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Symmetric Functions and Koszul Complexes

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The aim of this note is to give a representation-theoretic justification of some well-known identities in the theory of symmetric functions. Our main idea is to use basic properties of Koszul complexes.

To begin with let us consider a vector space U over a field K of characteristic zero and the symmetric algebra $S.(U)$ on U. It is well known that the Koszul complex corresponding to the canonical inclusion $U \hookrightarrow S.(U)$ gives rise to a free resolution of the residue field K over $S.(U)$, i.e., we have an exact sequence

$$\cdots \to \Lambda^p(U) \otimes S.(U) \to \cdots \to U \otimes S.(U) \to S.(U) \to K \to 0.$$

By taking the trace of the obvious action of $C \in Gl(U)$ on both $S.(U)$ and $\Lambda^{\cdot}(U)$ we get

$$\left(\sum_{i \geqslant 0} s_i\right)\left(\sum_{j=0}^{rk\,U} (-1)^j \lambda^j\right) = 1$$

where s_i is the ith complete homogeneous symmetric function in the eigenvalues of C and λ^j is the jth elementary symmetric function. By taking U of arbitrarily high rank we get the Newton identities

$$\sum_{j=0}^{n} (-1)^j \lambda^j(x)\, s_{n-j}(x) = 0, \qquad n \geqslant 1, \tag{1}$$

for symmetric functions in infinitely many indeterminates $x = (x_1, x_2, \ldots)$.

The same idea applies to other representations of $Gl(U)$ and gives representation-theoretic interpretation of the following classical formulas. As above all symmetric functions involved depend on infinitely many indeterminates.

Replacing U by $\Lambda^2(U)$ we have

$$\left(\sum_I s_I\right)\left(\sum_J (-1)^{|J|/2} s_J\right) = 1 \tag{2}$$

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where $|I|$ is the weight of the partition I, $\tilde{I}$ is its conjugate partition and I ranges over all partitions such that $\tilde{I}$ has even parts; J ranges over all partitions of the form $(p_1,..., p_r | p_1 + 1,..., p_r + 1)$ in Frobenius notation and s_I is the Schur function corresponding to the partition I. The left-hand factor in (2) is the trace of the action of $Gl(U)$ on $S.(\Lambda^2(U))$ and the right-hand one is the alternating sum corresponding to the trace of $\Lambda^{\cdot}(\Lambda^2(U))$ (see Littlewood [3]). Therefore the formula (2) implies another identity

$$\prod_{i<j} (1 - x_i x_j) = \sum_J (-1)^{|J|/2} s_J$$

where J ranges over the same set of partitions as in (2). Similarly replacing U by $S_2(U)$ we have

$$\left(\sum_I s_I\right)\left(\sum_J (-1)^{|J|/2} s_J\right) = 1 \qquad (3)$$

where I ranges over all partitions with even parts and J ranges over all partitions $(p_1 + 1,..., p_r + 1 | p_1,..., p_r)$. Consequently

$$\prod_i (1 - x_i^2) \prod_{i<j} (1 - x_i x_j) = \sum_J (-1)^{|J|/2} s_J$$

where J ranges over the same set of partitions as in (3). The above identities are due to Littlewood [3, p. 238], see also Macdonald [4, p. 46] for another proof using Weyl's identity for root-systems of classical groups.

Note that both (2) and (3) split into a family of identities similar to (1) because the Koszul complex splits into a sum of homogeneous components, e.g., the formula (2) gives rise to identities indexed by natural numbers

$$\sum_{(|I| + |J|)/2 = n} (-1)^{|J|/2} s_I s_J = 0 \qquad \text{for } n \geqslant 1,$$

where I and J range as in (2).

In the same way we can interpret the Koszul complex of the canonical inclusion $E \otimes F \hookrightarrow S.(E \otimes F)$ over the symmetric algebra $S.(E \otimes F)$ where E, F are vector spaces. In this case we have by the Cauchy formula

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_I s_I(x)\, s_I(y)$$

summed over all partitions I. Therefore the inverse is

$$\prod_{i,j} (1 - x_i y_j) = \sum_J (-1)^{|J|} s_J(x)\, s_{\tilde{J}}(y)$$

where the summation ranges over all partitions J.

This note was inspired by a question raised by R. Stanley about a representation-theoretic interpretation of another Littlewood formula

$$\prod_i (1-x_i) \prod_{i<j} (1-x_i x_j) = \sum_I (-1)^{(|I|+r(I))/2} s_I \tag{4}$$

where the summation ranges over all self-conjugate partitions I and $r(I)$ is the rank of I, i.e., the length of the main diagonal of its diagram.

The inverse of the product in (4) can be expressed (see [3, p. 238] and a note in [4, p. 54]) in the form

$$\prod_i (1-x_i)^{-1} \prod_{i<j} (1-x_i x_j)^{-1} = \sum_I s_I$$

summed over all partitions I. Observe that this product is the trace of the action of $Gl(U)$ on the algebra $S.(U+\Lambda^2(U))$. By taking the Koszul complex associated to the canonical inclusion $U+\Lambda^2(U) \hookrightarrow S.(U+\Lambda^2(U))$ over $S.(U+\Lambda^2(U))$ we get the formula

$$\left(\sum_I s_I\right)\left(\sum_{k,J} (-1)^{k+|J|/2} \lambda^k s_J\right) = 1$$

where the summation ranges over all non-negative integers k and all partitions J from a set A. The set A consists of all partitions of the form $(p_1,\dots,p_r|p_1+1,\dots,p_r+1)$, $p_1>p_2>\cdots>p_r>0$, and the zero partition 0; note that $s_0=1$. Hence

$$\prod_i (1-x_i) \prod_{i<j} (1-x_i x_j) = \sum_{k,J} (-1)^{k+|J|/2} \lambda^k s_J \tag{5}$$

with k, J as above. We shall prove that the right-hand side of this formula is in fact the right-hand side of (4).

When dealing with partitions we use the same conventions and terminology as in [2]. In particular, partitions are weakly decreasing sequences of non-negative integers and the diagram of a partition $I=(i_1, i_2,\dots, i_m)$ starts with the bottom row of length i_1 and goes upward. For example,

$$\begin{array}{lll} \times & & \\ \times & \times & \\ \times & \times & \\ \times & \times & \times \end{array} \tag{6}$$

is the diagram of the partition (3, 2, 2, 1).

Using the Pieri formula we can express the right-hand side of (5) as an alternating sum of Schur functions. We prove that

(i) each Schur function corresponding to a self-conjugate partition can be obtained in one way only from products of type $\lambda^k s_J$, $J \in A$, and

(ii) other summands of $\lambda^k s_J$, $J \in A$, can be grouped in pairs in such a way that they cancel in the right-hand side of (5).

LEMMA. *The Schur function $s_{(a|b)}$ is a summand of some $\lambda^k s_J$, $J \in A$, in the following cases only*:

(0) $b = a > 0$, *a summand of* $\lambda s_{(a-1|b)}$,

(1) $b = a + p, p \geqslant 1, a > 0$, *a summand of* $\lambda^{p-1} s_{(a|b-p+1)}$,

(1*) $b = a + p, p \geqslant 1, a > 0$, *a summand of* $\lambda^{p+1} s_{(a-1|b-p)}$,

(2) $b = p \geqslant 2$, $a = 0$, *a summand of* $\lambda^{p-2} s_{(0|1)}$,

(2*) $b = p \geqslant 2, a = 0$, *a summand of* $\lambda^p s_0$.

A proof of the lemma follows immediately from the Pieri formula and the definition of A.

If I is a hook we say that a summand s_I of $\lambda^k s_J$, $J \in A$, is of type 0, 1, 1*, 2, 2*, respectively, depending on a case listed in the lemma. Now the lemma implies that the only Schur functions indexed by hooks that do appear in the right-hand side of (5) are of type 0 (self-conjugate) and that summands indexed by the same hooks of type 1, 1*, and 2, 2*, respectively, cancel in (5).

In order to pass from hooks to arbitrary partitions we define a type of the summand s_I of $\lambda^k s_J$, $J \in A$, for any I. By the Pieri formula the diagram of I is obtained from that of J by adding k squares in such a way that there is at most one added in each row. If we view I as a sequence of its hooks we are adding squares in this process to consecutive hooks thus determining types of these hooks according to the list in the lemma. We define the type of s_I in $\lambda^k s_J$, $J \in A$, to be the sequence of types of its hooks.

For example, $s_{(4,3,0|6,3,2)}$ is a summand of $\lambda^5 s_{(3,2,0|4,3,1)}$ and is obtained as illustrated in the following diagram:

□
□
x x □
x x x (7)
x x x
x x x x □
x x x x □

Its type is [1*, 0, 2].

Suppose that s_I is a summand of $\lambda^k s_J$, $J \in A$, of type $[p_1,\dots,p_r]$, $p_i = 0$, 1, 1*, 2, 2*. We adopt the convention $1^{**} = 1$, $2^{**} = 2$. Let us consider the case when at least one p_i is different from zero and let t be the smallest index such that $p_t \neq 0$. We claim that there exists a summand s_I of $\lambda^{k'} s_{J'}$, $J' \in A$, of type $[p_1,\dots,p_{t-1}, p_t^*, p_{t+1},\dots,p_r]$. Indeed, it is enough to replace the procedure p_t by p_t^* in the tth hook of s_I in $\lambda^k s_J$. A simple inspection shows that it is always possible. The above correspondence defines an involutive endomorphism on the set of all summands of $\lambda^k s_J$, $J \in A$, of type different from $(0, 0,\dots, 0)$. Since the signs by corresponding summands are opposite these summands cancel in the sum (5).

For example, to the summand (7) of type $[1^*, 0, 2]$ appearing in (5) with the sign "minus" there corresponds the summand of type $[1, 0, 2]$ determined by the diagram

□
x
x x □
x x x
x x x
x x x x □
x x x x x

and appearing with the sign "plus" in (5).

Summands of type $(0,\dots, 0)$ in (5) are indexed by self-conjugate partitions and each appears only once as a summand of some $\lambda^k s_J$, $J \in A$. More precisely, if $I = (i_1,\dots, i_r | i_1,\dots, i_r)$ and $i_r \neq 0$ then s_I is a summand of $\lambda^r s_{(i_1-1,\dots,i_r-1 | i_1,\dots,i_r)}$ and its sign in (5) is determined by $r + \sum i_k = r(I) + (|I| - r(I))/2 = (r(I) + |I|)/2$. If $I = (i_1,\dots, i_{r-1}, 0 | i_1,\dots, i_{r-1}, 0)$ then s_I is a summand of $\lambda^r s_{(i_1-1,\dots,i_{r-1}-1 | i_1,\dots,i_{r-1})}$ and appears in (5) with the same sign.

Observe that unlike in the previous formulas the right-hand side of (4) cannot be interpreted as an alternating sum of traces corresponding to some exact complex over $S.(U + \Lambda^2(U))$. Although we start with the Koszul complex some summands cancel on the function level but this cannot be done on the level of representations as is easy to prove by simple combinatorial arguments.

Finally, we would like to prove by similar methods one more formula which seems to be new. We claim that

$$\prod_i (1 + x_i)^{-1} \prod_{i \leqslant j} (1 + x_i x_j)^{-1} = \sum_{I \in B} (-1)^{(|I| + p(I))/2} s_I \tag{8}$$

where $p(I)$ is the number of odd parts of I and B is the set of all partitions I satisfying

$$i_k - i_{k+1} \equiv 0 \text{ or } 1 \pmod 4 \qquad \text{for } i_k \text{ even,}$$
$$i_k - i_{k+1} \equiv 1 \text{ or } 2 \pmod 4 \qquad \text{for } i_k \text{ odd.}$$

We set $i_s = 0$ for $s >$ length I.

Our starting point is an observation that $\prod_i (1+x_i) \prod_{i \leqslant j} (1+x_i x_j)$ is the trace associated with the exterior algebra $\Lambda^{\cdot}(U + S_2(U))$. We are going to construct a free resolution of K over $\Lambda^{\cdot}(U+S_2(U))$ to compute its inverse. This is a construction that is very similar to that of the Koszul complex. This time, however, our basic ring is not commutative but skew-commutative. For any vector space W over a field K of characteristic zero we have an infinite complex of free $\Lambda^{\cdot}(W)$-modules

$$\cdots \to S_p(W) \otimes \Lambda^{\cdot}(W) \xrightarrow{d_p} \cdots \to S_2(W) \otimes \Lambda^{\cdot}(W) \xrightarrow{d_2} W \otimes \Lambda^{\cdot}(W) \xrightarrow{d_1} \Lambda_{\cdot}(W)$$

which is a minimal free $\Lambda^{\cdot}(W)$-resolution of K. The differential d_p is defined by the formula

$$d_p(w_1 \cdots w_p \otimes a) = \sum_{i=1}^{p} w_1 \cdots \hat{w}_i \cdots w_p \otimes w_i a$$

for $w_i \in W$, $a \in \Lambda^{\cdot}(W)$ where the sign $\hat{}$ means that the corresponding element is to be omitted. One can easily check that the complex is indeed acyclic. We send the reader to [1, p. 226] for a proof in the context of Tate resolutions of commutative graded algebras.

Taking $W = U + S_2(U)$ and passing to traces we obtain

$$\prod_i (1+x_i)^{-1} \prod_{i \leqslant j} (1+x_i x_j)^{-1} = \sum_{k,J} (-1)^{k+|J|/2} s_k s_J \tag{9}$$

where J ranges over all partitions appearing in the decomposition of $S_{\cdot}(S_2(U))$, i.e., once again by one of Littlewood formulas J ranges over all partitions with even parts.

We prove that the right-hand sides of (8) and (9) are equal. To this end we compute products of the type $s_k s_J$ where J has even parts. By the Pieri formula such a product is a sum of Schur functions indexed by partitions obtained from J by adding k squares in such a way that there is at most one added in each column. Let us start with a one-row partition. If the length of the row is of the form $4m$ or $4m+1$ then the corresponding Schur function appears as a summand in products $s_k s_J$ exactly $2m+1$ times, e.g., the Schur function s_5 is a summand of $s_1 s_4$, $s_3 s_2$ and $s_5 s_0$. All these summands but one cancel in the right-hand side of (9). The corresponding Schur function appears in (9) with coefficient ∓ 1. If the length of the row of a one-row partition is of the form $4m+2$ or $4m+3$, then the corresponding Schur function appears as a summand in products $s_k s_j$ exactly $2m+2$ times. In this case all the summands cancel in the right-hand side of (9).

A simple combinatorial analysis similar to the one explained for one-row partitions shows that the Schur function s_I appears in the right-hand side of (9) an odd number of times if and only if $I \in B$. All the other Schur functions either appear an even number of times and cancel in (9) or do not appear at all.

Let us illustrate this on an example of a two-row partition $I = (9, 4)$. We must count in how many ways we can remove squares from the diagram of I in such a way that at most one in each column is removed and the resulting diagram has even rows. Here are all possibilities.

$$
-\left\{\begin{array}{lllllllll}
\mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & & & & & \\
\mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \square
\end{array}\right.
$$

$$
+\left\{\begin{array}{lllllllll}
\mathrm{x} & \mathrm{x} & \square & \square & & & & & \\
\mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \square \\
\mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & & & & & \\
\mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \square & \square & \square
\end{array}\right.
$$

$$
-\left\{\begin{array}{lllllllll}
\square & \square & \square & \square & & & & & \\
\mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \square \\
\mathrm{x} & \mathrm{x} & \square & \square & & & & & \\
\mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \square & \square & \square \\
\mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & & & & & \\
\mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \square & \square & \square & \square & \square
\end{array}\right.
$$

$$
+\left\{\begin{array}{lllllllll}
\square & \square & \square & \square & & & & & \\
\mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \square & \square & \square \\
\mathrm{x} & \mathrm{x} & \square & \square & & & & & \\
\mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \square & \square & \square & \square & \square
\end{array}\right.
$$

$$
-\left\{\begin{array}{lllllllll}
\square & \square & \square & \square & & & & & \\
\mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \square & \square & \square & \square & \square
\end{array}\right.
$$

All together we have 9 possibilities which can be grouped according to the number of squares that are removed. These groups appear in the right-hand side of (9) with the indicated signs and all the summands but the first cancel. Notice that the number 9 is obtained here as a product corresponding to 3 possibilities of removing squares from the first row and 3 possibilities of removing squares from the second one in such a way that there are no overlaps in columns (and this depends only on the difference $i_1 - i_2$.) For general $I \in B$ the number of times s_I appears in (9) is odd and is a product (indexed by rows) of numbers expressing all possibilities of removing squares from the part of the mth row that does not overlap with the $(m+1)$th row (so depending on $i_m - i_{m+1}$ squares), for all m.

If $I \in B$ then s_I is a summand of $s_p s_J$ where $p = p(I)$ is the number of odd parts of I and J is obtained from I by removing one square from each row of I of odd length. This is always possible since for $I \in B$ $i_m \neq i_{m+1}$ if i_m is odd. Only summands of this type stay in (9) after cancellation of other terms. Therefore the sign by s_I is equal to $p(I) + (|I| - p(I))/2 = (|I| + p(I))/2$.

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