\hat{u} = \delta E \equiv (\delta, \ldots, \delta)$, where $\delta > 0$ is sufficiently small, are coupled upper and lower solutions of (5.16) if

$$\sum_{j \neq 1}^N b_{ij} + \sum_{j=1}^N c_{ij} < 1, \quad i = 1, \ldots, N. \quad (5.22)$$

The same is true for the Neumann boundary problem (5.14), (1.2) where $\alpha_i = 1$, $\beta_i = 0$.

This observation leads to the following

**Corollary 5.1.** Under the condition (5.22) all the conclusions in (i)–(iii) of Theorem 5.3 hold true for the positive $T$-periodic solutions (or quasisolutions) $\bar{u}(t)$, $\mathbf{u}(t)$, $\mathbf{u}^*(t)$ and $\mathbf{u}(t; \eta)$ of (5.16). These results hold true also for the Neumann boundary problem (5.14), (1.2) where $\alpha_i = 1$, $\beta_i = 0$ for all $i$.

**Remark 5.1.** In Theorems 5.1–5.3 the requirement of $u_i(0, x) \neq 0$ is necessary, for if $u_i(0, x) \equiv 0$ for some $i$ then the $i$th component $u_i(t, x)$ is identically zero, and therefore the solution $\mathbf{u}(t, x; \eta)$ cannot be bounded from below by a positive lower solution.

**References**


