Note

A short proof of existence of disjoint hypercyclic operators

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ABSTRACT

We give a short proof of existence of disjoint hypercyclic tuples of operators of any given length on any separable infinite dimensional Fréchet space. Similar argument provides disjoint dual hypercyclic tuples of operators of any length on any infinite dimensional Banach space with separable dual.

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1. Introduction

All vector spaces in this article are over the field K, being either the field C of complex numbers or the field R of real numbers. As usual, Z_+ is the set of non-negative integers and N is the set of positive integers. Symbol L(X) stands for the space of continuous linear operators on a topological vector space X, while X' is the space of continuous linear functionals on X. If X is a normed space, then X' is assumed to carry the standard norm topology. For each T ∈ L(X), the dual operator T' : X' → X' is defined as usual: \((T' f)(x) = f(T x)\) for f ∈ X' and x ∈ X. An F-space is a complete metrizable topological vector space. A locally convex F-space is called a Fréchet space. Recall that x ∈ X is called a hypercyclic vector for T ∈ L(X) if \(\{T^nx : n ∈ Z_+\}\) is dense in X. T is called hypercyclic if it has a hypercyclic vector. We refer to the book [1] and references therein for further information on hypercyclic operators. Recently Bés and Peris [5] and Bernal-González [2] independently introduced the following concept.

Definition 1.1. Let X be a topological vector space, m ∈ N and T = (T_1, ..., T_m) ∈ L(X)^m. The m-tuple T is called disjoint hypercyclic if there exists x ∈ X such that

\[ O(x, T) = \{ (T_1^n x, ..., T_m^n x) : n ∈ Z_+ \} \text{ is dense in } X^m. \]

Obviously, T is disjoint hypercyclic if and only if the operator \(T_1 ⊕ ... ⊕ T_m\) has a hypercyclic vector of the shape \((x, ..., x)\) for some x ∈ X. Salas [9] introduced the following concept.

Definition 1.2. Let X be a Banach space, m ∈ N and T = (T_1, ..., T_m) ∈ L(X)^m. Then T is called disjoint dual hypercyclic if both T and T' = (T'_1, ..., T'_m) ∈ L(X')^m are disjoint hypercyclic.

In [5,2] interesting examples of disjoint hypercyclic m-tuples are provided. Bés, Martin and Peris [3] and Salas [9] independently demonstrated that for any m ∈ N, any separable infinite dimensional Banach space supports a disjoint hypercyclic m-tuple of continuous linear operators. Moreover, the construction in [3] works also for separable infinite dimensional...
Fréchet spaces, while the construction in [9] provides a dual disjoint hypercyclic \( m \)-tuple on any infinite dimensional Banach space with separable dual. The following theorems summarize these results.

**Theorem D.** Let \( X \) be a separable infinite dimensional Fréchet space and \( m \in \mathbb{N} \). Then there exists a disjoint hypercyclic tuple \( T = (T_1, \ldots, T_m) \in L(X)^m \).

**Theorem S.** Let \( X \) be an infinite dimensional Banach space with separable dual \( X' \) and \( m \in \mathbb{N} \). Then there exists a disjoint dual hypercyclic tuple \( T = (T_1, \ldots, T_m) \in L(X)^m \).

It is worth noting that Theorem D generalizes the result of Bonet and Peris [6], who demonstrated that each separable infinite dimensional Fréchet space \( X \) supports a hypercyclic operator \( T \). Grivaux [7] observed that hypercyclic operators \( T \) constructed in [6] are in fact mixing and therefore hereditarily hypercyclic. That is, for each infinite subset \( A \) of \( \mathbb{Z}_+ \) there is \( x \in X \) such that \( \{T^nx: n \in A\} \) is dense in \( X \). According to Bès and Peris [4], the direct sums of finitely many copies of \( T \) are also hypercyclic. The following proposition summarizes this observation.

**Proposition 1.3.** For any separable infinite dimensional Fréchet space \( X \), there is \( T \in L(X) \) such that the direct sum \( T \oplus \cdots \oplus T = T^\oplus m \) of \( m \) copies of \( T \) is a hypercyclic operator on \( X^m \) for each \( m \in \mathbb{N} \).

Salas [8] proved that each infinite dimensional Banach space \( X \) with separable dual supports a dual hypercyclic operator \( T \). It is worth noting that the operator \( T \) constructed by Salas has an extra property. Namely, both \( T \) and \( T' \) satisfy the hypercyclicity criterion [4] and therefore the direct sums of finitely many copies of \( T \) as well as the direct sums of finitely many copies of \( T' \) are hypercyclic. The following proposition summarizes this observation.

**Proposition 1.4.** For any infinite dimensional Banach space \( X \) with separable \( X' \), there is \( T \in L(X) \) such that the direct sum \( T^\oplus m \) of \( m \) copies of \( T \) is a hypercyclic operator on \( X^m \) and the direct sum \( T'^\oplus m \) of \( m \) copies of \( T' \) is a hypercyclic operator on \( (X')^m \) for each \( m \in \mathbb{N} \).

Although constructions in [3,9] are interesting in their own right, it turns out that there is a short and simple reduction of Theorems D and S to the already known Propositions 1.3 and 1.4. The objective of this short note is to show this.

### 2. Reduction of Theorems D and S to Propositions 1.3 and 1.4

Let \( X \) be a topological vector space. Throughout this section \( GL(X) \) stands for the set of invertible linear operators \( T: X \to X \) such that both \( T \) and \( T^{-1} \) are continuous.

**Lemma 2.1.** Let \( m \in \mathbb{N} \), \( X \) be a topological vector space, \( T_1, \ldots, T_m \in L(X) \) and \( (x_1, \ldots, x_m) \in X^m \) be a hypercyclic vector for \( T_1 \oplus \cdots \oplus T_m \). Assume also that \( S_1, \ldots, S_m \in GL(X) \) and \( x \in X \) are such that \( S_jx_j = x \) for \( 1 \leq j \leq m \). Then \( (x, \ldots, x) \in X^m \) is a hypercyclic vector for \( R_1 \oplus \cdots \oplus R_m \), where \( R_j = S_jT_jS_j^{-1} \). In particular, \( (R_1, \ldots, R_m) \) is disjoint hypercyclic.

**Proof.** Since the orbit \( O \) of \( (x_1, \ldots, x_m) \) with respect to \( T_1 \oplus \cdots \oplus T_m \) is dense in \( X^m \) and \( \Lambda = S_1 \oplus \cdots \oplus S_m \in GL(X^m) \), \( \Lambda(O) \) is also dense in \( X^m \). Standard similarity argument shows that \( \Lambda(O) \) is exactly the orbit of \( (x, \ldots, x) \) with respect to \( R_1 \oplus \cdots \oplus R_m \). \( \square \)

The following lemma is an elementary and well-known fact. We include the proof for the sake of completeness.

**Lemma 2.2.** Let \( X \) be a locally convex topological vector space. Then the group \( GL(X) \) acts transitively on \( X \setminus \{0\} \). That is, for each non-zero \( x, y \in X \), there is \( S \in GL(X) \) such that \( Sx = y \).

**Proof.** Let \( x \) and \( y \) be non-zero vectors in \( X \). If \( x \) and \( y \) are not linearly independent, there exists non-zero \( \lambda \in \mathbb{K} \) such that \( y = \lambda x \). Clearly \( S = \lambda I \in GL(X) \) and \( Sx = y \). Assume now that \( x \) and \( y \) are linearly independent. Using the Hahn–Banach theorem, we can find \( f, g \in X' \) such that \( f(x) = g(y) = 1 \) and \( f(y) = g(x) = 0 \). Now we define \( S \in L(X) \) by the formula \( Su = u + (g - f)(u)x + (f - g)(u)y \). It is easy to see that \( S \in GL(X) \) and \( Sx = y \). \( \square \)

**Proof of Theorem D.** Let \( X \) be a separable infinite dimensional Fréchet space and \( m \in \mathbb{N} \). By Proposition 1.3, there is \( T \in L(X) \) such that \( T^\oplus m \in L(X^m) \) is hypercyclic. Let \( (x_1, \ldots, x_m) \) be a hypercyclic vector for \( T^\oplus m \). By Lemma 2.2, we can pick \( S_1, \ldots, S_m \in GL(X) \) and \( x \in X \) such that \( S_jx_j = x \) for \( 1 \leq j \leq m \). By Lemma 2.1, \( R_1 \oplus \cdots \oplus R_m \) is disjoint hypercyclic, where \( R_j = S_jT_jS_j^{-1} \). \( \square \)

In order to prove Theorem S, we need a slightly more sophisticated version of Lemma 2.2. First, we need the following elementary lemma.
Lemma 2.3. Let $\alpha$, $\beta$, $\gamma \in \mathbb{K}$ be such that $(\alpha, \beta) \neq (0,0)$ and $(\beta, \gamma) \neq (0,0)$. Then there exist $a, b, c \in \mathbb{K}$ such that $a\alpha + b\beta = -1$, $b\alpha + c\beta = 1$ and $a\beta + b\gamma \neq 0$.

Proof. If $\beta \neq 0$ and $\gamma \neq 0$, we set $a = 0$, $b = -\beta^{-1}$ and $c = (\alpha + \beta)\beta^{-2}$. If $\beta \neq 0$ and $\gamma = 0$, we set $a = 1$, $b = -(1 + \alpha)\beta^{-1}$ and $c = (\alpha + \beta + \alpha^2)\beta^{-2}$. If $\beta = 0$, then $\alpha \neq 0$ and $\gamma \neq 0$, and we set $a = -\alpha^{-1}$, $b = \alpha^{-1}$ and $c = 0$. It remains to observe that the required conditions are satisfied.

Lemma 2.4. Let $X$ be a topological vector space, $x, y \in X$ and $f, g \in X'$ be such that $f(y) = g(x)$ and $f(x)g(y) \neq f(y)g(x)$. Then there exists $S \in GL(X)$ such that $Sx = y$ and $S'f = g$.

Proof. Let $\alpha = f(x)$, $\beta = g(x)$ and $\gamma = g(y)$. Since $f(x)g(y) \neq f(y)g(x) = g(x)^2$, $(\alpha, \beta) \neq (0, 0)$ and $(\beta, \gamma) \neq (0, 0)$. By Lemma 2.3, we can find $a, b, c \in \mathbb{K}$ such that

$$af(x) + bg(x) = -1, \quad bf(x) + cg(x) = 1 \quad \text{and} \quad ag(x) + bg(y) \neq 0. \quad (2.1)$$

Now we consider $S \in L(X)$ defined by the formula

$$Su = u + af(u)x + bf(u)y + bg(u)x + cg(u)y. \quad (2.2)$$

We shall demonstrate that $S$ satisfies all required conditions. It is easy to see that the dual operator $S'$ acts according to the formula

$$S'\varphi = \varphi + a\varphi(x)f + b\varphi(y)f + b\varphi(x)g + c\varphi(y)g. \quad (2.3)$$

Substituting $u = x$ into (2.2) and using the equalities in (2.1), we see that $Sx = y$. Substituting $\varphi = f$ into (2.3) and using the equalities in (2.1), we see that $S'f = g$. It remains to show that $S \in GL(X)$.

Since $f(x)g(y) \neq f(y)g(x)$, we see that $f, g$ are linearly independent, $x, y$ are linearly independent and $X = L \oplus N$, where $L = \ker f \cap \ker g$ and $N = \text{span}[x, y]$. Moreover, from (2.2) it follows that $L$ and $N$ are both invariant for $S$ and that $Su = u$ for each $u \in L$. Thus, in order to show that $S \in GL(X)$, it suffices to demonstrate that $S|_N$ is an invertible operator on the 2-dimensional space $N$. The matrix $M$ of $S|_N$ in the basis $[x, y]$ has shape

$$M = \begin{pmatrix} af(x) + bg(x) + 1 & af(y) + bg(y) \\ bf(x) + cg(x) & bf(y) + cg(y) + 1 \end{pmatrix}. \quad (2.4)$$

Using the equalities in (2.1) and $f(y) = g(x)$, we can rewrite the last matrix:

$$M = \begin{pmatrix} 0 & ag(x) + bg(y) \\ 1 & bg(x) + cg(y) + 1 \end{pmatrix}. \quad (2.5)$$

Hence $\det M = -(ag(x) + bg(y)) \neq 0$ according to (2.1). Thus $S|_N$ is an invertible operator on $N$ and therefore $S \in GL(X)$.

Proof of Theorem 5. Let $X$ be an infinite dimensional Banach space such that $X'$ is separable. By Proposition 1.4, there exists $T \in L(X)$ such that $T^{d\text{im}} \in L(X^m)$ and $T^{\text{im}} \in L((X')^m)$ are hypercyclic. Pick any $x \in X$ and $f \in X'$ such that $f(x) = 1$. Since the set of hypercyclic vectors of any hypercyclic operator is dense [1], we can find a hypercyclic vector $f_1, \ldots, f_m$ for $T^{d\text{im}}$ such that functionals $f, f_1, \ldots, f_m$ are linearly independent. Once again, using the density of the set of hypercyclic vectors of a hypercyclic operator, we can find a hypercyclic vector $y_1, \ldots, y_m$ for $T^{\text{im}}$ such that $f_j(y_j) \neq 0$ and $f_j(y_j) f_j(x) \neq f_j(x)f_j(y_j)$ for $1 \leq j \leq m$. Next, let $x_j = c_j y_j$, where $c_j = \frac{1}{f_j(y_j)}$. Since $c_j$ are non-zero constants, $(x_1, \ldots, x_n)$ is also a hypercyclic vector for $T^{\text{im}}$. Moreover, $f(x_j)f_j(x) \neq f(x)f_j(x)$ and $f(x) = f_j(x)$ for $1 \leq j \leq m$. By Lemma 2.4, we can find $S \in GL(X)$ such that $Sx_j = x$ and $S_j f = f_j$ for $1 \leq j \leq m$. By Lemma 2.1, $(R_1, \ldots, R_m) \in L(X)^m$ is disjoint hypercyclic, where $R_j = S_j T S_j^{-1}$. Since $(R_j')^{-1} T S_j' (S_j')^{-1} \in GL(X')$ and $(S_j')^{-1} f_j = f_j$, Lemma 2.1 implies that $(R_1', \ldots, R_m') \in L(X')$ is disjoint hypercyclic. That is, $(R_1, \ldots, R_m)$ is disjoint dual hypercyclic.

References