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Dynamical Methods for Polar Decomposition and Inversion of Matrices

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ABSTRACT

We show how to obtain polar decomposition as well as inversion of fixed and time-varying matrices using a class of nonlinear continuous-time dynamical systems. First we construct a dynamical system that causes an initial approximation of the inverse of a time-varying matrix to flow exponentially toward the true time-varying inverse. Using a time-parametrized homotopy from the identity matrix to a fixed matrix with unknown inverse, and applying our result on the inversion of time-varying matrices, we show how any positive definite fixed matrix may be dynamically inverted by a prescribed time without an initial guess at the inverse. We then construct a dynamical system that solves for the polar decomposition factors of a time-varying matrix given an initial approximation for the inverse of the positive definite symmetric part of the polar decomposition. As a by-product, this method gives another method of inverting time-varying matrices. Finally, using homotopy again, we show how

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dynamic polar decomposition may be applied to fixed matrices with the added benefit that this allows us to dynamically invert any fixed matrix by a prescribed time.

1. INTRODUCTION

In [1–3] we presented a continuous-time dynamic methodology, called *dynamic inversion*, for the solution of finite-dimensional time-dependent inverse problems of the form

$$F(\theta, t) = 0. \tag{1}$$

Dynamic inversion shows how to construct a dynamical system whose state θ is attracted asymptotically to a continuous isolated solution $\theta_*(t)$ of (1).

One may also pose functions of a time-varying matrix as solutions $\Gamma_*(t)$ of problems of the form $F(\Gamma, t) = 0$. For instance, if $A(t) \in \mathbb{R}^{n \times n}$ is invertible, then $\Gamma_*(t) = A(t)^{-1}$ is the unique solution to (1), where $F(\Gamma, t) := A(t)\Gamma - I$. Motivated by this realization, in the present paper we will investigate the use of dynamic inversion to construct dynamical systems that perform matrix inversion as well as polar decomposition.

1.1. *Previous Work*

Continuous-time dynamic methods of solving matrix equations have appeared previously in the literature. Any dynamical system on a matrix space with an asymptotically stable equilibrium may be considered to be a dynamic inverter that solves for its equilibrium. For example, continuous-time dynamic methods for determining eigenvalues date back at least as far as Rutishauer [4, 5]. Brockett [6, 7] has shown how one can use matrix differential equations to perform computation often thought of being intrinsically discrete. Bloch [8, 9] has shown how Hamiltonian systems may be used to solve principal-component and linear programming problems. Symes [10] and Chu [11] have studied the Toda flow as a continuous-time analog of the QR algorithm. Chu [12] and Chu and Drissel [13] have explored the use of differential equations in solving linear-algebra problems. Smith [14] and Helmke et al [15] have constructed dynamical systems that perform singular-value decomposition. Dynamic methods of matrix inversion have also appeared in the artificial-neural-network literature [16, 17]. For a review of some dynamic methods as well as a comprehensive list of references for dynamic approaches to optimization, see [18].

A dynamic decomposition related to polar decomposition of fixed matrices has appeared in Helmke and Moore [18], though, as the authors point out, their gradient-based method does not guarantee the positive definiteness of the symmetric component of the polar decomposition. We discuss this method further in Section 6. Using dynamic inversion, we will derive a system that produces the desired inverse and polar decomposition products at any fixed time $t_1 > 0$ with guaranteed positive definiteness of the symmetric component.

As far as we know, all prior continuous-time dynamic approaches to inversion of matrix equations use gradient flows. In contrast, dynamic inversion is not restricted to gradient methods.

1.2. *Main Results*

The main results of this paper are as follows: We will construct dynamical systems that

1. invert time-dependent matrices asymptotically,
2. invert constant matrices from a spectrally restricted set (including positive definite matrices) by a prescribed time,
3. invert and decompose any *time*-dependent invertible matrix into its polar decomposition factors,
4. invert and decompose any constant nonsingular matrix into its polar decomposition factors by a prescribed time.

Results 2 and 4 will be otained from results 1 and 3, respectively, using homotopy.

1.3. *Overview*

In Section 2 we give a brief review of the main relevant points of dynamic inversion. In Section 3 we examine the application of dynamic inversion to the problem of inverting time-varying matrices, assuming a good initial guess for the matrix inverse at time $t = 0$. In Section 4 we consider the problem of inverting constant (time-independent) matrices. By using a matrix homotopy from the identity we will use the results of Section 3 to produce exact inversion of a restricted class of constant matrices, including positive definite matrices, by a prescribed time. To remove the spectral limitations on the class of fixed matrices which we may invert in finite time, in Section 5 we will consider the polar decomposition of a time-varying matrix. We will show how, starting from a good guess at the initial value of the inverse of the positive definite part of the polar decomposition, we may construct a dynamical system that produces an estimate that exponentially converges to the inverse of the positive definite symmetric part. From this estimate and the original matrix we will obtain the decomposition products as well as the inverse through multiplication. In Section 6 we will revisit the problem of constant matrix inversion and show how, combining homotopy with dynamic polar decomposition, we may dynamically produce the polar decomposition factors as well as the inverse of *any* constant matrix by a prescribed time *without* requiring an initial guess.

Our objective in this paper is to present a methodology for the construction of analog computational paradigms for solving inverse problems. We leave open the issue of how dynamic inverters may best be realized by physical computing systems, and so we will avoid discussion of numerical considerations such as condition number and numerical stability. However, since our dynamic inverters are *stable* integrators, the primary issue to be faced in their implementation is the realization of integration.

1.4. Notation

Here, for easy reference, we define some of the notation used in the sequel.

We are concerned with problems of the following form: Given a time-dependent map $F(\Gamma, t)$, find $\Gamma_*(t)$ satisfying $F(\Gamma_*(t), t) = 0$ for all $t \geqslant 0$. Thus $\Gamma_*(t)$, which we sometimes refer to as Γ_* for brevity, denotes the *exact* solution of the inverse problem. We will use Γ to denote the first argument of F, as well as an estimator for $\Gamma_*(t)$.

$\mathbb{R}_+$: The set from which we draw our values of time, t, is $\mathbb{R}_+ := \{t \in \mathbb{R} \mid t \geqslant 0\}$.

$\underline{k}$: For any integer $k \geqslant 1$, let $\underline{k}$ denote the set of integers $\{1, 2, \dots, k\}$.

$L(A, B)$: For vector spaces A and B, $L,(A, B)$ is the set of all linear maps from A to B.

$\mathrm{GL}(n)$: The group of nonsingular $n \times n$ matrices having real-valued entries, $\{M \in \mathbb{R}^{n \times n} \mid \det M \neq 0\}$.

$\mathrm{O}(n)$: The group of orthogonal $n \times n$ matrices having real-valued entries, $\{M \in \mathbb{R}^{n \times n} \mid M^T M = I\}$.

$\mathrm{S}(n)$: The vector of space of symmetric $n \times n$ matrices having real-valued entries, $\{M \in \mathbb{R}^{n \times n} \mid M^T = M\}$.

$s(n)$: The dimension of $\mathrm{S}(n)$, i.e. $s(n) := \frac{1}{2}n(n+1)$.

$[f(M_{ij})]_{i, j \in \underline{n}}$: The $n \times n$ matrix with $f(M_{ij})$ in row i and column j.

$\|x\|_2$: The 2-norm of x, $\|x\|_2 := \sqrt{x^T x}$.

$\sigma(M)$: The *spectrum* of $M \in \mathrm{GL}(n)$, i.e. the set of eignvalues of M.

M^R, M^L: If $M \in \mathbb{R}^{m \times n}$, $m \leqslant n$, is full-rank, then $M^R := M^T(MM^T)^{-1} \in \mathbb{R}^{n \times m}$ is the *right inverse* of M. Note that $MM^R = I \in \mathbb{R}^{m \times m}$. If $M \in \mathbb{R}^{m \times n}$, $m \geqslant n$, is full-rank, then $M^L := (M^T M)^{-1} M^T \in \mathbb{R}^{n \times m}$ is the *left inverse* of M, and $M^L M = I \in \mathbb{R}^{m \times m}$.

$D_k F(a^1, a^2, \dots, a^n)$: For any map $F(a^1, a^2, \dots, a^n)$, the partial derivative of F with respect to a^k.

$\mathscr{B}_r$: For each dimension n we define the open ball $\mathscr{B}_r := \{x \in \mathbb{R}^n : \|x\| < r\}$. The choice of a particular norm $\|\cdot\|$ will be apparent from context. In order to emphasize the dimension of $\mathscr{B}_r$ we will often specify the set having the same dimension as $\mathscr{B}_r$ for which $\mathscr{B}_r$ is a subset, e.g. $\mathscr{B}_r \subset \mathbb{R}^n$.

2. DYNAMIC INVERSION

Given an inverse problem $F(\theta, t) = 0$, dynamic inversion specifies how one can construct a system of nonlinear ordinary differential equations whose solution $\theta(t)$ converges asymptotically to a continuous isolated solution $\theta_*(t)$ of the inverse problem.

A key element of dynamic inversion is the notion of a dynamic inverse $G[w, \theta, t]$ of a nonlinear map $F(\theta, t)$. The dynamic inverse is nonunique, and is defined in terms of the *unknown* root of a map F.

DEFINITION 2.1. For $F : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$, $(\theta, t) \mapsto F(\theta, t)$, let $\theta_*(t)$ be a continuous isolated solution of $F(\theta, t) = 0$. A map $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$, $(w, \theta, t) \mapsto G[w, \theta, t]$ is called a *dynamic inverse* of F on the ball $\mathscr{B}_r := \{z \in \mathbb{R}^n \mid \|z\| \leqslant r\}$, $r > 0$, if

1. $G[0, z + \theta_*(t), t] = 0$ for all $t \geqslant 0$ and $z \in \mathscr{B}_r$,
2. the map $G[F(\theta, t), \theta, t]$ is Lipschitz in θ and piecewise continuous in t, and
3. there is a real constant β with $0 < \beta < \infty$, such that

(*dynamic inverse criterion*)

$$z^T G[F(z + 0_*(t), t), z + \theta_*(t), t] \geqslant \beta \|z\|_2^2 \tag{2}$$

for all $z \in \mathscr{B}_r$.

As shown in [2, 1], if $D_1 F(\theta_*(t), t)^{-1}$ exists, then any matrix $M(\theta, t)$ such that $M(\theta, t) D_1 F(\theta_*(t), t)$ is positive definite may be used to form a dynamic inverse $G[w, \theta, t] := M(\theta, t) \cdot w$ of F. Examples of such $M(\theta, t)$ include $D_1 F(\theta, t)^T$ and $D_1 F(\theta, t)^{-1}$ for θ sufficiently close to $\theta_*(t)$.

The key dynamic inversion theorem [2, Theorem 3.5] which we will rely upon in the sequel is as follows.

THEOREM 2.2 (Dynamic inversion theorem—vanishing error). *Let $\theta_*(t)$ be a continuous isolated solution of $F(\theta, t) = 0$, with $F : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$, $(\theta, t) \mapsto F(\theta, t)$. Assume that $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$, $(w, \theta, t) \mapsto G[w, \theta, t]$, is a dynamic inverse of $F(\theta, t)$ for all θ satisfying $\theta - \theta_*(t) \in \mathscr{B}_r$, and for some finite $\beta > 0$. Let $E : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$, $(\theta, t) \mapsto E(\theta, t)$ be locally Lipschitz in θ and continuous in t. Assume that for some constant $\kappa \in (0, \infty)$, $E(\theta, t)$ satisfies*

$$\left\| E(z + \theta_*(t), t) - \dot{\theta}_*(t) \right\|_2 \leqslant \kappa \|z\|_2 \tag{3}$$

for all $z \in \mathscr{B}_r$. Let $\theta(t)$ denote the solution to the dynamical system

$$(\text{dynamic inverter}) \qquad \dot{\theta} = -\mu G[F(\theta, t), \theta, t] + E(\theta, t) \tag{4}$$

with initial condition $\theta(0)$ satisfying $\theta(0) - \theta_*(0) \in \mathscr{B}_r$. Then

$$\| \theta(t) - \theta_*(t) \|_2 \leqslant \| \theta(0) - \theta_*(0) \|_2 e^{-(\mu\beta - \kappa)t} \tag{5}$$

for all $t \in \mathbb{R}_+$, and in particular, if $\mu > \kappa/\beta$, then $\theta(t)$ converges to $\theta_*(t)$ exponentially as $t \to \infty$.

2.1. *Constructing a Derivative Estimator*

The map $E(\theta, t)$ may be regarded as an estimator for $\dot{\theta}_*$. A straightforward method of obtaining such an estimator is by differentiating $F(\theta_*, t) = 0$ with respect to t,

$$D_1 F(\theta_*, t) \dot{\theta}_* + D_2 F(\theta, t) = 0 \tag{6}$$

solving for $\dot{\theta}_*$,

$$\dot{\theta}_* = -D_1 F(\theta_*, t)^{-1} D_2 F(\theta_*, t) \tag{7}$$

and replacing θ_* by its estimator θ to get

$$E(\theta, t) := -D_1 F(\theta, t)^{-1} D_2 F(\theta, t). \tag{8}$$

As illustrated in the next section, if an asymptotic estimator Γ of $D_1 F(\theta_*, t)^{-1}$ is available, then for Γ sufficiently close to $D_1 F(\theta_*, t)^{-1}$ we may instead use

$$E(\Gamma, \theta, t) := -\Gamma D_2(\theta, t). \tag{9}$$

3. INVERTING TIME-VARYING MATRICES

Consider the problem of estimating the inverse $\Gamma_*(t) \in \mathbb{R}^{n\times n}$ of a time-varying matrix $A(t) \in GL(n)$. Assume that we have representations for both $A(t)$ and $\dot{A}(t)$, and that $A(t)$ is C^1 in t. Let Γ be an element of $\mathbb{R}^{n\times n}$, and let $F: \mathbb{R}^{n\times n} \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, $(\Gamma, t) \mapsto F(\Gamma, t)$ be defined by

$$F(\Gamma, t) := A(t)\Gamma - I. \tag{10}$$

For $\Gamma_*(t)$ to be the inverse of $A(t)$, $\Gamma_*(t)$ must be a solution of $A(t)\Gamma - I = 0$.

An estimator $E(\Gamma, t)$ for $\dot{\Gamma}_*(t)$ is given by (9) where we replace θ by Γ:

$$E(\Gamma, t) := -\Gamma\dot{A}(t)\Gamma. \tag{11}$$

Differentiate $F(\Gamma, t)$ with respect to Γ to get

$$D_1F(\Gamma, t) = A(t), \tag{12}$$

whose inverse is Γ_*. Thus a choice of dynamic inverse is

$$G[W, \Gamma] := \Gamma W \tag{13}$$

for Γ sufficiently close to $\Gamma_* = A^{-1}(t)$ and for $W \in \mathbb{R}^{n\times n}$. The dynamic inverter for this problem then takes the form

$$\dot{\Gamma} = -\mu G[F(\Gamma, t), \Gamma] + E(\Gamma, t); \tag{14}$$

i.e., according to (10), (11), and (13),

$$\dot{\Gamma} = -\mu\Gamma(A(t)\Gamma - I) - \Gamma\dot{A}(t)\Gamma \tag{15}$$

and we choose as initial conditions $\Gamma(0) \approx \Gamma_*(0) = A^{-1}(0)$. Theorem 2.2 guarantees that for sufficiently large μ, and for $\Gamma(0)$ sufficiently close to $A^{-1}(0)$, Equation (15) will produce an estimator $\Gamma(t)$ whose error $\Gamma(t) - \Gamma_*(t)$ decays exponentially to zero at a rate determined by our choice of μ. We summarize the observations above in the following consequence of Theorem 2.2.

THEOREM 3.1 (Dynamic inversion of time-varying matrices). *Let $A(t) \in GL(n)$ be C^1 in t, with $A(t)$, $A(t)^{-1}$, and $\dot{A}(t)$ bounded on $[0, \infty)$. Let $G[W, \Gamma, t]$ be a dynamic inverse (see Definition 2.1) of $F(\Gamma, t) = A(t)\Gamma - I$ for all $t \in \mathbb{R}_+$, and for all Γ such that $\Gamma - \Gamma_*$ is in $\mathscr{B}_r$. Let $\Gamma(t) \in \mathbb{R}^{n\times n}$ be the solution to*

$$\dot{\Gamma} = -\mu G[A(t)\Gamma - I, \Gamma, t] - \Gamma \dot{A}(t)\Gamma \tag{16}$$

with $\|\Gamma(0) - \Gamma_*(0)\| \leqslant r < \infty$. Then for sufficiently small r, there exists a $\tilde{\mu} > 0$, $k_1 > 0$, and $k_2 > 0$ such that for all $\mu > \tilde{\mu}$, and for all $t \geqslant 0$,

$$\|\Gamma(t) - \Gamma_*(t)\|_2 \leqslant k_1 \|\Gamma(0) - \Gamma_*(0)\| e^{-k_2 t}. \tag{17}$$

In particular $\lim_{t\to\infty}\Gamma(t) = A(t)^{-1}$.

EXAMPLE 3.2 (A dynamic inverter for a time-varying matrix). Let

$$G[W, t] := A(t)^T W. \tag{18}$$

By Theorem 3.1, for sufficiently large constant $\mu > 0$, and for $\Gamma(0)$ sufficiently close to $A(0)^{-1}$, the solution $\Gamma(t)$ of

$$\dot{\Gamma} = -\mu A(t)^T (A(t)\Gamma - I) - \Gamma \dot{A}(t)\Gamma \tag{19}$$

approaches $A(t)^{-1}$ exponentially as $t \to \infty$.

See also (15), where the dynamic inverse $G[W, \Gamma] = \Gamma W$ is used instead of $G[W, t] = A(t)^T W$.

EXAMPLE 3.3 (Dynamic inversion of a mass matrix). Consider a finite-dimensional mechanical system modeled by the second-order differential equation

$$M(q)\ddot{q} + N(q, \dot{q}) = 0. \tag{20}$$

Usually the matrix $M(q)$ is positive definite and symmetric for all q, since the kinetic energy, $\frac{1}{2}\dot{q}^T M(q)\dot{q}$, is normally greater than zero for all $\dot{q} \neq 0$. It is often convenient to express such systems in an explicit form, with $\ddot{q}$ alone on the left side of a second-order ordinary differential equation. To do so, we will invert $M(q)$ dynamically.

Let Γ be a symmetric estimator for $M(q)^{-1}$. Suppose we know $M^{-1}(q(0))$ approximately. If our approximation is sufficiently close to the true value of $M^{-1}(q(0))$, then setting $\Gamma(0)$ to that approximation and letting $\mu > 0$ be sufficiently large allows us to apply Theorem 3.1. Then the system

$$\begin{aligned} \dot{\Gamma} &= -\mu\Gamma(M(q)\Gamma - I) - \Gamma\left[\frac{\partial M_{i,j}(q)}{\partial q}\dot{q}\right]_{i,j\in \underline{n}} \cdot \Gamma, \\ \ddot{q} &= \Gamma N(q,\dot{q}) \end{aligned} \tag{21}$$

provides an exponentially convergent estimate of $\ddot{q}$ for all t. Furthermore, if $\Gamma(0) = M(q(0))^{-1}$, then $\Gamma(t) = M^{-1}(q(t))$ for all $t \geqslant 0$.

REMARK 3.4 (Symmetry and the choice of dynamic inverse). In Example 3.3, $M(q)$ is symmetric, as is its inverse $M(q)^{-1}$. The right-hand side of (21) is also symmetric; hence if $\Gamma(0)$ is symmetric, so is $\Gamma(t)$ for all t. If we had chosen $G[W, q] := M(q)^T W$ as a dynamic inverse (see, for instance, Example 3.2), we would not have had this symmetry. The symmetry allows us to cast the top equation of (21) on the space $S(n)$ of symmetric $n \times n$ matrices, thereby reducing the complexity of the dynamic inverter; what would otherwise be n^2 equations (21) is reduced to $s(n) := n(n+1)/2$ equations.

3.1. *Left and Right Inversion of Time-Varying Matrices*

Consider a matrix $A(t) \in \mathbb{R}^{m\times n}$. Assume that $A(t)$ is of full rank for all $t \geqslant 0$. We consider two cases:

(1) If $m \leqslant n$, then $A(t)$ has a right inverse $\Gamma_*(t) \in \mathbb{R}^{n\times m}$ satisfying

$$F(\Gamma, t) := A(t)\Gamma - I = 0. \tag{22}$$

It is easily verified that

$$G[W] := \Gamma W. \tag{23}$$

is a dynamic inverse for $F(\Gamma, t)$ when Γ is sufficiently close to $\Gamma_* = A(t)^T(A(t)A(t)^T)^{-1}$. Differentiate $F(\Gamma_*, t) = 0$ with respect to t, solve for $\dot{\Gamma}_*$, and replace Γ_* by Γ to get the derivative estimator

$$E(\Gamma, t) := -\Gamma\dot{A}(t)\Gamma. \tag{24}$$

Thus a *dynamic inverter for right inversion of a time-varying matrix* is

$$\dot{\Gamma} = -\mu\Gamma(A(t)\Gamma - I) - \Gamma\dot{A}(t)\Gamma. \tag{25}$$

The form of this dynamic inverter is seen to be identical to (15). Alternatively we may use Theorem 3.1 to invert $A(t)A(t)^T$, constructing the right inverse as $A(t)^T\Gamma(t)$.

(2) In the case that $m \geqslant n$, $A(t)$ has a left inverse $\Gamma_*(t)$ which satisfies

$$F(\Gamma, t) := \Gamma A(t) - I = 0 \tag{26}$$

We may use the dynamic inverter (25) with $A(t)$ replaced by $A(t)^T$, and $\dot{A}(t)$ replaced by $\dot{A}(t)^T$, to approximate the left inverse of $A(t)$.

4. INVERSION OF CONSTANT MATRICES

In this section we consider two methods for the dynamic inversion of constant (time-independent) matrices: one for asymptotic inversion, and the other for inversion in finite time. In Section 6 we consider another more complex, but also more general approach to the same problem.

Constant matrices may be inverted in a manner similar to the inversion of time-varying matrices as described in the last section. Let

$$F(\Gamma) := M\Gamma - I. \tag{27}$$

Let $\Gamma(t)$ denote the estimator for the inverse of a constant matrix M, with $\Gamma_* = M^{-1}$ as the solution of $F(\Gamma) = 0$. Since M is constant, $\dot{\Gamma}_*$ is zero. As a consequence, if $\Gamma(0)$ is sufficiently close to Γ_*, then a dynamic inverse of $F(\Gamma)$ is $G[W, \Gamma] := \Gamma W$, and we can use the dynamic inverter

$$(\textit{dynamic inverter for constant invertible matrices}) \quad \dot{\Gamma} = -\mu\Gamma(M\Gamma - I). \tag{28}$$

Choosing $\Gamma(0)$ sufficiently close to Γ_* assures us that as $t \to \infty$, $\Gamma(t)$ flows to $\Gamma_* = M^{-1}$, and Γ will not intersect the set of singular matrices.

4.1. A Comment on Gradient Methods

As shown in Example 3.2, the function $G[W, \Gamma] := \Gamma W$ is not our only choice of a dynamic inverse $G[W, \Gamma, t]$ which is linear in W. It is easily verified that $G[W] = M^T W$, $W \in \mathbb{R}^{n\times n}$, is also a dynamic inverse for $F(\Gamma) := M\Gamma - I$, and that for this choice of dynamic inverse we do not need to worry about the dynamic inverse becoming singular; it is valid under any choice of initial condition and leads to the dynamic inverter

$$(\textit{gradient dynamic inverter for matrices}) \quad \dot{\Gamma} = -\mu M^T(M\Gamma - I), \tag{29}$$

for which $\Gamma \to M^{-1}$ as $t \to \infty$.

REMARK 4.1 (Left and right inverses of constant matrices). If M has full row rank, with $M \in \mathbb{R}^{m\times n}$, $m \leqslant n$, then the equilibrium solution Γ_* of (29) is the right inverse $M^R := M^T(MM^T)^{-1}$ of M. If instead we choose $F(\Gamma) := \Gamma M - I$ and $G[W] := WM^T$, and if $M \in \mathbb{R}^{m\times n}$, $m \geqslant n$, has full column rank, then the solution Γ_* to the resulting dynamic inverter

$$(\textit{dynamic left-inverter for constant matrices}) \qquad \dot{\Gamma} = -\mu(\Gamma M - I)M^T \tag{30}$$

converges to the left inverse $M^L := (M^T M)^{-1} M^T$ of M as $t \to \infty$.

The dynamic inverter (29) is the standard least-squares gradient flow (see [18, section 1.6]) for the function $\Phi : \mathbb{R}^n \to \mathbb{R}$, $\Gamma \mapsto \Phi(\Gamma)$, where

$$\Phi(\Gamma) := \tfrac{1}{2}\| M\Gamma - I \|_2^2. \tag{31}$$

It is also the neural-network constant matrix inverter of Wang [17]. Of course other gradient schemes may have the same solution as (29), though they may start from gradients of functions other than (31) (see, for instance, [16]). In general, artificial neural networks are constructed to dynamically solve for the minimum of an energy function having a unique (at least locally) minimum, i.e., they realize gradient flows.

4.1.1. Connecting Gradient Methods with Dynamic Inversion. In general a dynamic inverter (4) is made up of a given F and choices of G, E, and μ as described in Section 2. The function $F(\Gamma, t)$ is the implicit function to be inverted, $G[w, \theta, t]$ is a dynamic inverse for $F(\Gamma, t)$, and $E(\theta, t)$ is an estimator for the derivative with respect to t of the root Γ_* of $F(\Gamma, t) = 0$. To relate gradient methods to dynamic inversion, we consider the decomposition of a gradient flow system into an E, F, G, and μ making up a dynamic inverter. For instance, let $H : \mathbb{R}^{n \times n} \times \mathbb{R} \to \mathbb{R}$ be a smooth function. A gradient system for finding critical points of H with respect to Γ is

$$(\textit{gradient system}) \qquad \dot{\Gamma} = -\nabla H(\Gamma, t) + \frac{\partial}{\partial t} H(\Gamma, t), \tag{32}$$

where ∇ denotes the Γ-gradient of $H(\Gamma, t)$. We may always identify gradient systems with dynamic inversion through the trivial dynamic inverse $G[W] = W$. Then $F(\Gamma, t) = \nabla H(\Gamma, t)$ and $E(\Gamma, t) = (\partial/\partial t) H(\Gamma, t)$. Let $\mu = 1$. Then

$$\dot{\Gamma} = -G[F(\Gamma, t)] + E(\Gamma, t) \tag{33}$$

is the same as (32). Thus we have identified the gradient system (32) with a dynamic inverter.

It is more interesting, however, to find a dynamic inverse G such that if G were changed to the identity map, then the desired root would still be the solution to $F(\Gamma, t) = 0$, but the resulting dynamic inverter would not converge to the desired root. For example, identifying $F(\Gamma) = M\Gamma - I$, $G[W] = M^T W$, and $E = 0$ decomposes the gradient flow (29) into a dynamic inverter. For arbitrary $M \in \mathrm{GL}(n)$, the stability properties of $\dot{\Gamma} = -\mu F(\Gamma)$ are unknown. But with G defined as $G[W] = M^T W$, $\dot{\Gamma} = -\mu G[F(\Gamma)]$ has an asymptotically stable equilibrium at $\Gamma_* = M^{-1}$. For a system of the form (29) such a decomposition is straightforward. For more complicated gradient systems, however, we have no general methodology for decomposition into E, F, and G.

4.2. *Dynamic Inversion of Constant Matrices by a Prescribed Time*

The constant-matrix dynamic inverters (28) and (29) above have the potential disadvantage of producing an exact inverse only asymptotically as $t \to \infty$. One may, however, wish to obtain the inverse by a prescribed time. To this end we now consider another method. If we could create a time-varying matrix $H(t)$ that is invertible by inspection at $t = 0$, and that equals M at some known finite time $t > 0$, say $t = 1$, then perhaps we could use the technique of Section 3 for the inversion of time-varying matrices to invert $H(t)$. If $\Gamma(0) = H(0)^{-1}$, then the solution of the dynamic inverter at time

$t = 1$ will be M^{-1}. We require, of course, that $H(t)$ remain in GL(n) as t goes from 0 to 1. One ideal candidate for the inital value of the time-varying matrix is the identity matrix I, since it is its own inverse.

EXAMPLE 4.2 (Constant-matrix inversion by a prescribed time using homotopy). Let M be a constant matrix in $\mathbb{R}^{n \times n}$. Consider the t-dependent matrix

$$(\textit{matrix homotopy}) \qquad H(t) = (1-t)I + tM. \tag{34}$$

In the space of $n \times n$ matrices, $t \mapsto H(t)$ describes a t-parametrized curve, or *homotopy*, of matrices from the identity to $M = H(1)$ as indicated in Figure 1; in fact this curve (34) is a straight line. From Theorem 3.1 we know how to dynamically invert a time-varying matrix given that we have an approximation of its inverse at time $t = 0$. Since we know the exact inverse at time $t = 0$, we may use the dynamic inverter of Theorem 3.1 to track the exact inverse of the time-varying matrix for all $t \geqslant 0$. We may invert $H(t)$ by substituting $H(t)$ for $A(t)$, and $\dot{H}(t) = M - I$ for $\dot{A}(t)$, in (16), setting $\Gamma(0) = I$. Since our initial conditions are a precise inverse of $H(0)$, Theorem 3.1 tells us that the matrix Γ at $t = 1$ is the precise inverse of M, as shown schematically in Figure 2—that is, of course, if $H(t)$ remains nonsingular as t goes from 0 to 1. If $H(t)$ should become singular for any $\bar{t} \in [0, 1]$, then linear mappings such as $W \mapsto \Gamma \cdot W$ and $W \mapsto H(t)^T \cdot W$ fail to be dynamic inverses of $F(\Gamma, t) = H(t)\Gamma - I$ at $\bar{t}$.

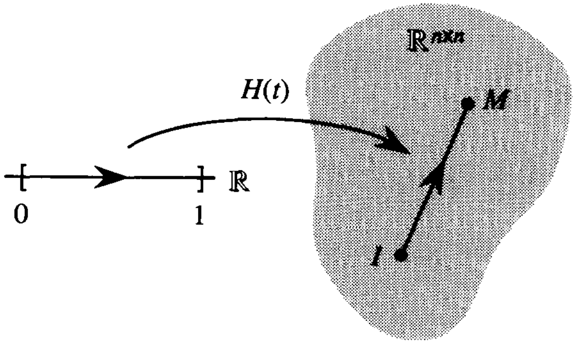

FIG. 1. The matrix homotopy $H(t)$.

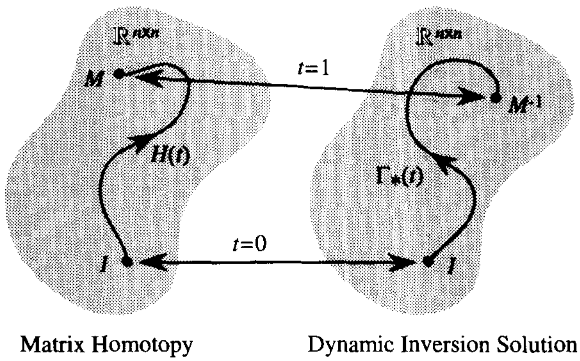

FIG. 2. The matrix homotopy $H(t)$ from I to M with the corresponding solution $\Gamma_*(t)$, the inverse of $H(t)$.

For a dynamic inverter for this example let

$$\begin{aligned} F(\Gamma, t) &:= ((1-t)I + tM)\Gamma - I, \\ G[W, \Gamma] &:= \Gamma W, \\ E(\Gamma) &:= -\Gamma(M-1)\Gamma. \end{aligned} \tag{35}$$

Then a dynamic inverter is $\dot{\Gamma} = -\mu G[F(\Gamma, t), \Gamma] + E(\Gamma)$ with $\Gamma(0) = I$. Expanded, this is

$$\dot{\Gamma} + -\mu\Gamma(((1-t)I + tM)\Gamma - I) - \Gamma(M - I)\Gamma. \tag{36}$$

Another choice of linear dynamic inverse is $G[W, t] := ((1-t)I + tM)^T W$, giving

$$\dot{\Gamma} = -\mu H(t)^T (H(t)\Gamma - I) - \Gamma(M - I)\Gamma \tag{37}$$

as an alternative choice of prescribed-time dynamic inverter for constant matrices.

Homotopy-based methods, also called *continuation methods*, for solving sets of linear and nonlinear equations have been around for quite some time.

For a review of developments prior to 1980 see Allgower and Georg [19]. The general idea is that one starts with a problem with a known solution (e.g. the inverse of the identity matrix) and smoothly transform that problem to a problem with an unknown solution, transforming the known solution in a corresponding manner until the unknown solution is reached. Often it is considerably easier to transform a known solution to a problem into an unknown solution to a closely related problem than to calculate the new solution from scratch. Solution of the roots of nonlinear polynomial equations (see Dunyak et al. [20] and Watson [21] for examples) is a typical example with broad engineering application.

Now we deal with the fact that the scheme of Example 4.2 requires that there be no $t \in [0, 1]$ for which $H(t)$ given by (34) is singular. To do this recall that there are two maximal connected open subsets which constitute $\mathrm{GL}(n)$, namely $\mathrm{GL}^+(n) = \{M \in \mathbb{R}^{n\times n} \mid \det M > 0\}$ and $\mathrm{GL}^-(n) = \{M \in \mathbb{R}^{n\times n} \mid \det M < 0\}$. These two sets are disjoint and are separated by the variety of singular $n \times n$ matrices $\{M \in \mathbb{R}^{n\times n} \mid \det M = 0\}$. The identity I is in $\mathrm{GL}^+(n)$. For the curve $t \mapsto H(t)$ to be invertible, it must never leave $\mathrm{GL}^+(n)$ (see Figure 3). For our particular choice of $H(t)$, since $H(0) = I$, and I is in $\mathrm{GL}^+(n)$, the homotopy $H(t)$ should remain in $\mathrm{GL}^+(n)$ to be invertible for all $t \in [0, 1]$. The following lemma specifies sufficient conditions on M for $H(t)$ (34) to remain in $\mathrm{GL}^+(n)$ as t goes from 0 to 1.

LEMMA 4.3 (Matrix homotopy lemma). *If $M \in \mathrm{GL}(n)$ has no eigenvalues in $(-\infty, 0)$, then for each $t \in [0, 1]$, $H(t) = (1 - t)I + tM$ is in* $\mathrm{GL}(n)$.

REMARK 4.4 (Inversion of positive definite symmetric constant matrices). If M is a positive definite symmetric matrix, then the assumption of Lemma 4.3 holds.

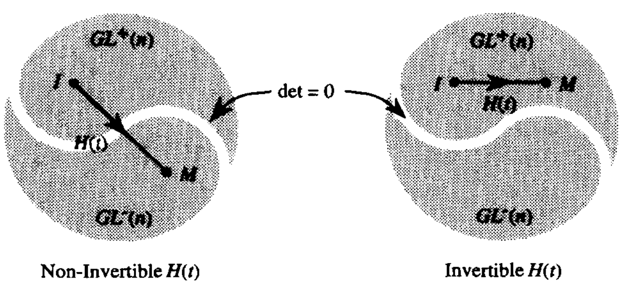

FIG. 3. The homotopy from I to M must remain in $\mathrm{GL}^+(n)$ to be invertible.

REMARK 4.5 (Subset starlike about I). Let $\sigma(M)$ be the spectrum of M. Lemma 4.3 tell us that the subset of $\mathrm{GL}(n)$ consisting of all $M \subset \mathrm{GL}(n)$ such that $\sigma(M) \cap (-\infty, 0) = \varnothing$ is starlike about I, i.e., for each M in this subset, the straight line segment from I to M remains in the subset.

Proof of Lemma 4.3. Suppose that $H(t) = (1-t)I + tM$ is singular for some $\bar{t} \in [0,1]$. The identity I is nonsingular, as is M by assumption, so $\bar{t} \notin \{0,1\}$. Thus there exists a nonzero $v \in \mathbb{R}^n$ such that

$$((1-\bar{t})I + \bar{t}M)v = 0 \tag{38}$$

Since $\bar{t} \neq 0$, we can divide (38) by $-\bar{t}$ to obtain

$$\left(\frac{\bar{t}-1}{\bar{t}} I - M\right)v = 0 \tag{39}$$

But $\bar{t}$ can only satisfy (39) if $\lambda(\bar{t}) := (\bar{t}-1)/\bar{t}$ is an eigenvalue of M. As t ranges over $(0,1)$, $\lambda(t)$ ranges over $(-\infty, 0)$. But by assumption M has no eigenvalues in $(-\infty, 0)$; hence no such $\bar{t}$ exists in $(0,1)$, and so $H(t)$ is nonsingular on $[0,1]$. ■

We may obtain the exact inverse of M at *any* prescribed time $t_1 > 0$ by replacing $H(t)$ with $H(t/t_1)$ in (36) or (37). We summarize our results of this section in the following theorem.

THEOREM 4.6 (Dynamic inversion of constant matrices by a prescribed time). *For any constant $M \in \mathrm{GL}(n)$, and for any prescribed $t_1 > 0$, if $\sigma(M) \cap (-\infty, 0) = \varnothing$, then the solution $\Gamma(t)$ of the dynamic inverter*

(*prescribed-time dynamic inverter for constant matrices*)

$$\dot{\Gamma} = -\mu\Gamma\left(\left(\left(1 - \frac{t}{t_1}\right)I + \frac{t}{t_1}M\right)\Gamma - 1\right) - \Gamma(M - I)\Gamma, \tag{40}$$

with $\Gamma(0) = I$, satisfies $\Gamma(t_1) = M^{-1}$.

REMARK 4.7 (Preservation of symmetry). If M is symmetric, then the right-hand side of (40) is also symmetric. Thus if $\Gamma(0)$ is symmetric, then $\Gamma(t)$ is symmetric for all $t \in [0, t_1]$. Note that this symmetry need not hold for (37), where $G[W,t] = H(t)^T W$ is used as the dynamic inverse.

EXAMPLE 4.8 (Right and left inverses of constant matrices by a prescribed time). Let $A \in \mathbb{R}^{m \times n}$ be a constant matrix with $m \leqslant n$, and assume that A has full rank. The right inverse of A is given by $A^R := A^T(AA^T)^{-1}$. To obtain A^R at time t_1, we may apply Theorem 4.6, replacing M by AA^T, which is positive definite. Then $A^T(AA^T)^{-1} = A^T\Gamma(t_1)$, and we have

(*prescribed-time dynamic right inversion of a constant matrix*)

$$\dot{\Gamma} = -\mu\Gamma\left(\left(\left(1 - \frac{t}{t_1}\right)I + \frac{t}{t_1}AA^T\right)\Gamma - I\right) - \Gamma(AA^T - I),$$
$$A^T\Gamma(t_1) = A^R. \tag{41}$$

If a constant A has full column rank, then since A^TA is positive definite, the left inverse $A^L := (A^TA)^{-1}A^T$ may be obtained by substituting A^TA for M in Theorem 4.6. Then $A^L = \Gamma(t_1)A^T$.

Theorem 4.6 is limited in its utility by the necessity that M have a spectrum which does not intersect $(-\infty, 0)$. By appealing to the polar decomposition in Section 6 below, we will show that we may, at the cost of a slight increase in complexity, use dynamic inversion to produce an exact inverse of *any* invertible constant M, irrespective of its spectrum, by any prescribed time $t_1 \geqslant 0$.

5. POLAR DECOMPOSITION FOR TIME-VARYING MATRICES

In this section we will show how dynamic inversion may be used to perfom polar decomposition [22] and inversion of a time-varying matrix. We will assume that $A(t) \in \mathrm{GL}(n)$, and that $A(t)$, $\dot{A}(t)$, and $A(t)^{-1}$ are bounded for $t \in \mathbb{R}_+$.

Though polar decomposition will be used here largely as a path to inversion, polar decomposition finds substantial utility in its own right. In particular it is used widely in the study of stress and strain in continuous media. See, for instance, Marsden and Hughes [23].

First consider the polar decomposition of a *constant* matrix $M \in \mathrm{GL}(n)$, $M = PU$, where U is in the space of $n \times n$ orthogonal matrices with real entries, $\mathrm{O}(n)$, and P is the symmetric positive definite square root of MM^T. Regarding M as a linear operator $\mathbb{R}^n \to \mathbb{R}^n$, the polar decomposition expresses the action of M on a vector as a rotation (possibly with a reflection) followed by a scaling along the eigenvectors of MM^T. If $M \in \mathrm{GL}(n)$, then P and U are unique.

Now consider the case of a t-dependent nonsingular square matrix $A(t)$. Since $A(t)$ is nonsingular for all $t \geqslant 0$, $A(t)A(t)^T$ is positive definite for all $t \geqslant 0$. For any $t \geqslant 0$, the unique positive definite solution to $XA(t)A(t)^T X - I = 0$ is $X_*(t) = P(t)^{-1}$. Thus if we know $X_*(t)$, then from $A(t) = P(t)U(t)$ we can get the orthogonal factor $U(t)$ of the polar decomposition by $U(t) = X_*(t)A(t)$, as well as the symmetric positive definite part $P(t) = X_*(t)A(t)A(t)^T$. We can also obtain the inverse of $A(t)$ as $A(t)^{-1} = U(t)^T X_*(t)$.

Since $P(t)$ is a symmetric $n \times n$ matrix, it is parametrized by $s(n) := n(n+1)/2$ elements, as is its inverse $P^{-1}(t)$. We will construct the dynamic inverter that produces $P^{-1}(t)$.

Let

$$\Lambda(t) := A(t)A(t)^T. \tag{42}$$

Let $F : \mathrm{S}(n) \times \mathbb{R}_+ \to \mathrm{S}(n)$, $(X, t) \mapsto F(X, t)$ be defined by

$$F(X,t) := X\Lambda(t)X - I. \tag{43}$$

Let X_* be a solution of $F(X, t) = 0$. Then $X_*(t)$ is a symmetric square root of $\Lambda(t)$.

Nothing in the form of $F(X, t)$ enforces the *positive definiteness* of the solution $X_*(t)$. For instance, for each solution $X_*(t)$ of $F(x, t) = 0$, $-X_*(t)$ is also a solution. Each solution $t \mapsto X_*(t)$ is, however, *isolated* as long as $Y \mapsto D_1F(X_*, t) \cdot Y$, where $F(X, t)$ is defined by (43), is nonsingular. We will show in the next subsection, Section 5.1, that the nonsingularity of $A(t)$ implies the nonsingularity of $Y \mapsto D_1F(X_*, t) \cdot Y$.

5.1. *The Lyapunov Map*

We will use a linear dynamic inverse for $F(X, t)$ in (43) based upon the inverse of the linear map from $\mathrm{S}(n)$ to $\mathrm{S}(n)$, $Y \mapsto D_1F(X_*, t) \cdot Y$. We will estimate this inverse using dynamic inversion. It is not obvious, however, that $Y \mapsto D_1F(X_*, t) \cdot Y$ is invertible, so we deal with this issue first.

Differentiate (43) with respect to X to get

$$D_1F(X,t) : Y \mapsto D_1F(X,t) \cdot Y := Y\Lambda(t)X + X\Lambda(t)Y. \tag{44}$$

We refer to a map of the form

$$L_M : Y \mapsto L_M Y := YM + MY \tag{45}$$

with Y and M in $\mathbb{R}^{n\times n}$ as a *Lyapunov map*, due to its relation to the *Lyapunov equation* $YB + MC = Q$, which arises in the study of the stability of linear control systems (see e.g. Horn and Johnson [24, Chapter 4]). It may be proven that L_M is an invertible linear map if no two eigenvalues of M add up to zero (see e.g. [24, Theorem 4.4.6, p. 270]).

Now note that $\Lambda(t)X_* = X_*\Lambda(t) = P(t)$, which is positive definite and symmetric, having only real-valued and strictly positive eigenvalues. Thus no pair of eigenvectors of $\Lambda(t)X_*$ sum to zero. Therefore $D_1(X,t)\cdot Y$ is nonsingular. Since $D_1F(X,t)\cdot Y$ is continuous in X, it follows that $D_1F(X,t)\cdot Y$ remains invertible for all X in a sufficiently small neighborhood of X_*.

5.2. *Dynamic Polar Decomposition*

The estimator for the map $W \mapsto D_1F(X_*,t)^{-1}\cdot W$ will be denoted $\Gamma \in L(\mathrm{S}(n),\mathrm{S}(n))$ so that

$$\Gamma_* \cdot W = D_1F(x_*,t)^{-1}\cdot W. \tag{46}$$

Using Γ, we may define a dynamic inverse for $F(X,t)$. Let $G:\mathrm{S}(n)\times L(\mathrm{S}(n),\mathrm{S}(n)) \to \mathrm{S}(n)$, $(W,\Gamma)\mapsto G[W,\Gamma]$ be defined by

$$G[W,\Gamma] := D_1F(X_*,t)^{-1}\big|_{\Gamma_*=\Gamma}\cdot W = \Gamma\cdot W \tag{47}$$

for $W \in \mathrm{S}(n)$. This makes $G[W,\Gamma]$ a dynamic inverse for $F(X,t) = X\Lambda(t)X - I$, as long as Γ is sufficiently close to Γ_*.

To construct an estimator $E(X,\Gamma,t) \in \mathrm{S}(n)$ for $\dot{X}_*$, first differentiate $F(x_*,t) = 0$,

$$D_1F(X_*,t)\cdot\dot{X}_* + D_2F(X_*,t) = 0 \tag{48}$$

and then solve for $\dot{X}_*$,

$$\dot{X}_* = -D_1F(X_*,t)^{-1}\cdot D_2F(X_*,t) = -\Gamma_*\cdot D_2F(X_*,t). \tag{49}$$

Note that $D_2F(X_*,t) = X_*\dot{\Lambda}(t)X_*$. Now substitute X and Γ for X_* and Γ_* to obtain

$$E(X,\Gamma,t) := -\Gamma\cdot\big(X\dot{\Lambda}(t)X\big). \tag{50}$$

To obtain Γ dynamically, let $F^{\gamma} : \mathrm{S}(n) \times L(\mathrm{S}(n), \mathrm{S}(n)) \times \mathbb{R}_+ \to L(\mathrm{S}(n), \mathrm{S}(n))$, $(X, \Gamma, t) \mapsto F^{\gamma}(X, \Gamma, t)$ be defined by

$$F^{\gamma}(X, \Gamma, t) := D_1 F(X, t) \cdot \Gamma - \mathrm{Id}, \tag{51}$$

where Id denotes the identity mapping from $L(\mathrm{S}(n), \mathrm{S}(n))$ to $L(\mathrm{S}(n), \mathrm{S}(n))$. A linear dynamic inverse for $F^{\gamma}(X, \Gamma, t)$ is $G^{\gamma} : L(\mathrm{S}(n), \mathrm{S}(n)) \times L(\mathrm{S}(n), \mathrm{S}(n)) \to L(\mathrm{S}(n), \mathrm{S}(n))$, $(W, \Gamma) \mapsto G^{\gamma}[W, \Gamma]$ defined by

$$G^{\gamma}[W, \Gamma] := \Gamma \cdot W. \tag{52}$$

For an estimator $E^{\gamma}(X, \Gamma, t)$ for $\dot{\Gamma}_*$, we differentiate $F^{\gamma}(X_*, \Gamma_*, t) = 0$ with respect to t, solve for $\dot{\Gamma}_*$, and substitute X and Γ for X_* and Γ_* respectively to get

$$E^{\gamma}(X, \Gamma, t) := -\Gamma \cdot \left(\frac{d}{dt} D_1 F(X, t) \right) \bigg|_{\dot{X}_* = E(X, \Gamma, t)} \cdot \Gamma. \tag{53}$$

Combining the E's, F's, and G's from (50), (43), (47), (53), (51), and (52), we obtain the dynamic inverter

$$\begin{aligned} \dot{X} &= -\mu G[F(X, t), \Gamma] + E(\Gamma, X, t), \\ \dot{\Gamma} &= -\mu G^{\gamma}[F^{\gamma}(X, \Gamma, t), \Gamma] + E^{\gamma}(X, \Gamma, t), \end{aligned} \tag{54}$$

or in an expanded form

(*dynamic polar decomposition for time-varying matrices*)

$$\begin{aligned} \dot{X} &= -\mu \Gamma \cdot (X \Lambda(t) X - I) - \Gamma \cdot \left(X \dot{\Lambda}(t) X \right), \\ \dot{\Gamma} &= -\mu \Gamma \cdot (D_1 F(X, t) \cdot \Gamma - \mathrm{Id}) - \Gamma \cdot \left(\frac{d}{dt} D_1 F(X, t) \right) \bigg|_{\dot{X} = -\Gamma \cdot (X \dot{\Lambda} X)} \cdot \Gamma. \end{aligned} \tag{55}$$

In this scheme we have

$$XA(t) \to U(t), \qquad XA(t) A(t)^T \to P(t) \qquad A(t)^T X^2 \to A^{-1}(t) \tag{56}$$

as $t \to \infty$.

Inital conditions for the dynamic inverter (55) may be set so that $X(0) = P(0)^{-1}$ and $\Gamma(0) = D_1 F(P(0)^{-1}, 0)^{-1}$. Under these conditions $\Gamma(t) \equiv P(t)^{-1}$ for all $t \geqslant 0$.

Combining the results above with the dynamic inversion theorem, Theorem 2.2, gives the following theorem.

THEOREM 5.1 (Dynamic polar decomposition of time-varying matrices). *Let $A(t)$ be in $\mathrm{GL}(n)$ for all $t \in \mathbb{R}_+$. Let the polar decomposition of $A(t)$ be $A(t) = P(t)U(t)$ with $P(t) \in \mathrm{S}(n)$ the positive definite symmetric square root of $\Lambda(t) := A(t)A(t)^T$, and $U(t) \in \mathrm{O}(n)$ for all $t \in \mathbb{R}_+$. Let X be in $\mathrm{S}(n)$, and let Γ be in $L(\mathrm{S}(n), \mathrm{S}(n))$. Let $(X(t), \Gamma(t))$ denote the solution of the dynamic inverter (55) where $F(X, t)$ is given by (43). Then there exists a $\tilde{\mu}$ such that if the dynamic inversion gain μ satisfies $\mu > \tilde{\mu}$, and $(X(0), \Gamma(0) \cdot W)$ is sufficiently close to $(P(0)^{-1}, D_1 F(P(0)^{-1}, t)^{-1} \cdot W)$ for all $W \in S(n)$, then*

1. *$\Lambda(t)X(t)$ exponentially converges to $P(t)$,*
2. *$X(t)A(t)$ exponentially converges to $U(t)$, and*
3. *$A(t)X(t)^2$ exponentially converges to $A(t)^{-1}$.*

5.3. *A Numerical Example*

Though numerical inversion of the Lyapunov map has long been a topic of interest in the context of control theory [25, 26], we do not know of any *matrix* map $L^{-1} : \mathrm{S}(n) \to \mathrm{S}(n)$, taking *matrices* to *matrices*, which inverts $Y \mapsto D_1 F(X, t) \cdot Y = X\Lambda(t)Y + Y\Lambda(t)X$. By converting $D_1 F(X, t) \cdot Y$ to an $s(n) \times s(n)$ matrix, however, and representing elements of $\mathrm{S}(n)$ as vectors, the inverse $D_1 F(X, t)^{-1} \cdot Y$ as a mapping between vector spaces $\mathbb{R}^{s(n)} \to \mathbb{R}^{s(n)}$ can be obtained through matrix inversion. For the purposes of the example below, we will resort to vector notation in referring to elements of $\mathrm{S}(n)$.

REMARK 5.2 (Vector notation for symmetric matrices). We will adopt a notation that allows us to switch between matrix representation and vector representation of elements of $\mathrm{S}(n)$.

Chose an ordered basis $\beta = \{\beta_1, \dots, \beta_{s(n)}\}$ for $\mathrm{S}(n)$. To any $x \in \mathbb{R}^{s(n)}$ there corresponds a unique matrix $\hat{x} \in \mathrm{S}(n)$ where the correspondence is through the expansion of $\hat{x}$ in the ordered basis β,

$$\hat{x} \equiv (x)^{\wedge} := \sum_{i \in \underline{s(n)}} x^i \beta_i \in \mathrm{S}(n). \tag{57}$$

Conversely, for any $X \in \mathrm{S}(n)$, let $\check{X}$ denote the vector of the expansion coefficients of

$$X = \sum_{i \in \underline{s(n)}} x^i \beta_i \tag{58}$$

in the basis β, so that $\check{X} \equiv (X)^{\vee} = x$. Then $(\check{X})^{\wedge} = X$ and $(\hat{x})^{\vee} = x$.

Using the notation of Remark 5.2, and letting $\Gamma \in \mathbb{R}^{s(n) \times s(n)}$ be the estimator for the matrix representation of $D_1 F(X, t)^{-1}$, the dynamic inverter (55) takes the form

$$\begin{aligned} \dot{x} &= -\mu\Gamma(\hat{x}\Lambda(t)\hat{x} - I)^{\vee} - \Gamma(\hat{x}\dot{\Lambda}(t)\hat{x})^{\vee}, \\ \dot{\Gamma} &= -\mu\Gamma(D_1 F(x,t)\Gamma - I) - \Gamma\left(\frac{d}{dt} D_1 F(x,t)\Big|_{\dot{x} = -\Gamma(\hat{x}\dot{\Lambda}(t)\hat{x})^{\vee}}\right)\Gamma, \end{aligned} \tag{59}$$

and again we have

$$\hat{x}A(t) \to U(t), \qquad \hat{x}A(t)A(t)^T \to P(t), \qquad A(t)^T(\hat{x})^2 \to A^{-1}(t) \tag{60}$$

as $t \to \infty$.

An example of the polar decomposition of a 2×2 matrix will illustrate application of Theorem 5.1 and the equations (59).

EXAMPLE 5.3 (Polar decomposition of a time-varying matrix). Let

$$A(t) := \begin{bmatrix} 10 + \sin 10t & \cos t \\ -t & 1 \end{bmatrix}. \tag{61}$$

In this case $x \in \mathbb{R}^3$ and $\Gamma \in \mathbb{R}^{3 \times 3}$. We will perform polar decomposition and inversion of $A(t)$ over $t \in [0, 8]$, an interval over which $A(t)$ is nonsingular. We will estimate $P(t)$ and $U(t)$ such that $A(t) = P(t)U(t)$, with $P(t) \in \mathrm{S}(2)$ being the positive definite symmetric square root of $A(t)A(t)^T$, and with $U(t) \in \mathrm{O}(2)$.

Let

$$\Lambda(t) = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{bmatrix} = A(t)\,A(t)^T. \tag{62}$$

We choose the ordered basis β of S(2) to be

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}. \tag{63}$$

In this basis we have

$$F(x,t) = (\hat{x}\Lambda(t)\hat{x} - I)^{\check{}} = \begin{bmatrix} \lambda_1 x_1^2 + 2\lambda_2 x_1 x_2 + \lambda_3 x_2^2 - 1 \\ \lambda_1 x_1 x_2 + \lambda_2 x_2^2 + \lambda_2 x_1 x_3 + \lambda_3 x_2 x_3 \\ \lambda x_2^2 + 2\lambda_2 x_2 x_3 + \lambda_3 x_3^2 - 1 \end{bmatrix}. \tag{64}$$

Then

$$D_1 F(x,t) = \begin{bmatrix} 2(\lambda_1 x_1 + \lambda_2 x_2) & 2(\lambda_2 x_1 + \lambda_3 x_2) & 0 \\ \lambda_1 x_2 + \lambda_2 x_3 & \lambda_1 x_1 + 2\lambda_2 x_2 + \lambda_3 x_3 & \lambda_2 x_1 + \lambda_3 x_2 \\ 0 & 2(\lambda_1 x_2 + \lambda_2 x_3) & 2(\lambda_2 x_2 + \lambda_3 x_3) \end{bmatrix}. \tag{65}$$

For an estimator for $\dot{x}$ we have from the coordinate from of (50)

$$E(x,\Gamma,t) = -\Gamma \begin{bmatrix} \dot{\lambda}_1 x_1^2 + 2\dot{\lambda}_2 x_1 x_2 + \dot{\lambda}_3 x_2^2 \\ \dot{\lambda}_1 x_1 x_2 + \dot{\lambda}_2 x_2^2 + \dot{\lambda}_2 x_1 x_3 + \dot{\lambda}_3 x_2 x_3 \\ \dot{\lambda}_1 x_2^2 + 2\dot{\lambda}_2 x_2 x_3 + \dot{\lambda}_3 x_3^2 \end{bmatrix}. \tag{66}$$

The estimator E^{γ} for $\dot{\Gamma}_*$ is given by (53), where

$$\frac{d}{dt} D_1 F(x,t)\Big|_{\dot{x} = E(x,\Gamma,t)} = -\Gamma\big(\hat{x}(t)\dot{\Lambda}(t)\hat{x}\big)^{\check{}} = \begin{bmatrix} \dot{L}_{11} & \dot{L}_{12} & 0 \\ \dot{L}_{21} & \dot{L}_{22} & \dot{L}_{23} \\ 0 & \dot{L}_{32} & \dot{L}_{33} \end{bmatrix} \tag{67}$$

with

$$
\begin{aligned}
\dot{L}_{11} &= 2\dot{\lambda}_1 x_1 + 2\lambda_1 E_1(x,\Gamma,t) + 2\dot{\lambda}_2 x_2 + 2\lambda_2 E_2(x,\Gamma,t),\\
\dot{L}_{12} &= 2\dot{L}_{23},\\
\dot{L}_{21} &= \dot{\lambda}_1 x_2 + \lambda_1 E_2(x,\Gamma,t) + \dot{\lambda}_2 x_3 + \lambda_2 E_3(x,\Gamma,t),\\
\dot{L}_{22} &= \dot{\lambda}_1 x_1 + \lambda_1 E_1(x,\Gamma,t) + 2\dot{\lambda}_2 x_2 + 2\lambda_2 E_2(x,\Gamma,t) + \dot{\lambda}_3 x_3\\
&\qquad + \lambda_3 E_3(x,\Gamma,t),\\
\dot{L}_{23} &= \dot{\lambda}_2 x_1 + \lambda_2 E_1(x,\Gamma,t) + \dot{\lambda}_3 x_2 + \lambda_3 E_2(x,\Gamma,t),\\
\dot{L}_{32} &= 2\dot{L}_{21},\\
\dot{L}_{33} &= 2\dot{\lambda}_2 x_2 + 2\lambda_2 E_2(x,\Gamma,t) + 2\dot{\lambda}_3 x_3 + 2\lambda_3 E_3(x,\Gamma,t)
\end{aligned}
\tag{68}
$$

Dynamic inversion using the equations (55) was simulated using the adaptive-step-size Runge-Kutta integrator `ode45` from Matlab, with the default tolerance of 10^{-6}. The inital conditions were set so that

$$
\begin{aligned}
\hat{x}(0) &= \Lambda(0)^{1/2} + \hat{e}_x,\\
\Gamma(0) &= D_1 F(x(0),t)^{-1},
\end{aligned}
\tag{69}
$$

where $e_x = [-0.55, 0.04, -2.48]^{\mathrm{T}}$ is an error that has been deliberately added to demonstrate the error transient of the dynamic inverter. The value of μ was set to 10.

The graph of Figure 4 shows the values of the individual elements of $A(t)$. The top graph of Figure 5 shows the elements of $x(t)$, the estimator for $P(t)^{-1}$, and the bottom graph shows the elements of $\Gamma(t)$.

Figure 6 shows $\log_{10}(\|\hat{x}(t)\Lambda(t)\hat{x}(t) - I\|_\infty)$ indicating the extent to which $\hat{x}$, the estimator for $P(t)^{-1}$, fails to be the square root of $\Lambda(t) = A(t)A(t)^T$.

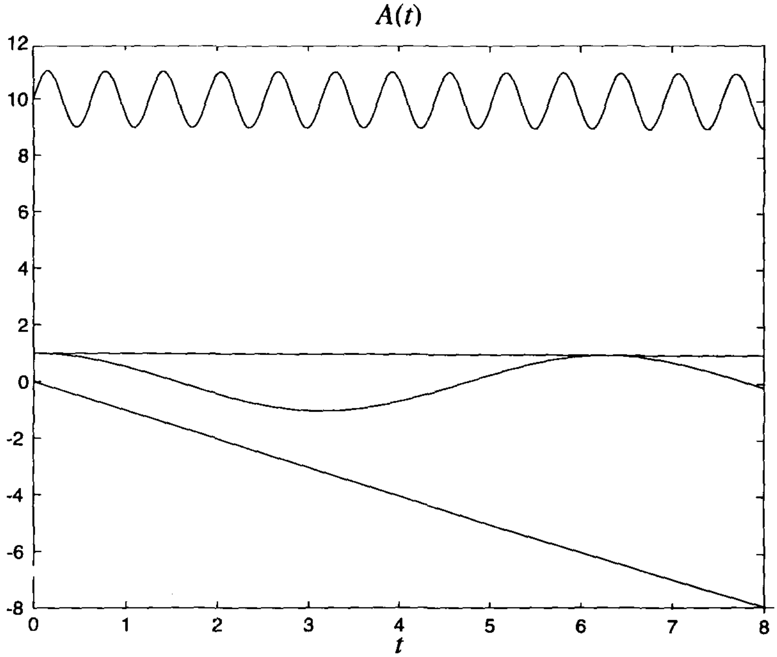

FIG. 4. Elements of $A(t)$ [see (61)].

For estimates of $P(t)$, $U(t)$, and $A(t)^{-1}$ we have

$$\hat{x}(t)\,A(t)\,A(t)^{T} \to P(t), \qquad \hat{x}(t)\,A(t) \to U(t), \quad \text{and}$$

$$A(t)^{T}\hat{x}(t)^{2} \to A(t)^{-1}. \tag{70}$$

REMARK 5.4 (Symmetry of the dynamic inverter). It is interesting to note that $P(t)^{-1}$, besides being a solution to $X\Lambda(t)X - I = 0$, is also a solution to $\Lambda(t)X^{2} - I = 0$ as well as $X^{2}\Lambda(t) - I = 0$. But $\Lambda(t)X^{2} - I$ and $X^{2}\Lambda(t) - I$ are not, in general, symmetric even when $\Lambda(t)$ and X are symmetric. Though exponential convergence is still guaranteed when using these forms, the flow $X(t)$ is not, in general, confined to $\mathrm{S}(n)$. Using these forms would increase the number of equations in the dynamic inverter by $n(n-1)/2 + n^{2} - s(n)^{2}$, since not only would the right-hand side of the top equation of (55) no longer be symmetric, but the matrix representation of Γ would be $n^{2} \times n^{2}$ rather than $s(n) \times s(n)$.

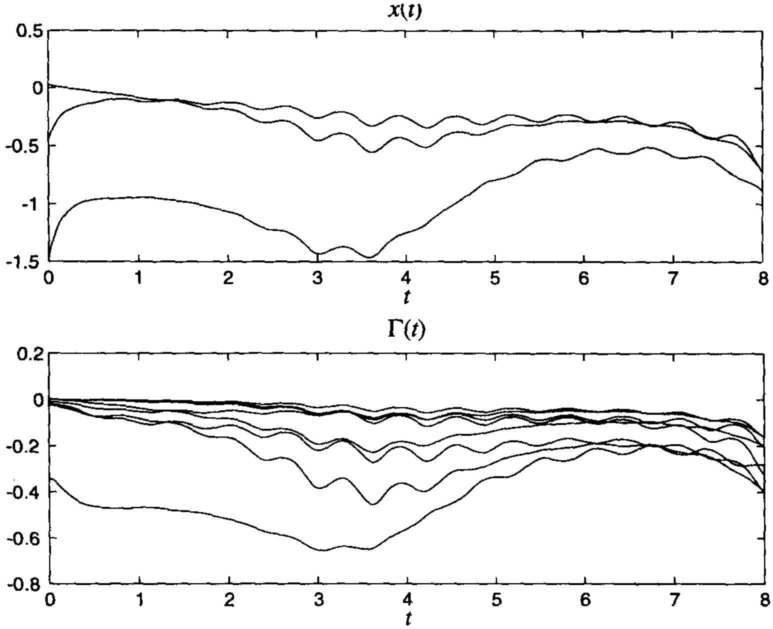

FIG. 5. Elements of x (top) and Γ (bottom). See Example 5.3.

6. POLAR DECOMPOSITION AND INVERSION OF CONSTANT MATRICES

In the dynamic inversion techniques of Sections 3 and 5 we assumed that we had available an approximation of $A^{-1}(0)$ with which to set $\Gamma(0)$ in the dynamic inversion of $A(t)$. Thus we would need to invert at least one constant matrix, $A(0)$, in order to start the dynamic inverter. Methods of constant matrix inversion presented in Section 4 have the potential disadvantage either of producing exact inversion only asymptotically as $t \to \infty$, or of working only on matrices with no eigenvalues in the interval $(-\infty, 0)$. The question naturally arises, then, how we might use dynamic inversion to invert *any* constant matrix so that the exact inverse is available by a prescribed time. In this section, by appealing to both homotopy and polar decomposition, we give an answer to this question.

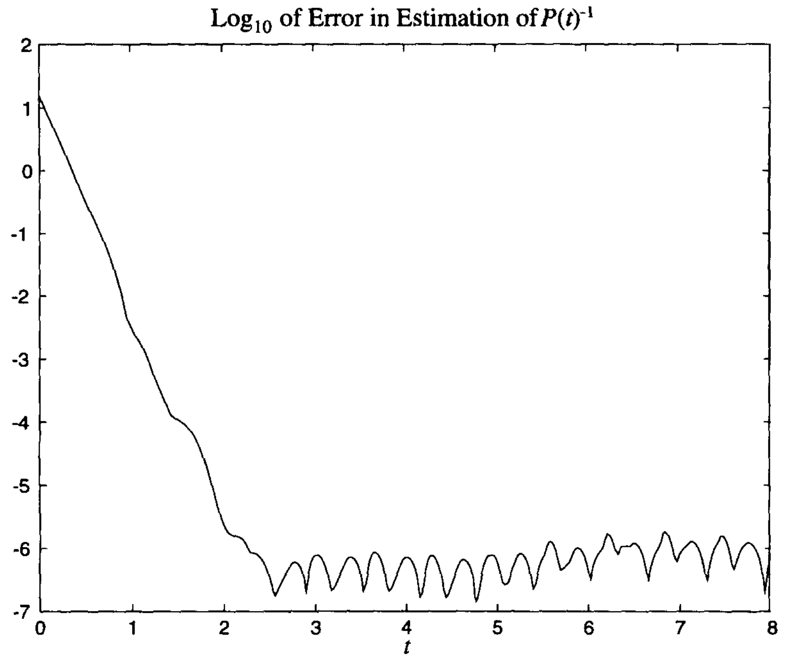

FIG. 6. The error $\log_{10}(\|\hat{x}(t)\Lambda(t)\hat{x}(t) - I\|_\infty)$ indicating the extent to which x fails to satisfy $\hat{x}\Lambda(t)\hat{x} - I = 0$. The ripple from $t \approx 1.8$ to $t = 8$ is due to numerical noise. See Example 5.3.

Let M be in $\mathrm{GL}(n)$ with $P = P^T > 0$, $UU^T = I$, and $M = PU$. Helmke and Moore (see [18, pp. 150–152]) have described a gradient flow for the function $\|A - UP\|^2$ (the square of the Frobenius norm),

$$\begin{aligned} \dot{\overline{U}} &= \overline{U}\overline{P}M^T\overline{U} - M\overline{P}, \\ \dot{\overline{P}} &= -2\overline{P} + M^T\overline{U} + \overline{U}^T M, \end{aligned} \tag{71}$$

where $\overline{P}$ and $\overline{U}$ are meant to approximate P and U respectively. Asymptotically, this system produces factors P_* and U_* satisfying $M - P_* U_* = 0$ for almost all initial conditions $\overline{P}(0)$, $\overline{U}(0)$ as $t \to \infty$. A difficulty with this approach, as the authors point out, is that positive definiteness of the approximator $\overline{P}$ is not guaranteed.

In this section we describe a dynamical system that provides polar decomposition of *any* nonsingular constant matrix by any prescribed time, with the positive definiteness of the estimator of P guaranteed. This will be accomplished by applying Theorem 5.1 on dynamic polar decomposition of time-varying matrices to the homotopy

$$\Lambda(t) := (1 - t)I + tMM^T. \tag{72}$$

Unlike the homotopy $H(t) = (1 - t)I + tM$ of Section 4, the homotopy $\Lambda(t)$ is guaranteed to have a spectrum which avoids $(-\infty, 0)$ for *any* nonsingular M, since $\Lambda(t)$ is a positive definite symmetric matrix for all $t \in [0, 1]$. The situation is depicted in Figure 7.

Recall that M is in $\mathrm{GL}(n)$. For $\Lambda(t)$ as defined in (72) note that $\Lambda(0) = I$, $\Lambda(1) = MM^T$, and for all $t \in [0, 1]$, $\Lambda(t)$ is positive definite and symmetric. Let $P(t)$ denote the positive definite symmetric square root of $\Lambda(t)$. Let the estimator of $P^{-1}(t)$ be $X \in \mathrm{S}(n)$. Differentiate $\Lambda(t)$ with respect to t to get

$$\dot{\Lambda}(t) = MM^T - I. \tag{73}$$

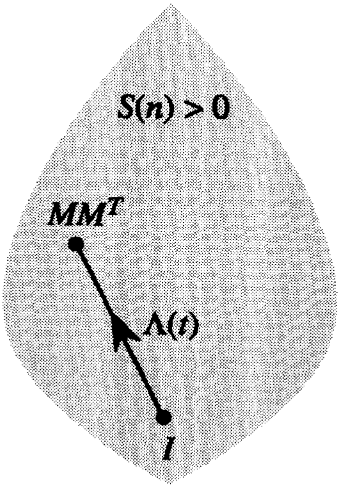

FIG. 7. $\Lambda(t)$ is positive definite and symmetric for all $t \in [0, 1]$.

Now we may apply the dynamic inverter of Section 5 in order to perform the polar decomposition of M. As in (43), let

$$F(X,t) := X\Lambda(t)X - I. \tag{74}$$

By inspection it may be verified that $X_*(0) = I$ and $\Gamma_*(0) = \frac{1}{2}I$. If we set $X(0) = I$ and $\Gamma(0) = \frac{1}{2}I$, then Theorem 2.2 and the results of the last section assure us that $X(t) \equiv P^{-1}(t)$ for all $t \geqslant 0$, and thus $X(1) = P^{-1}$. Consequently

$$\begin{aligned} X(1) &= P^{-1}, & \Lambda(1)X(1) &= MM^TX(1) = P, \\ X(1)M &= U, & M^TX(1)^2 &= M^{-1}. \end{aligned} \tag{75}$$

Note that $\dot{\Lambda}(t) = MM^T - I = 0$ if and only if M is unitary, in which case $M^{-1} = M^T$.

Combining the results of this section with the results of the last section gives the following theorem.

THEOREM 6.1 (Dynamic polar decomposition of constant matrices by a prescribed time). *Let M be in* GL(n). *Let the polar decomposition of M be $M = PU$ with $P \in$ S(n) the positive definite symmetric square root of MM^T and $U \in$ O(n). Let X be in* S(n), *and let Γ be in* $L($S(n), S$(n))$. *Let $X(0) = I$ and $\Gamma(0) = \frac{1}{2}$*Id. *Let $(X(t), \Gamma(t))$ denote the solution of the prescribed-time dynamic inverter for constant matrices*:

$$\begin{aligned} \dot{\Gamma} &= -\mu G^{\gamma}[F^{\gamma}(\Gamma, X, t), \Gamma] + E^{\gamma}(\Gamma, X), \\ \dot{X} &= -\mu G[F(X,t), \Gamma] + E(\Gamma, X); \\ \Lambda(t) &= (1-t)I + tMM^T, \\ F(X,t) &= X\Lambda(t)X - I, \\ G[W,\Gamma] &= \Gamma \cdot W, \\ E(\Gamma, X) &= -\Gamma \cdot (X(MM^T - I)X), \\ F^{\gamma}(\Gamma, X, t) &= D_1F(X,t)\cdot\Gamma - \mathrm{Id}, \\ G^{\gamma}[W,\Gamma] &= \Gamma \cdot W, \\ E^{\gamma}(X,\Gamma) &= -\Gamma\left(\frac{d}{dt}D_1F(X,t)\right)\bigg|_{\dot{X}=E(\Gamma,X)}\cdot\Gamma. \end{aligned} \tag{76}$$

Then for any $\mu > 0$,

$$MM^TX(1) = P, \qquad X(1)M = U, \quad \textit{and} \quad M^TX(1)^2 = M^{-1}. \tag{77}$$

REMARK 6.2 (Polar decomposition by any prescribed time). As in Theorem 4.6, we can force X to equal P^{-1} at any time $t_1 > 0$ by substituting t/t_1 for t in $\Lambda(t)$, and proceeding with the derivation of the dynamic inverter as above. Then $X(t_1) = P^{-1}$.

EXAMPLE 6.3. A numerical simulation of a dynamic inverter for the polar decomposition of a constant 2-by-2 matrix was performed. The integration was performed in Matlab [27] using `ode45`, an adaptive-step-size Runge-Kutta routine, using the default tolerance of 10^{-6}. The matrix M was chosen (randomly) to be

$$M = \begin{bmatrix} 7 & -3 \\ -24 & -3 \end{bmatrix}. \tag{78}$$

The value of μ was set to 10. The evolution of the elements of $x(t)$ and $\Gamma(t)$ is shown in Figure 8. Figure 9 shows the base-10 log of $\|F(x,t)\|_\infty = \|\hat{x}(t)MM^T\hat{x}(t) - I\|_\infty$, indicating to the extent to which x, the estimator for P^{-1}, fails to be the square root of $\Lambda(t) = MM^T$.

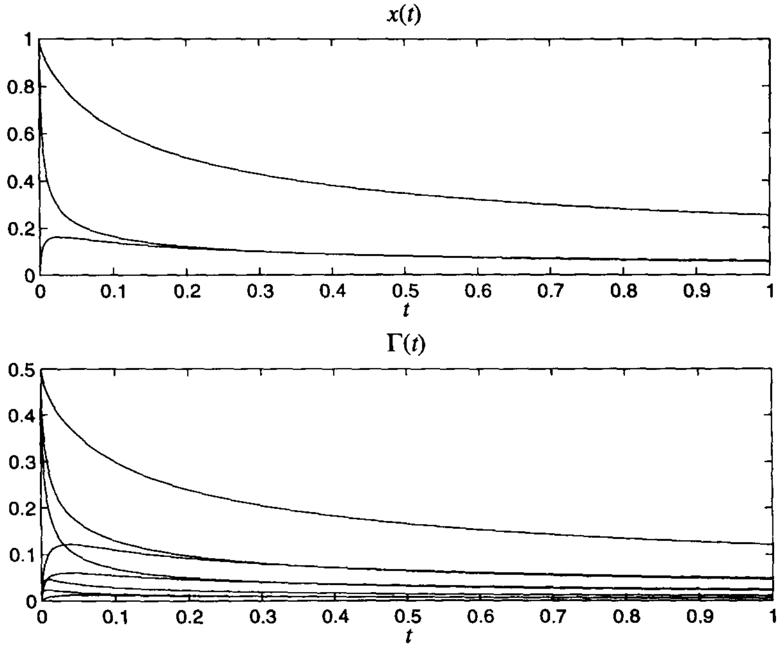

FIG. 8. Elements of $x(t)$ (top) and $\Gamma(t)$ (bottom), for Example 6.3.

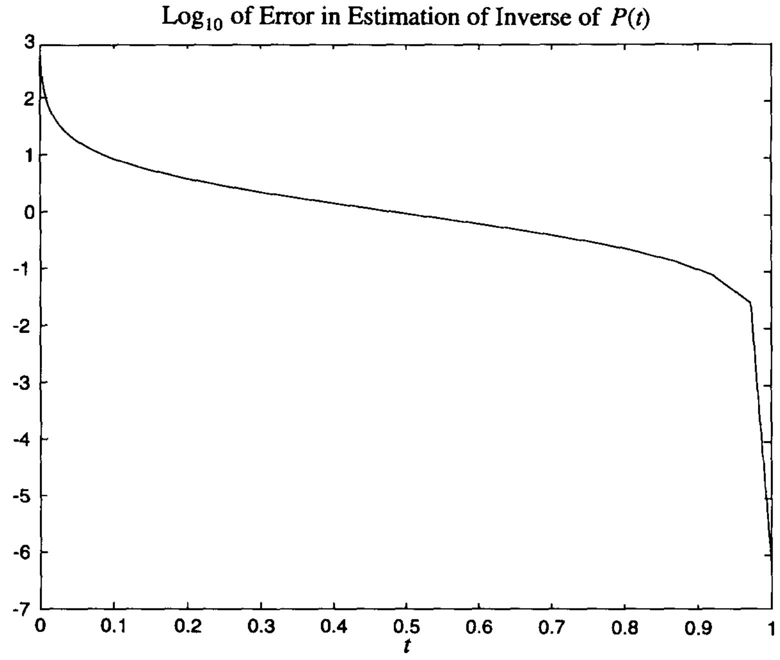

FIG. 9. The base-10 log of the error $\|\hat{x}(t)MM^T\hat{x}(t) - I\|_\infty$ for Example 6.3.

The final value ($t = 1$) of the error $\|\hat{x}(t)MM^T\hat{x}(t) - I\|_\infty$ was

$$\|\hat{x}(1)\Lambda(1)\hat{x}(1) - I\|_\infty = 1.0611 \times 10^{-6}. \tag{79}$$

The final values of P, U, and M^{-1} were, to four decimal places,

$$P = MM^T\hat{x}(1) = \begin{bmatrix} 5.2444 & -5.5223 \\ -5.5223 & 23.5479 \end{bmatrix},$$

$$U = \hat{x}(1)M = \begin{bmatrix} 0.3473 & -0.9377 \\ -0.9377 & -0.3473 \end{bmatrix}, \tag{80}$$

$$M^{-1} = M^T\hat{x}(1)^2 = \begin{bmatrix} 0.0323 & -0.0323 \\ -0.2581 & -0.0753 \end{bmatrix}.$$

7. SUMMARY

We have seen how the polar decomposition and inversion of time-varying and constant matrices may be accomplished by continuous-time dynamical systems. Our results are easily modified to provide solutions for time-varying and time-invariant linear equations of the form $A(t)x = b(t)$. We have also seen that dynamic inversion can provide a general conceptual framework through which to view other methods of dynamic computation such as gradient flow methods.

Dynamic inversion is showing promise in the context of the control of physical systems. For instance, in some control problems, dynamic inversion may provide essential signals which can be incorporated into controllers for nonlinear dynamical systems [28, 1]. In those same problems it may also be used for matrix inversion. For example, dynamic inversion has been incorporated into a controller for robotic manipulators in [29, 1], where the dynamic inverter produces inverse kinematic solutions necessary for the control law. If inversion of, say, a time-varying mass matrix is also required in the same problem, a dynamic inverter may be augmented to provide that capability too, without interfering with other inversions within the same problem.

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