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# Approximate controllability of fractional stochastic evolution equations

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ABSTRACT

A class of dynamic control systems described by nonlinear fractional stochastic differential equations in Hilbert spaces is considered. Using fixed point technique, fractional calculations, stochastic analysis technique and methods adopted directly from deterministic control problems, a new set of sufficient conditions for approximate controllability of fractional stochastic differential equations is formulated and proved. In particular, we discuss the approximate controllability of nonlinear fractional stochastic control system under the assumptions that the corresponding linear system is approximately controllable. The results in this paper are generalization and continuation of the recent results on this issue. An example is provided to show the application of our result. Finally as a remark, the compactness of semigroup is not assumed and subsequently the conditions are obtained for exact controllability result.



## 1. Introduction

Control theory is an important area of application oriented mathematics which deals with the design and analysis of control systems. In particular, the concept of controllability plays an important role in both the deterministic and the stochastic control theory. In recent years, controllability problems for various types of nonlinear dynamical systems in infinite dimensional spaces by using different kinds of approaches have been considered in many publications. An extensive list of these publications can be found (see [1–5] and the references therein). Moreover, the exact controllability enables to steer the system to arbitrary final state while approximate controllability means that the system can be steered to arbitrary small neighborhood of final state. Klamka [6] derived a set of sufficient conditions for the exact controllability of semilinear systems. Further, approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications. The approximate controllability of systems represented by nonlinear evolution equations has been investigated by several authors [7–9], in which the authors effectively used the fixed point approach. Fu and Mei [10] studied the approximate controllability of semilinear neutral functional differential systems with finite delay. The conditions are established with the help of semigroup theory and fixed point technique under the assumption that the linear part of the associated nonlinear system is approximately controllable. Recently, Sakthivel et al. [11] established a set of sufficient conditions for obtaining the approximate controllability of semilinear fractional differential systems in Hilbert spaces.

Stochastic differential equations have many applications in economics, ecology and finance. In recent years, the controllability problems for stochastic differential equations have become a field of increasing interest, (see [12,13] and references therein). The extensions of deterministic controllability concepts to stochastic control systems have been discussed only in a limited number of publications. More precisely, there are less number of papers on the approximate

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controllability of the nonlinear stochastic systems [14,15]. Klamka [16,17] studied stochastic relative exact and approximate controllability problems for finite dimensional linear stationary dynamical systems with single time-variable point delay in the control by implementing the open mapping theorem. A set of necessary and sufficient conditions are established for the exact and approximate stochastic controllability of linear system with state delays in [18].

The concept of non-integral derivative and integral is used to study the behavior of real world problems in science and engineering [19,20]. In various problems of physics, mechanics, electrochemistry, diffusion processes and viscoelasticity, fractional derivatives describe certain physical phenomena more accurately than integer order derivatives. Recently, the existence and uniqueness results of initial and boundary value problem for fractional differential equations have been reported in (see [21–24] and the references therein). Zhou and Jiao [25] discussed the existence of mild solutions to fractional neutral evolution equations in an arbitrary Banach space. More recently, Wang et al. [26] investigated nonlocal problems for a class of fractional integrodifferential equations via fractional operators and optimal controls in Banach spaces in which the mild solution is introduced with the Caputo fractional derivative in terms of some probability density function and operator semigroup.

However, to the best of our knowledge, the approximate controllability problem for nonlinear fractional stochastic system in Hilbert spaces has not been investigated yet. Motivated by this consideration, in this paper we will study the approximate controllability problem for nonlinear fractional stochastic systems, which are natural generalizations of controllability concepts well known in the theory of infinite dimensional deterministic control systems. Specifically, we study the approximate controllability of nonlinear fractional control systems under the assumption that the associated linear system is approximately controllable. In fact, the results in this paper are motivated by the recent work of [15] and the fractional differential equations discussed in [25,27]. The main tools used in this paper are stochastic analysis techniques, fractional calculations and Banach contraction principle. Moreover, without assuming the compactness of semigroup, the results are established for the exact controllability of fractional stochastic systems.

## 2. Preliminaries

In this section, we provide definitions, lemmas and notations necessary to establish our main results. Throughout this paper, we use the following notations. Let $(\Omega, \Gamma, P)$ be a complete probability space equipped with a normal filtration $\Gamma_t, t \in J = [0, b]$ satisfying the usual conditions (i.e., right continuous and $\Gamma_0$ containing all $P$-null sets). We consider three real separable spaces $X, E$ and $U$, and $Q$-Wiener process on $(\Omega, \Gamma_b, P)$ with the linear bounded covariance operator $Q$ such that $trQ < \infty$. We assume that there exists a complete orthonormal system $\{e_n\}_{n\geq 1}$ in $E$, a bounded sequence of non-negative real numbers $\{\lambda_n\}$ such that $Qe_n = \lambda_n e_n,\ n = 1, 2, \ldots$ and a sequence $\{\beta_n\}_{n\geq 1}$ of independent Brownian motions such that

$$\langle w(t), e\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\langle e_n, e\rangle \beta_n(t), \quad e \in E,\ t \in [0, b]$$

and $\Gamma_t = \Gamma_t^w$, where $\Gamma_t^w$ is the sigma algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(Q^{1/2}E; X)$ be the space of all Hilbert–Schmidt operators from $Q^{1/2}E$ to $X$ with the inner product $\langle \psi, \pi\rangle L_2^0 = \mathrm{tr}[\psi Q\pi^*]$. Let $L^2(\Gamma_b, X)$ be the Banach space of all $\Gamma_b$-measurable square integrable random variables with values in the Hilbert space $X$. Let $E(\cdot)$ denotes the expectation with respect to the measure $P$. Let $C([0, b]; L^2(\Gamma, X))$ be the Banach space of continuous maps from $[0, b]$ into $L^2(\Gamma, X)$ satisfying $\sup_{t\in J} E\|x(t)\|^2 < \infty$. Let $H_2([0, b]; X)$ is a closed subspace of $C([0, b]; L^2(\Gamma, X))$ consisting of measurable and $\Gamma_t$-adapted $X$-valued process $x \in C([0, b]; L^2(\Gamma, X))$ endowed with the norm $\|x\|_{H_2} = (\sup_{t\in J} E\|x(t)\|_X^2)^{1/2}$. For details, we refer the reader to ([28,29] and references therein).

The purpose of this paper is to investigate the approximate controllability for a class of nonlinear fractional stochastic differential equation of the form

$$^cD_t^q x(t) = Ax(t) + Bu(t) + f(t, x(t)) + \sigma(t, x(t))\frac{dw(t)}{dt}, \quad t \in J, \tag{1}$$

$$x(0) = x_0, \tag{2}$$

where $0 < q < 1$; $^cD_t^q$ denotes the Caputo fractional derivative operator of order $q$; $x(\cdot)$ takes its values in the Hilbert space $X$; $A$ is the infinitesimal generator of a compact semigroup of uniformly bounded linear operators $\{S(t),\ t \geq 0\}$; the control function $u(\cdot)$ is given in $L_\Gamma^2([0, b], U)$ of admissible control functions, $U$ is a Hilbert space. $B$ is a bounded linear operator from $U$ into $X$; $f : J \times X \to X$ and $\sigma : J \times X \to L_2^0$ are appropriate functions; $x_0$ is $\Gamma_0$ measurable $X$-valued random variables independent of $w$.

Let us recall the following known definitions. For more details see [20]

**Definition 2.1.** The fractional integral of order $\beta$ with the lower limit 0 for a function $f$ is defined as

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{f(s)}{(t-s)^{1-\beta}}ds, \quad t > 0,\ \beta > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

**Definition 2.2.** Riemann–Liouville derivative of order $\beta$ with lower limit zero for a function $f : [0,\infty) \to R$ can be written as

$$^{L}D^{\beta}f(t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\beta+1-n}}ds, \quad t>0,\ n-1<\beta<n.$$

**Definition 2.3.** The Caputo derivative of order $\beta$ for a function $f : [0,\infty) \to R$ can be written as

$$^{c}D^{\beta}f(t) = {}^{L}D^{\beta}\left(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \quad t>0,\ n-1<\beta<n.$$

**Remark 2.4.** (a) If $f(t) \in C^n[0,\infty)$, then

$$^{c}D^{\beta}f(t) = \frac{1}{\Gamma(n-\beta)}\int_0^t \frac{f^n(s)}{(t-s)^{\beta+1-n}}ds = I^{n-\beta}f^n(s), \quad t>0,\ 0\le n-1<\beta<n.$$

(b) The Caputo derivative of a constant is equal to zero.
(c) If $f$ is an abstract function with values in $E$, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

The following results will be used throughout this paper.

**Lemma 2.5** ([7]). *Let $G : [0,b]\times\Omega \to L_2^0$ be a strongly measurable mapping such that $\int_0^b E\|G(t)\|_{L_2^0}^p dt < \infty$. Then*

$$E\left\|\int_0^t G(s)dw(s)\right\|^p \le L_G\int_0^t E\|G(s)\|_{L_2^0}^p ds$$

*for all $0\le t\le b$ and $p\ge 2$, where $L_G$ is the constant involving $p$ and $b$.*

Now, we present the mild solution of the problem (1)–(2).

**Definition 2.6** ([30]). A stochastic process $x \in H_2([0,b],X)$ is a mild solution of (1)–(2) if for each $u \in L_\Gamma^2([0,b],U)$, it satisfies the following integral equation,

$$x(t) = \mathscr{T}(t)x_0 + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)[Bu(s)+f(s,x(s))]ds + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\sigma(s,x(s))dw(s),$$

where $\mathscr{T}(t) = \int_0^\infty \xi_q(\theta)S(t^q\theta)d\theta$; $\mathscr{S}(t) = q\int_0^\infty \theta\xi_q(\theta)S(t^q\theta)d\theta$; $S(t)$ is a $C_0$-semigroup generated by a linear operator $A$ on $X$; $\xi_q$ is a probability density function defined on $(0,\infty)$, that is $\xi_q(\theta)\ge 0,\ \theta\in(0,\infty)$ and $\int_0^\infty \xi_q(\theta)d\theta = 1$.

**Lemma 2.7** ([31]). *The operators $\{\mathscr{T}(t)\}_{t\ge 0}$ and $\{\mathscr{S}(t)\}_{t\ge 0}$ are strongly continuous, i.e., for $x\in X$ and $0\le t_1<t_2\le b$, we have $\|\mathscr{T}(t_2)x-\mathscr{T}(t_1)x\|\to 0$ and $\|\mathscr{S}(t_2)x-\mathscr{S}(t_1)x\|\to 0$ as $t_1\to t_2$.*

We impose the following conditions on data of the problem:

(i) For any fixed $t\ge 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are bounded linear operators, i.e., for any $x\in X$,

$$\|\mathscr{T}(t)x\| \le M\|x\|, \qquad \|\mathscr{S}(t)x\| \le \frac{Mq}{\Gamma(q+1)}\|x\|.$$

(ii) The function $f : J\times X\to X$ and $\sigma : J\times X \to L_2^0$ satisfy linear growth and Lipschitz conditions. Moreover, there exist positive constants $N>0$, $\tilde{N}>0$, $L>0$ and $\tilde{L}>0$ such that

$$\|f(t,x)-f(t,y)\|^2 \le N\|x-y\|^2, \qquad \|f(t,x)\|^2 \le \tilde{N}(1+\|x\|^2)$$
$$\|\sigma(t,x)-\sigma(t,y)\|_{L_2^0}^2 \le L\|x-y\|^2, \qquad \|\sigma(t,x)\|_{L_2^0}^2 \le \tilde{L}(1+\|x\|^2).$$

(iii) The linear stochastic system is approximately controllable on $[0,b]$.

For each $0\le t<b$, the operator $\alpha(\alpha I+\Psi_0^b)^{-1}\to 0$ in the strong operator topology as $\alpha\to 0^+$, where $\Psi_0^b = \int_0^b (b-s)^{2(q-1)}\mathscr{S}(b-s)BB^*\mathscr{S}^*(b-s)ds$ is the controllability Gramian, here $B^*$ denotes the adjoint of $B$ and $\mathscr{S}^*(t)$ is the adjoint of $\mathscr{S}(t)$. Observe that linear fractional deterministic control system

$$D_t^q x(t) = Ax(t) + (Bu)(t), \quad t\in[0,b], \tag{3}$$
$$x(0) = x_0$$

corresponding to (1)–(2) is approximately controllable on $[0, b]$ iff the operator $\alpha(\alpha I+\Psi_0^b)^{-1} \to 0$ strongly as $\alpha \to 0^+$. The approximate controllability for linear fractional deterministic control system (3) is a natural generalization of approximate controllability of linear first order control system [12, Theorem 2].

**Definition 2.8.** System (1)–(2) is approximately controllable on $[0, b]$ if $\overline{\Re(b)} = L^2(\Omega, \Gamma_b, X)$, where

$$\Re(b) = \{x(b) = x(b, u) : u \in L^2_\Gamma([0, b], U)\},$$

here $L^2_\Gamma([0, b], U)$, is the closed subspace of $L^2_\Gamma([0, b] \times \Omega; U)$, consisting of all $\Gamma_t$ adapted, $U$-valued stochastic processes.

The following lemma is required to define the control function. The reader can refer to [32] for the proof.

**Lemma 2.9.** *For any $\tilde{x_b} \in L^2(\Gamma_b, X)$, there exists $\tilde{\phi} \in L^2_\Gamma(\Omega; L^2(0, b; L^0_2))$ such that $\tilde{x_b} = E\tilde{x_b} + \int_0^b \tilde{\phi}(s)dw(s)$.*

Now for any $\alpha > 0$ and $\tilde{x_b} \in L^2(\Gamma_b, X)$, we define the control function in the following form

$$\begin{aligned} u^\alpha(t, x) = {} & B^*(b-t)^{q-1}\mathscr{S}^*(b-t)\left[(\alpha I + \Psi_0^b)^{-1}(E\tilde{x_b} - \mathscr{T}(b)x_0) + \int_0^t (\alpha I + \Psi_0^b)^{-1}\tilde{\phi}(s)dw(s)\right] \\ & - B^*(b-t)^{q-1}\mathscr{S}^*(b-t)\int_0^t (\alpha I + \Psi_0^b)^{-1}(b-s)^{q-1}\mathscr{S}(b-s)f(s, x(s))ds \\ & - B^*(b-t)^{q-1}\mathscr{S}^*(b-t)\int_0^t (\alpha I + \Psi_0^b)^{-1}(b-s)^{q-1}\mathscr{S}(b-s)\sigma(s, x(s))dw(s). \end{aligned}$$

**Lemma 2.10.** *There exists a positive real constant $\hat{M}$ such that for all $x, y \in H_2$, we have*

$$E\|u^\alpha(t, x) - u^\alpha(t, y)\|^2 \leq \frac{\hat{M}}{\alpha^2}\int_0^t E\|x(s) - y(s)\|^2 ds, \tag{4}$$

$$E\|u^\alpha(t, x)\|^2 \leq \frac{\hat{M}}{\alpha^2}\left(1 + \int_0^t E\|x(s)\|^2 ds\right). \tag{5}$$

**Proof.** First, we will provide the proof of inequality (4), since (5) can be established in a similar way. Let $x, y \in H_2$. From the Holders inequality, Lemma 2.5 and the assumption on the data, we obtain

$$\begin{aligned} & E\|u^\alpha(t, x) - u^\alpha(t, y)\|^2 \\ & \leq 2E\left\|B^*(b-t)^{q-1}\mathscr{S}^*(b-t)\int_0^t (\alpha I + \Psi_0^b)^{-1}(b-s)^{q-1}\mathscr{S}(b-s)[f(s, x(s)) - f(s, y(s))]ds\right\|^2 \\ & \quad + 2E\left\|B^*(b-t)^{q-1}\mathscr{S}^*(b-t)\int_0^t (\alpha I + \Psi_0^b)^{-1}(b-s)^{q-1}\mathscr{S}(b-s)[\sigma(s, x(s)) - \sigma(s, y(s))]dw(s)\right\|^2 \\ & \leq \frac{2}{\alpha^2}\|B\|^2(b)^{2q-2}\left(\frac{Mq}{\Gamma(q+1)}\right)^4 \frac{b^{2q-1}}{(2q-1)}[N + L]\int_0^t E\|x(s) - y(s)\|^2 ds \\ & \leq \frac{1}{\alpha^2}\hat{M}\int_0^t E\|x(s) - y(s)\|^2 ds, \end{aligned}$$

where $\hat{M} = 2\|B\|^2(b)^{2q-2}\left(\frac{Mq}{\Gamma(q+1)}\right)^4 \frac{b^{2q-1}}{(2q-1)}[N + L]$. The proof of the inequality (5) is similar to that of (4) and hence it is omitted. □

## 3. Controllability results

Now, let us present the main result of this paper. In this section, we formulate and prove conditions for approximate controllability of the fractional stochastic dynamical control system (1)–(2) using the contraction mapping principle. In particular, we establish approximate controllability of nonlinear fractional stochastic control system (1)–(2) under the assumptions that the corresponding linear system is approximately controllable. For any $\alpha > 0$, define the operator $F_\alpha : H_2 \to H_2$ by

$$F_\alpha x(t) = \mathscr{T}(t)x_0 + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)[f(s, x(s)) + Bu^\alpha(s, x)]ds + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\sigma(s, x(s))dw(s). \tag{6}$$

Now, we state and prove the following lemma, which will be used in the proof of main result.

**Lemma 3.1.** *For any $x \in H_2$, $F_\alpha(x)(t)$ is continuous on $[0, b]$ in $L^2$-sense.*

**Proof.** Let $0 \le t_1 < t_2 \le b$. Then for any fixed $x \in H_2$, from Eq. (6), we have

$$\mathbb{E}\|(F_\alpha x)(t_2) - (F_\alpha x)(t_1)\|^2 \le 4[E\|(\mathscr{T}(t_2) - \mathscr{T}(t_1))x_0\|^2 + \sum_{i=1}^{3} \mathbb{E}\|\Pi_i^x(t_2) - \Pi_i^x(t_1)\|^2].$$

The strong continuity of $\mathscr{T}(t)$, the first term on the R.H.S goes to zero as $t_2 - t_1 \to 0$. Next, it follows from Holder's inequality and assumptions on the theorem that

$$\begin{aligned}
\mathbb{E}\|\Pi_1^x(t_2) - \Pi_1^x(t_1)\|^2 &\le 3E\left\|\int_0^{t_1}(t_1 - s)^{q-1}(\mathscr{S}(t_2 - s) - \mathscr{S}(t_1 - s))f(s, x(s))ds\right\|^2 + 3E\left\|\int_0^{t_1}((t_2 - s)^{q-1}\right.\\
&\quad \left.- (t_1 - s)^{q-1})\mathscr{S}(t_2 - s)f(s, x(s))ds\right\|^2 + 3E\left\|\int_{t_1}^{t_2}(t_2 - s)^{q-1}\mathscr{S}(t_2 - s)f(s, x(s))ds\right\|^2\\
&\le 3\frac{t_1^{2q-1}}{2q-1}\int_0^{t_1} E\|(\mathscr{S}(t_2 - s) - \mathscr{S}(t_1 - s))f(s, x(s))ds\|^2 + 3\left(\frac{Mq}{\Gamma(1+q)}\right)^2\\
&\quad \times \left(\int_0^{t_1}((t_2 - s)^{q-1} - (t_1 - s)^{q-1})^2 ds\right)\left(\int_0^{t_1} E\|f(s, x(s))\|^2 ds\right)\\
&\quad + 3\frac{(t_2 - t_1)^{2q-1}}{1 - 2q}\left(\frac{Mq}{\Gamma(1+q)}\right)^2\int_{t_1}^{t_2} E\|f(s, x(s))\|^2 ds.
\end{aligned}$$

Further, we obtain

$$\begin{aligned}
\mathbb{E}\|\Pi_2^x(t_2) - \Pi_2^x(t_1)\|^2 &\le 3E\left\|\int_0^{t_1}(t_1 - s)^{q-1}(\mathscr{S}(t_2 - s) - \mathscr{S}(t_1 - s))Bu^\alpha(s, x)ds\right\|^2 + 3E\left\|\int_0^{t_1}((t_2 - s)^{q-1}\right.\\
&\quad \left.- (t_1 - s)^{q-1})\mathscr{S}(t_2 - s)Bu^\alpha(s, x)ds\right\|^2 + 3E\left\|\int_{t_1}^{t_2}(t_2 - s)^{q-1}\mathscr{S}(t_2 - s)Bu^\alpha(s, x)ds\right\|^2\\
&\le 3\frac{t_1^{2q-1}}{2q-1}\int_0^{t_1} E\|(\mathscr{S}(t_2 - s) - \mathscr{S}(t_1 - s))Bu^\alpha(s, x)\|^2 ds + 3\left(\frac{Mq}{\Gamma(1+q)}\right)^2\|B\|^2\\
&\quad \times \left(\int_0^{t_1}((t_2 - s)^{q-1} - (t_1 - s)^{q-1})^2 ds\right)\left(\int_0^{t_1} E\|u^\alpha(s, x)\|^2 ds\right)\\
&\quad + 3\frac{(t_2 - t_1)^{2q-1}}{1 - 2q}\left(\frac{Mq}{\Gamma(1+q)}\right)^2\|B\|^2\int_{t_1}^{t_2} E\|u^\alpha(s, x)\|^2 ds.
\end{aligned}$$

Similarly, using Lemma 2.5 and assumptions on the theorem we get

$$\begin{aligned}
&\mathbb{E}\left\|\Pi_3^x(t_2) - \Pi_3^x(t_1)\right\|^2\\
&\le 3E\left\|\int_0^{t_1}(t_1 - s)^{q-1}(\mathscr{S}(t_2 - s) - \mathscr{S}(t_1 - s))\sigma(s, x(s))dw(s)\right\|^2 + 3E\left\|\int_0^{t_1}((t_2 - s)^{q-1} - (t_1 - s)^{q-1})\right.\\
&\quad \left.\times \mathscr{S}(t_2 - s)\sigma(s, x(s))dw(s)\right\|^2 + 3E\left\|\int_{t_1}^{t_2}(t_2 - s)^{q-1}\mathscr{S}(t_2 - s)\sigma(s, x(s))dw(s)\right\|^2\\
&\le 3L_\sigma\frac{t_1^{2q-1}}{2q-1}\int_0^{t_1} E\|(\mathscr{S}(t_2 - s) - \mathscr{S}(t_1 - s))\sigma(s, x(s))\|^2 ds + 3L_\sigma\left(\int_0^{t_1}((t_2 - s)^{q-1} - (t_1 - s)^{q-1})^2 ds\right)\\
&\quad \times \left(\int_0^{t_1}\|\mathscr{S}(t_2 - s)\sigma(s, x(s))\|^2 ds\right) + 3L_\sigma\frac{(t_2 - t_1)^{2q-1}}{1 - 2q}\left(\frac{Mq}{\Gamma(1+q)}\right)^2\int_{t_1}^{t_2} E\|\mathscr{S}(t_2 - s)\sigma(s, x(s))\|^2 ds.
\end{aligned}$$

Hence using the strong continuity of $\mathscr{S}(t)$ and Lebesgue's dominated convergence theorem, we conclude that the right-hand side of the above inequalities tends to zero as $t_2 - t_1 \to 0$. Thus we conclude $F_\alpha(x)(t)$ is continuous from the right in $[0, b)$. A similar argument shows that it is also continuous from the left in $(0, b]$. This completes the proof of this lemma. □

**Theorem 3.2.** *Assume hypotheses* (i) *and* (ii) *are satisfied. Then the system* (1)–(2) *has a mild solution on* $[0, b]$.

**Proof.** We prove the existence of a fixed point of the operator $F_\alpha$ by using the contraction mapping principle. First, we show that $F_\alpha(H_2) \subset H_2$. Let $x \in H_2$. From (6), we obtain

$$E\|F_\alpha x\|_{H_2}^2 \le 4\left[\sup_{0\le t\le b} E\|\mathscr{T}(t)x_0\|^2 + \sup_{0\le t\le b}\sum_{i=1}^{3} E\|\Pi_i^x(t)\|^2\right]. \quad (7)$$

Using assumptions (i)–(ii) and Lemma 2.10, and standard computations yield

$$\sup_{0\le t\le b} E\|\mathscr{T}(t)x_0\|^2 \le M^2\|x_0\|^2 \quad (8)$$

and

$$\sup_{0\le t\le b}\sum_{i=1}^{3} E\|\Pi_i^x(t)\|^2 \le 3\left(\frac{Mq}{\Gamma(q+1)}\right)^2\left[\frac{b^{2q-1}}{2q-1}\tilde{N} + \frac{b^{2q-1}}{2q-1}\tilde{L}L_\sigma\right](1+\|x\|_{H_2}^2)$$
$$+3\left(\frac{Mq}{\Gamma(q+1)}\right)^2\frac{b^{2q}}{2q-1}\|B\|^2\frac{\hat{M}}{\alpha^2}(1+b\|x\|_{H_2}^2). \quad (9)$$

Hence (8) and (9) together imply that $E\|F_\alpha x\|_{H_2}^2 < \infty$. By Lemma 3.1, $F_\alpha x \in H_2$. Thus for each $\alpha > 0$, the operator $F_\alpha$ maps $H_2$ into itself. Next, we use the Banach fixed point theorem to prove that $F_\alpha$ has a unique fixed point in $H_2$. We claim that there exists a natural $n$ such that $F_\alpha^n$ is a contraction on $H_2$. To see this, let $x, y \in H_2$ and we have

$$\begin{aligned} E\|(F_\alpha x)(t) - (F_\alpha y)(t)\|^2 &\le 3E\sum_{i=1}^{3}\|\Pi_i^x(t) - \Pi_i^y(t)\|^2 \\ &\le 3\left(\frac{Mq}{\Gamma(q+1)}\right)^2\left[\frac{b^{2q-1}}{2q-1}N + \frac{\hat{M}\|B\|^2}{\alpha^2}\frac{b^{2q-1}}{2q-1}b^2 + \frac{b^{2q-1}}{2q-1}LL_\sigma\right]\int_0^t E\|x(s)-y(s)\|^2 ds \\ &\le 3\left(\frac{Mq}{\Gamma(q+1)}\right)^2\left[\frac{b^{2q-1}}{2q-1}N + \frac{\hat{M}\|B\|^2}{\alpha^2}\frac{b^{2q+1}}{2q-1} + \frac{b^{2q-1}}{2q-1}LL_\sigma\right]\int_0^t E\|x(s)-y(s)\|^2 ds. \end{aligned}$$

Hence, we obtain a positive real constant $\gamma(\alpha)$ such that

$$E\|(F_\alpha x)(t) - (F_\alpha y)(t)\|^2 \le \gamma(\alpha)\int_0^t E\|x(s)-y(s)\|^2 ds \quad (10)$$

for all $t \in J$ and for any $x, y \in H_2$. For any natural number $n$, it follows from successive iteration of above inequality that, by taking the supremum over $[0, b]$,

$$\|(F_\alpha^n x)(t) - (F_\alpha^n y)(t)\|_{H_2}^2 \le \frac{(b\gamma(\alpha))^n}{n!}\|x-y\|_{H_2}^2. \quad (11)$$

For any fixed $\alpha > 0$, for sufficiently large $n$, $\frac{(b\gamma(\alpha))^n}{n!} < 1$. It follows from (11) that $F_\alpha^n$ is a contraction mapping, so that the contraction principle ensures that the operator $F_\alpha$ has a unique fixed point $x_\alpha$ in $H_2$, which is a mild solution of (1)–(2). □

**Theorem 3.3.** *Assume that the assumptions* (i)–(iii) *hold. Further, if the functions $f$ and $\sigma$ are uniformly bounded and $\{\mathscr{S}(t) : t \ge 0\}$ is compact, then the system* (1)–(2) *is approximately controllable on* $[0, b]$.

**Proof.** Let $x_\alpha$ be a fixed point of $F_\alpha$. By using the stochastic Fubini theorem, it can be easily seen that

$$x_\alpha(b) = \tilde{x}_b - \alpha(\alpha I + \Psi)^{-1}(E\tilde{x}_b - \mathscr{T}(b)x_0) + \alpha\int_0^b(\alpha I + \Psi_s^b)^{-1}(b-s)^{q-1}\mathscr{S}(b-s)f(s, x_\alpha(s))ds$$
$$+\alpha\int_0^b(\alpha I + \Psi_s^b)^{-1}[(b-s)^{q-1}\mathscr{S}(b-s)\sigma(s, x_\alpha(s)) - \tilde{\phi}(s)]dw(s).$$

It follows from the assumption on $f$ and $\sigma$ that there exists $\hat{D} > 0$ such that

$$\|f(s, x_\alpha(s))\|^2 + \|\sigma(s, x_\alpha(s))\|^2 \le \hat{D} \quad (12)$$

for all $(s, \omega) \in [0, b] \times \Omega$. Then there is a subsequence still denoted by $\{f(s, x_\alpha(s)), \sigma(s, x_\alpha(s))\}$ which converges to weakly to, say, $\{f(s), \sigma(s)\}$ in $X \times L_2^0$.

From the above equation, we have

$$\begin{aligned}E\|x_\alpha(b)-\tilde{x}_b\|^2 \le\ & 6\|\alpha(\alpha I+\Psi_0^b)^{-1}(E\tilde{x}_b-\mathscr{T}(b)x_0)\|^2+6E\left(\int_0^b (b-s)^{q-1}\|\alpha(\alpha I+\Psi_s^b)^{-1}\tilde{\phi}(s)\|_{L_2^0}^2 ds\right)\\ &+6E\left(\int_0^b (b-s)^{q-1}\|\alpha(\alpha I+\Psi_s^b)^{-1}\|\,\|\mathscr{S}(b-s)(f(s,x_\alpha(s))-f(s))\|ds\right)^2\\ &+6E\left(\int_0^b (b-s)^{q-1}\|\alpha(\alpha I+\Psi_s^b)^{-1}\mathscr{S}(b-s)f(s)\|ds\right)^2\\ &+6E\left(\int_0^b (b-s)^{q-1}\|\alpha(\alpha I+\Psi_s^b)^{-1}\|\,\|\mathscr{S}(b-s)(\sigma(s,x_\alpha(s))-\sigma(s))\|_{L_2^0}^2 ds\right)\\ &+6E\left(\int_0^b (b-s)^{q-1}\|\alpha(\alpha I+\Psi_s^b)^{-1}\mathscr{S}(b-s)\sigma(s)\|_{L_2^0}^2 ds\right).\end{aligned}$$

On the other hand, by assumption (iii), for all $0 \le s < b$ the operator $\alpha(\alpha I+\Psi_s^b)^{-1} \to 0$ strongly as $\alpha \to 0^+$ and moreover $\|\alpha(\alpha I+\Psi_s^b)^{-1}\| \le 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of $\mathscr{S}(t)$ implies that $E\|x_\alpha(b)-\tilde{x}_b\|^2 \to 0$ as $\alpha \to 0^+$. This gives the approximate controllability of (1)–(2). □

**Example 3.4.** To illustrate the theoretical result established in the preceding Theorem, we consider the following fractional stochastic control system of the form

$$\begin{aligned}{}^cD_t^q x(t,z) &= \frac{\partial^2 x(t,z)}{\partial z^2}+\mu(t,z)+\hat{f}(t,x(t,z))+\hat{\sigma}(t,x(t,z))\frac{d\hat{w}(t)}{dt}, \qquad (13)\\ x(t,0) &= x(t,1)=0, \quad t\in[0,b],\\ x(0,z) &= x_0(z), \quad z\in[0,1],\end{aligned}$$

where $b > 0$, $0 < q < 1$; $\hat{w}(t)$ is a two sided and standard one dimensional Brownian motion defined on the filtered probability space $(\Omega, \Gamma, P)$. To write the above system into the abstract form of (1), let $X = E = U = L^2[0,1]$. Define the operator $A : L^2[0,1] \to L^2[0,1]$ by $Aw = w''$ with domain

$$D(A) = \{w \in X;\ w, w' \text{ are absolutely continuous}, w'' \in X \text{ and } w(0)=w(1)=0\}.$$

$$Aw = \sum_{n=1}^{\infty} n^2(w, w_n)w_n, \quad w \in D(A),$$

where $w_n(s) = \sqrt{2}\sin(ns)$, $n = 1, 2, \ldots$ is the orthogonal set of eigenvectors in $A$. It is well known that $A$ generates a compact, analytic semigroup $\{S(t), t \ge 0\}$ in $X$ and

$$S(t)w = \sum_{n=1}^{\infty} e^{-n^2 t}(w, w_n)w_n, \quad \|S(t)\| \le e^{-t} \text{ for all } t \ge 0.$$

Especially, the operator $A^{1/2}$ is given by

$$A^{1/2}w = \sum_{n=1}^{\infty} n(w, w_n)w_n$$

with the domain $D(A^{1/2}) = \{w \in X : \sum_{n=1}^{\infty} n(w, w_n)w_n\}$.

Define $x(t)(z) = x(t,z), f(t,x(t))(z) = \hat{f}(t,x(t,z))$ and $\sigma(t,x(t))(z) = \hat{\sigma}(t,x(t,z))$. Define the bounded linear operator $B : U \to X$ by $Bu(t)(z) = \mu(t,z)$, $0 \le z \le 1$, $u \in U$. On the other hand, it can be easily seen that the deterministic linear fractional control system corresponding to (13) is approximately controllable on $[0,1]$ [33]. Therefore, with the above choices, the system (13) can be written to the abstract form (1)–(2) and all the conditions of Theorem 3.3 are satisfied. Thus by Theorem 3.3, fractional stochastic control system (13) is approximately controllable on $[0,1]$.

**Remark 3.5.** Theory and applications of delay differential equations form an important part of modern nonlinear dynamics. In recent years functional differential equations have been used to model processes in diverse areas such as population dynamics and ecology, physiology and medicine, economics and other natural sciences. However, in applied areas, deterministic systems fail to capture the essence of the fluctuations in the real situation, and one must instead consider models with stochastic processes. In many of the models the initial data and parameters are subject to random perturbations, or the dynamical systems themselves represent stochastic processes. This leads to consideration of stochastic delay differential equations. Thus, in real world problems, stochastic models with delays are important. However, to the best of our knowledge, no results yet exist on approximate controllability for fractional stochastic delay differential equations. Upon making some appropriate assumptions, by employing the ideas and techniques as in this paper, one can establish the controllability results for a class of fractional stochastic delay differential equations.

**Remark 3.6.** System (1)–(2) is exactly controllable on $[0, b]$ if $\Re(b) = L^2(\Omega, \Gamma_b, X)$. The linear fractional stochastic control system

$$\begin{aligned} &{}^cD_t^q x(t) = Ax(t) + (Bu)(t) + \sigma(t)\frac{dw(t)}{dt}, \quad t \in [0, b], \\ &x(0) = x_0 \end{aligned} \tag{14}$$

corresponding to (1)–(2) is exactly controllable on $[0, b]$. Note that, in this case the operator associated to linear stochastic system is defined by $\Pi_0^b(\cdot) = \int_0^b \mathscr{S}(b-t)BB^*\mathscr{S}^*(b-t)E\{\cdot|\Gamma_t\}dt$, is bounded invertible; that is there exists $\gamma > 0$ such that $E\|(\Pi_0^b)^{-1}\|^2 \leq \gamma^2$. It should be mention that in order to prove the exact controllability result, the compactness assumption on semigroup $\mathscr{S}(t)$ is not necessary.

**Corollary 3.7.** *Assume assumptions* (i) *and* (ii) *hold and the linear stochastic system is exactly controllable on all* $[0, t]$, $t > 0$. *If* $3\left(\frac{Mq}{\Gamma(q+1)}\right)^2 b\left[\frac{b^{2q-1}}{2q-1}N + \frac{\hat{M}\|B\|^2}{\gamma^2}\frac{b^{2q+1}}{2q-1} + \frac{b^{2q-1}}{2q-1}LL_\sigma\right] < 1$, *then the nonlinear fractional stochastic control system* (1)–(2) *is exactly controllable on* $[0, b]$.

**Proof.** By employing the steps in Theorem 3.2 with some changes and using Banach fixed point theorem one can easily show that the system (1)–(2) is exactly controllable. In order to prove the exact controllability result, the invertibility of the controllability operator will be considered. Now, we define the control function by

$$\begin{aligned} u(t, x) = B^*(b-t)^{q-1}\mathscr{S}^*(b-t)E\Big\{&(\Pi_0^b)^{-1}(\tilde{x_b} - \mathscr{T}(b)x_0 - \int_0^t (b-s)^{q-1}\mathscr{S}(b-s)f(s, x(s))ds \\ &- \int_0^t (b-s)^{q-1}\mathscr{S}(b-s)\sigma(s, x(s))dw(s))|\Gamma_t\Big\} \end{aligned}$$

and define the operator $F : H_2 \to H_2$ by

$$(Fx)(t) = \mathscr{T}(t)x_0 + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)[f(s, x(s)) + Bu(s, x)]ds + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\sigma(s, x(s))dw(s).$$

By considering the similar steps as in the proof of Theorem 3.2, we obtain

$$\begin{aligned} E\|(Fx)(t) - (Fy)(t)\|^2 &\leq 3E\sum_{i=1}^3 \|\Pi_i^x(t) - \Pi_i^y(t)\|^2 \\ &\leq 3\left(\frac{Mq}{\Gamma(q+1)}\right)^2\left[\frac{b^{2q-1}}{2q-1}N + \frac{\hat{M}\|B\|^2}{\gamma^2}\frac{b^{2q-1}}{2q-1}b^2 + \frac{b^{2q-1}}{2q-1}LL_\sigma\right]\int_0^t E\|x(s) - y(s)\|^2 ds \\ &\leq 3\left(\frac{Mq}{\Gamma(q+1)}\right)^2 b\left[\frac{b^{2q-1}}{2q-1}N + \frac{\hat{M}\|B\|^2}{\gamma^2}\frac{b^{2q+1}}{2q-1} + \frac{b^{2q-1}}{2q-1}LL_\sigma\right]\|x - y\|^2. \end{aligned}$$

Thus, assumption of this theorem allows us to conclude in view of the contraction mapping principle, that $F$ has a unique fixed point $x \in H_2$. Further $x(b) = \tilde{x_b}$. Thus, the system (1)–(2) is exactly controllable on $[0, b]$. □


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