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On the Calabi–Yau problem for maximal surfaces in $\mathbb{L}^3$ ☆

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ABSTRACT

In this paper we construct an example of a maximal surface in the Lorentz–Minkowski space $\mathbb{L}^3$, which is bounded by a hyperboloid and weakly complete in the sense explained by Umehara and Yamada [M. Umehara, K. Yamada, Maximal surfaces with singularities in Minkowski space, Hokkaido Math. J. 35 (2006) 13–40].

1. Introduction

A maximal hypersurface in a Lorentzian manifold is a spacelike hypersurface with zero mean curvature. Besides of their mathematical interest these hypersurfaces and more generally those having constant mean curvature have a significant importance in physics [9,10,12]. When the ambient space is the Minkowski space $\mathbb{L}^n$, one of the most important results is the proof of a Bernstein-type theorem for maximal hypersurfaces in $\mathbb{L}^n$. Calabi [4] proved that the only complete hypersurfaces with zero mean curvature in $\mathbb{L}^3$ (i.e. maximal surfaces) and $\mathbb{L}^4$ are spacelike hyperplanes, solving the so called Bernstein-type problem in dimensions 3 and 4. Cheng and Yau [6] extended this result to $\mathbb{L}^n$, $n \geqslant 5$. It is therefore meaningless to consider global problems on maximal and everywhere regular hypersurfaces in $\mathbb{L}^n$. In contrast, there exists a lot of results about existence of non-flat maximal surfaces with singularities [7,8,11].

It is well known the close relationship between maximal surfaces in $\mathbb{L}^3$ and minimal surfaces in $\mathbb{R}^3$ (see Remark 1 in Section 2.2). This fact lets us solve some problems on maximal surfaces by solving the analogous ones for minimal surfaces, and vice versa. This is not the case of the Calabi–Yau problem. In 1965 Calabi asked whether or not it is possible for a complete minimal surface in $\mathbb{R}^3$ to be bounded. Much work has been done on it over the past four decades. The most important result in this line was obtained by Nadirashvili [16], who constructed a complete minimal surface in the unit ball of $\mathbb{R}^3$. See [2] for more information about this topic. From a Nadirashvili's surface and using the relationship between maximal and minimal surfaces, we can obtain as most the existence of a weakly complete maximal surface contained in a cylinder of $\mathbb{L}^3$. Here, we use the concept of weakly completeness (see Definition 2 in Section 2.2) that was introduced by Umehara and Yamada [18].

In this paper, we construct an example of a weakly complete maximal surface in $\mathbb{L}^3$ with singularities, which is bounded by a hyperboloid. We would like to point out that our example does not have branch points, all the singularities are of lightlike type (see Definition 1 in Section 2.2).

More precisely, we prove the following existence theorem.

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Theorem 1. *There exists a weakly complete conformal maximal immersion with lightlike singularities of the unit disk into the set* $\{(x,y,z)\in\mathbb{L}^3 \mid x^2+y^2-z^2<-1\}$.

For several reasons, lightlike singularities of maximal surfaces in $\mathbb{L}^3$ are specially interesting. This kind of singularities are more attractive than branch points, in the sense that they have a physical interpretation [9,10]. At these points, the limit tangent plane is lightlike, the curvature blows up and the Gauss map has no well defined limit. However, as in the case of minimal surfaces, if we allow branch points, then proving the analogous result of Theorem 1 has less technical difficulties.

The fundamental tools used in the proof of this result (Runge's theorem and the López–Ros transformation) are those that Nadirashvili utilized to construct the first example of a complete bounded minimal surface in $\mathbb{R}^3$. Improvements of his technique have generated a lot of literature on the Calabi–Yau problem for minimal surfaces in $\mathbb{R}^3$ [3,14,15].

Similarly to the case of minimal surfaces, it would be stimulating to look for an additional property for a weakly complete bounded by a hyperboloid maximal surface: properness in the hyperboloid. In order to achieve it, the technique showed in this paper could be combined with the reasonings used in the construction [1] of a proper conformal maximal disk in $\mathbb{L}^3$, following the ideas of [13]. The main objection of this argument is that the sharpest result known about the convex hull property for maximal surfaces [5] needs the control of the image of the singularities of the surface.

2. Background and notation

2.1. The Lorentz–Minkowski three space

We denote by $\mathbb{L}^3$ the three dimensional Lorentz–Minkowski space $(\mathbb{R}^3,\langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle=dx_1^2+dx_2^2-dx_3^2$. The Lorentzian *norm* is given by $\|(x_1,x_2,x_3)\|^2=x_1^2+x_2^2-x_3^2$, and $\|x\|=\operatorname{sign}(\|x\|^2)\sqrt{|\|x\|^2|}$. We say that a vector $v\in\mathbb{R}^3\setminus\{(0,0,0)\}$ is spacelike, timelike or lightlike if $\|v\|^2$ is positive, negative or zero, respectively. The vector $(0,0,0)$ is spacelike by definition. A plane in $\mathbb{L}^3$ is spacelike, timelike or lightlike if the induced metric is Riemannian, non-degenerate and indefinite or degenerate, respectively.

In order to differentiate between $\mathbb{L}^3$ and $\mathbb{R}^3$, we denote $\mathbb{R}^3=(\mathbb{R}^3,\langle\cdot,\cdot\rangle_0)$, where $\langle\cdot,\cdot\rangle_0$ is the usual metric of $\mathbb{R}^3$, i.e., $\langle\cdot,\cdot\rangle_0=dx_1^2+dx_2^2+dx_3^2$. We also denote the Euclidean norm by $\|\cdot\|_0$.

By an (ordered) $\mathbb{L}^3$-orthonormal basis we mean a basis of $\mathbb{R}^3$, $\{u,v,w\}$, satisfying

- $\langle u,v\rangle=\langle u,w\rangle=\langle v,w\rangle=0$;
- $\|u\|=\|v\|=-\|w\|=1$.

Notice that u and v are spacelike vectors whereas w is timelike.

We call $\mathbb{H}^2:=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1^2+x_2^2-x_3^2=-1\}$ the hyperbolic sphere in $\mathbb{L}^3$ of constant intrinsic curvature -1. Notice that $\mathbb{H}^2$ has two connected components $\mathbb{H}^2_+:=\mathbb{H}^2\cap\{x_3\geqslant 1\}$ and $\mathbb{H}^2_-:=\mathbb{H}^2\cap\{x_3\leqslant -1\}$. The stereographic projection η for $\mathbb{H}^2$ from the point $(0,0,1)\in\mathbb{H}^2_+$ is the map $\eta:\mathbb{H}^2\to\mathbb{C}\cup\{\infty\}\setminus\{|z|=1\}$ given by

$$\eta(x_1,x_2,x_3)=\frac{x_1+\mathrm{i}x_2}{1-x_3},\qquad \eta(0,0,1)=\infty.$$

Notice that $\eta(\mathbb{H}^2_+)=\{|z|>1\}$ and $\eta(\mathbb{H}^2_-)=\{|z|<1\}$.

Given $r\geqslant 0$, we denote by $\mathrm{B}(r)$ as the lower convex domain determined by the set $\{\|x\|=-r\}$, i.e.,

$$\mathrm{B}(r)=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid \left\|(x_1,x_2,x_3)\right\|<-r,\ x_3<-r\right\}.$$

We also denote $\mathrm{b}(r)=\partial\mathrm{B}(r)$. Observe that $\mathrm{b}(1)=\mathbb{H}^2_-$. Moreover, if $r_1<r_2$, then $\overline{\mathrm{B}(r_2)}\subset\mathrm{B}(r_1)$ and $\mathrm{b}(r_1)\cap\mathrm{b}(r_2)=\emptyset$.

Finally, we define the maps $\mathcal{N}:\mathrm{B}(0)\to\mathbb{H}^2_+$ and $\mathcal{N}_0:\mathrm{B}(0)\to\mathbb{S}^2$ by the following way. Consider $p\in\mathrm{B}(0)$ and label $r=-\|p\|>0$. Let $\mathcal{N}^r:\mathrm{b}(r)\to\mathbb{H}^2_+$ and $\mathcal{N}_0^r:\mathrm{b}(r)\to\mathbb{S}^2$ be the outward pointing $\mathbb{L}^3$-normal Gauss map and the Euclidean outward pointing unit normal of $b(r)$, respectively. Then, we define

$$\mathcal{N}(p)=\mathcal{N}^r(p),\qquad \mathcal{N}_0(p)=\mathcal{N}_0^r(p).$$

Equivalently, $\mathcal{N}(p)=-p/\|p\|$ and $\mathcal{N}_0(p)=\mathcal{J}(p)/\|p\|_0$, where $\mathcal{J}(p_1,p_2,p_3)=(p_1,p_2,-p_3)$. Hence, both maps are differentiable and $\mathcal{N}_0(p)=-\mathcal{J}(\mathcal{N}(p))/\|\mathcal{N}(p)\|_0$.

2.2. Maximal surfaces

Any conformal maximal immersion $X:M\to\mathbb{L}^3$ is given by a triple $\Phi=(\Phi_1,\Phi_2,\Phi_3)$ of holomorphic 1-forms defined on the Riemann surface M, having no common zeros and satisfying

$$|\Phi_1|^2+|\Phi_2|^2-|\Phi_3|^2\neq 0; \tag{1}$$

$$\Phi_1^2+\Phi_2^2-\Phi_3^2=0; \tag{2}$$

and all periods of the Φ_j are purely imaginary. Here we consider Φ_i to be a holomorphic function times dz in a local parameter z. Then, the maximal immersion $X: M \to \mathbb{L}^3$ can be parameterized by $z \mapsto \operatorname{Re} \int^z \Phi$. The above triple is called the Weierstrass representation of the maximal immersion X. Usually, the second requirement (2) is guaranteed by the introduction of the formulas

$$\Phi_1 = \frac{\mathrm{i}}{2}(1-g^2)\eta, \qquad \Phi_2 = -\frac{1}{2}(1+g^2)\eta, \qquad \Phi_3 = g\eta$$

for a meromorphic function g with $|g(p)| \neq 1$, $\forall p \in M$ (the stereographically projected Gauss map) and a holomorphic 1-form η. We also call (g, η) or (g, Φ_3) the Weierstrass representation of X.

Remark 1. If (Φ_1, Φ_2, Φ_3) is the Weierstrass representation of a simply connected maximal surface, then $(\mathrm{i}\Phi_1, \mathrm{i}\Phi_2, \Phi_3)$ are the Weierstrass data of a simply connected minimal surface in $\mathbb{R}^3$ [17]. Moreover, both surfaces have the same meromorphic Gauss map g.

We are going to deal with maximal immersions with lightlike singularities, according with the following definition.

Definition 1. A point $p \in M$ is a lightlike singularity of the immersion X if it is not a branch point and $|g(p)| = 1$.

In this article, all the maximal immersions are defined on simply connected domains of $\mathbb{C}$, thus the Weierstrass 1-forms have no periods and so the only requirements are (1) at the points that are not singularities, and (2). In this case, the differential η can be written as $\eta = f(z)\,dz$. The metric of X can be expressed as

$$ds^2 = \frac{1}{2}\left(|\Phi_1|^2 + |\Phi_2|^2 - |\Phi_3|^2\right) = \left(\frac{1}{2}\left(1-|g|^2\right)|f||dz|\right)^2. \tag{3}$$

We use a concept of completeness that is less exigent than the classical one. The following definition was given by Umehara and Yamada [18].

Definition 2. A maximal immersion $X: M \to \mathbb{L}^3$ is weakly complete if the Riemann surface M is complete with the metric

$$d\sigma^2 = \frac{1}{2}\left(|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2\right) = \left(\frac{1}{2}\left(1+|g|^2\right)|f||dz|\right)^2. \tag{4}$$

The metric $d\sigma^2$ will be called the lift metric of X.

The Euclidean metric on $\mathbb{C}$ is denoted as $\langle,\rangle = |dz|^2$. Note that $ds^2 = (\lambda_X)^2|dz|^2$ and $d\sigma^2 = (\lambda_X^0)^2\,|dz|^2$ where the conformal coefficients λ_X and λ_X^0 are given by (3) and (4), respectively.

Remark 2. Observe that if X has a singularity of lightlike type in a point $z \in M$, then $\lambda_X(z) = 0$ but $\lambda_X^0(z) \neq 0$. On the other hand, if z is a branch point of X, one has $\lambda_X(z) = 0 = \lambda_X^0(z)$.

Throughout this paper, we use some $\mathbb{L}^3$-orthonormal bases. Given $X: \Omega \to \mathbb{L}^3$ a maximal immersion and S an $\mathbb{L}^3$-orthonormal basis, we write the Weierstrass data of X in the basis S as

$$\Phi_{(X,S)} = (\Phi_{(1,S)}, \Phi_{(2,S)}, \Phi_{(3,S)}), \qquad f_{(X,S)}, \qquad g_{(X,S)}, \qquad \eta_{(X,S)}.$$

In the same way, given $v \in \mathbb{R}^3$, we denote by $v_{(k,S)}$ the kth coordinate of v in S. We also represent by $v_{(*,S)} = (v_{(1,S)}, v_{(2,S)})$ the first two coordinates of v in the basis S.

Given a curve α in Ω, by length(α, ds) we mean the length of α with respect to the metric ds. Let $W \subset \Omega$ be a subset, then we define

- $\operatorname{dist}_{(W,ds)}(p,q) = \inf\{\text{length}(\alpha, ds) \mid \alpha : [0,1] \to W,\ \alpha(0) = p,\ \alpha(1) = q\}$, for any $p, q \in W$.
- $\operatorname{dist}_{(W,ds)}(U,V) = \inf\{\operatorname{dist}_{(W,ds)}(p,q) \mid p \in U,\ q \in V\}$, for any $U, V \subset W$.

Given a domain $D \subset \mathbb{C}$, we say that a function, or a 1-form, is harmonic, holomorphic, meromorphic, ... on $\overline{D}$, if it is harmonic, holomorphic, meromorphic, ... on a domain containing $\overline{D}$.

Let P be a simple closed polygonal curve in $\mathbb{C}$. By $\operatorname{Int} P$ we mean the bounded connected component of $\mathbb{C} \setminus P$. For a small enough $\xi > 0$, we denote by P^ξ as the parallel polygonal curve in $\operatorname{Int} P$, satisfying that the distance between parallel sides is equal to ξ. Whenever we write P^ξ we are assuming that ξ is small enough to define the polygon properly.

2.3. The López–Ros transformation

The proof of Lemma 1 exploits what has come to be call the López–Ros transformation. If (g, f) are the Weierstrass data of a maximal immersion $X : \Omega \to \mathbb{L}^3$ (being Ω simply connected), we define on Ω the data

$$\widetilde{g} = \frac{g}{h}, \qquad \widetilde{f} = fh,$$

where $h : \Omega \to \mathbb{C}$ is a holomorphic function without zeros. Observe that the new meromorphic data satisfy (1) at the regular points, and (2), so the new data define a maximal immersion (possibly with different lightlike singularities) $\widetilde{X} : \Omega \to \mathbb{L}^3$. This method provides us with a powerful and natural tool for deforming maximal surfaces. One of the most interesting properties of the resulting surface is that the third coordinate function is preserved.

3. Proof of Theorem 1

In order to prove Theorem 1 we will apply the following technical lemma. It will be proved later in Section 4.

Lemma 1. *Consider $r > 0$, P a polygon in $\mathbb{C}$ and $X : \overline{\text{Int}\, P} \to \mathbb{L}^3$ a conformal maximal immersion (possibly with lightlike singularities) satisfying*

$$X(\overline{\text{Int}\, P}) \subset \mathrm{B}(r). \tag{5}$$

Let ϵ and s be positive constants with $\sqrt{r^2 - 4s^2} - \epsilon > 0$. Then, there exist a polygon Q and a conformal maximal immersion (possibly with lightlike singularities) $Y : \overline{\text{Int}\, Q} \to \mathbb{L}^3$ such that

(L.1) $\overline{\text{Int}\, P^\epsilon} \subset \text{Int}\, Q \subset \overline{\text{Int}\, Q} \subset \text{Int}\, P$.
(L.2) $s < \text{dist}_{(\overline{\text{Int}\, Q}, d\sigma_Y^2)}(P^\epsilon, Q)$, *where $d\sigma_Y^2$ is the lift metric associated to the immersion Y.*
(L.3) $Y(\overline{\text{Int}\, Q}) \subset \mathrm{B}(R)$, *where $R = \sqrt{r^2 - 4s^2} - \epsilon$.*
(L.4) $\|Y - X\|_0 < \epsilon$ *in $\overline{\text{Int}\, P^\epsilon}$.*

Using this lemma, we construct a sequence of immersions $\{\psi_n\}_{n\in\mathbb{N}}$ that converges to an immersion ψ which proves Theorem 1, up to a reparametrization of its domain.

First of all, we consider a sequence of reals $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfying

$$\prod_{k=1}^{\infty} \alpha_k = \frac{1}{2}, \quad 0 < \alpha_k < 1, \ \forall k \in \mathbb{N}.$$

Moreover, we choose $r_1 > 1$ large enough so that the sequence $\{r'_n\}_{n\in\mathbb{N}}$ given by

$$r'_1 = r_1, \qquad r'_n = \sqrt{(r'_{n-1})^2 - (2/n)^2} - \frac{1}{n^2}$$

satisfies

$$r'_n > 1, \quad \forall n \in \mathbb{N}. \tag{6}$$

Now, we are going to construct a sequence $\{\Upsilon_n\}_{n\in\mathbb{N}}$, where the element

$$\Upsilon_n = \{P_n, \psi_n, \epsilon_n, \xi_n\}$$

is composed of a polygon P_n, a conformal maximal immersion $\psi_n : \overline{\text{Int}\, P_n} \to \mathbb{L}^3$, and $\epsilon_n < \frac{1}{n^2}$, and ξ_n are positive real numbers. We will choose ϵ_n and ξ_n so that the sequences $\{\epsilon_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ decrease to zero.

We construct the sequence in order to satisfy the following list of properties.

(A_n) $\overline{\text{Int}\, P_{n-1}^{\xi_{n-1}}} \subset \text{Int}\, P_{n-1}^{\epsilon_n} \subset \overline{\text{Int}\, P_{n-1}^{\epsilon_n}} \subset \text{Int}\, P_n^{\xi_n} \subset \overline{\text{Int}\, P_n^{\xi_n}} \subset \text{Int}\, P_n \subset \overline{\text{Int}\, P_n} \subset \text{Int}\, P_{n-1}$.
(B_n) $1/n < \text{dist}_{(\overline{\text{Int}\, P_n^{\xi_n}}, d\sigma_{\psi_n}^2)}(P_{n-1}^{\xi_{n-1}}, P_n^{\xi_n})$, where $d\sigma_{\psi_n}^2$ is the lift metric of the immersion ψ_n.
(C_n) $\psi_n(\overline{\text{Int}\, P_n}) \subset \mathrm{B}(r_n)$, where $r_n = \sqrt{r_{n-1}^2 - (2/n)^2} - \epsilon_n$. Notice that (6) guarantees that $\{r_n\}_{n\in\mathbb{N}}$ decreases to a real number $r_\infty > 1$.
(D_n) $\|\psi_n - \psi_{n-1}\|_0 < \epsilon_n$ in $\overline{\text{Int}\, P_{n-1}^{\epsilon_n}}$.
(E_n) $\lambda_{\psi_n}^0 \geqslant \alpha_n \cdot \lambda_{\psi_{n-1}}^0$ in $\overline{\text{Int}\, P_{n-1}^{\xi_{n-1}}}$.

The sequence $\{\Upsilon_n\}_{n\in\mathbb{N}}$ is constructed in a recursive way. The existence of a family Υ_1 satisfying assertion (C_1) is straightforward. The rest of the properties have no sense for $n = 1$.

Suppose that we have $\Upsilon_1, \ldots, \Upsilon_n$. We are going to construct Υ_{n+1}. We choose a decreasing sequence of positive reals $\{\varepsilon_m\}_{m\in\mathbb{N}} \searrow 0$ with $\varepsilon_m < \min\{1/(n+1)^2, \epsilon_n\}$ for all $m \in \mathbb{N}$. For each m, we consider the polygon Q_m and the conformal maximal immersion $Y_m : \overline{\mathrm{Int}\, Q_m} \to \mathbb{L}^3$ given by Lemma 1 for the following data:

$$r = r_n, \qquad P = P_n, \qquad X = X_n, \qquad \epsilon = \varepsilon_m, \qquad s = \frac{1}{n+1}.$$

For a large enough m, (L.1) in Lemma 1 guarantees that $\overline{\mathrm{Int}\, P_n^{\xi_n}} \subset \mathrm{Int}\, Q_m$. Moreover, from property (L.4), we deduce that the sequence $\{Y_m\}_{m\in\mathbb{N}}$ uniformly converges to ψ_n in $\overline{\mathrm{Int}\, P_n^{\xi_n}}$. Then, taking into account that Y_m is a harmonic map and that its Weierstrass data are given by its derivatives, we conclude that the sequence $\{\lambda^0_{Y_m}\}_{m\in\mathbb{N}}$ uniformly converges to $\lambda^0_{\psi_n}$ in $\overline{\mathrm{Int}\, P_n^{\xi_n}}$. Hence, there exists $m_0 \in \mathbb{N}$ satisfying

$$\overline{\mathrm{Int}\, P_n^{\xi_n}} \subset \mathrm{Int}\, P_n^{\varepsilon_{m_0}} \subset \overline{\mathrm{Int}\, P_n^{\varepsilon_{m_0}}} \subset \mathrm{Int}\, Q_{m_0}, \tag{7}$$

$$\lambda^0_{Y_{m_0}} \geqslant \alpha_{n+1} \cdot \lambda^0_{\psi_n}, \quad \text{in } \overline{\mathrm{Int}\, P_n^{\xi_n}}. \tag{8}$$

In order to obtain (8) we have taken into account that the immersion ψ_n has no branch points, it only has singularities of lightlike type (see Remark 2).

At this point, we define $P_{n+1} = Q_{m_0}$, $\psi_{n+1} = Y_{m_0}$ and $\epsilon_{n+1} = \varepsilon_{m_0}$. From (L.2) in Lemma 1, we conclude that $1/(n+1) < \mathrm{dist}_{(\overline{\mathrm{Int}\, P_{n+1}},\, d\sigma^2_{\psi_{n+1}})}(P_n^{\epsilon_{n+1}}, P_{n+1})$. Therefore, taking into account (7) we can take ξ_{n+1} small enough so that (A_{n+1}) and (B_{n+1}) hold. Properties (C_{n+1}) and (D_{n+1}) are consequence of (L.3) and (L.4), respectively, whereas (8) implies (E_{n+1}). This concludes the construction of the sequence $\{\Upsilon_n\}_{n\in\mathbb{N}}$.

Now, define $\Delta := \bigcup_{n\in\mathbb{N}} \mathrm{Int}\, P_n^{\epsilon_{n+1}} = \bigcup_{n\in\mathbb{N}} \mathrm{Int}\, P_n^{\xi_n}$. Since (A_n), the set Δ is an expansive union of simply connected domains resulting in Δ being simply connected. Moreover, Δ is bounded since properties (A_n), $n \in \mathbb{N}$, so it is biholomorphic to a disk. On the other hand, from (D_n) we obtain that $\{\psi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, uniformly on compact sets of Δ. Then, Harnack's Theorem guarantees the existence of a harmonic map $\psi : \Delta \to \mathbb{L}^3$ such that $\{\psi_n\}_{n\in\mathbb{N}} \to \psi$, uniformly on compact sets of Δ. Then, ψ has the following properties.

- *ψ is maximal and conformal.* These facts are consequence of that ψ is harmonic.
- *ψ has no branch points.* For any $z \in \Delta$ there exists $n \in \mathbb{N}$ so that $z \in \mathrm{Int}\, P_n^{\xi_n}$. Given $k > n$ and using (E_j), $j = n+1, \ldots, k$, one has $\lambda^0_{\psi_k}(z) \geqslant \alpha_k \cdots \alpha_1 \lambda^0_{\psi_n}(z)$. Hence, taking the limit as $k \to \infty$, we infer that

$$\lambda^0_{\psi}(z) \geqslant \frac{1}{2}\lambda^0_{\psi_n}(z) > 0,$$

and so, ψ has no branch points. Notice that the last inequality holds because of ψ_n has no branch points.

Remark 3. Observe that this argument does not work if we use the conformal coefficients λ_{ψ_k} instead of $\lambda^0_{\psi_k}$. This fact is implied by the possible existence of singularities of lightlike type.

- *ψ is weakly complete.* This fact follows from properties (B_n), (E_n), $n \in \mathbb{N}$, and the fact that the sum $\sum_{n=1}^{\infty} 1/n$ diverges.
- $\psi(\Delta) \subset \mathrm{B}(1)$. Let $z \in \Delta$ and $n \in \mathbb{N}$ such that $z \in \mathrm{Int}\, P_n^{\xi_n}$. For each $k \geqslant n$, property (C_k) guarantees that $\psi_k(z) \in \mathrm{B}(r_k) \subset \mathrm{B}(r_\infty)$. Taking limit as $k \to \infty$, we obtain $\psi(z) \in \overline{\mathrm{B}(r_\infty)} \subset \mathrm{B}(1)$.

This completes the proof of Theorem 1.

4. Proof of Lemma 1

The first step of the proof consists of the construction of a labyrinth on $\mathrm{Int}\, P$ which depends on the polygon P and a positive integer N. Let ℓ be the number of sides of P. From now on, N is a positive multiple of ℓ. Although N is fix, we will assume throughout the proof of the lemma that we have taken it large enough so that some inequalities hold. Without loss of generality, we assume $0 \in \mathrm{Int}\, P^{\epsilon}$.

Remark 4. Throughout the proof of the lemma, a set of positive real constants depending on the data of the lemma, i.e., r, P, X, ϵ and s, will appear. The symbol "const" will denote these different constants. It is important to note that the choice of these constants does not depend on N.

First of all, consider $\zeta_0 \in\,]0, \epsilon[$. Therefore, P^{ζ_0} is well defined and $\overline{\mathrm{Int}\, P^{\epsilon}} \subset \mathrm{Int}\, P^{\zeta_0}$. We also assume that N satisfies $2/N < \zeta_0$.

Let $v_1, \ldots, v_{2N}$ be a set of points in the polygon P (containing the vertices of P) which divides each side of P into $2N/\ell$ equal parts. Let $v'_1, \ldots, v'_{2N}$ be the points resulting from transferring the above partition to the polygon $P^{2/N}$. Then, we define the following sets.

- L_i is the segment that joins v_i and v'_i, $i = 1, \ldots, 2N$.
- $\mathcal{G}_i = P^{i/N^3}$, $i = 0, \ldots, 2N^2$.

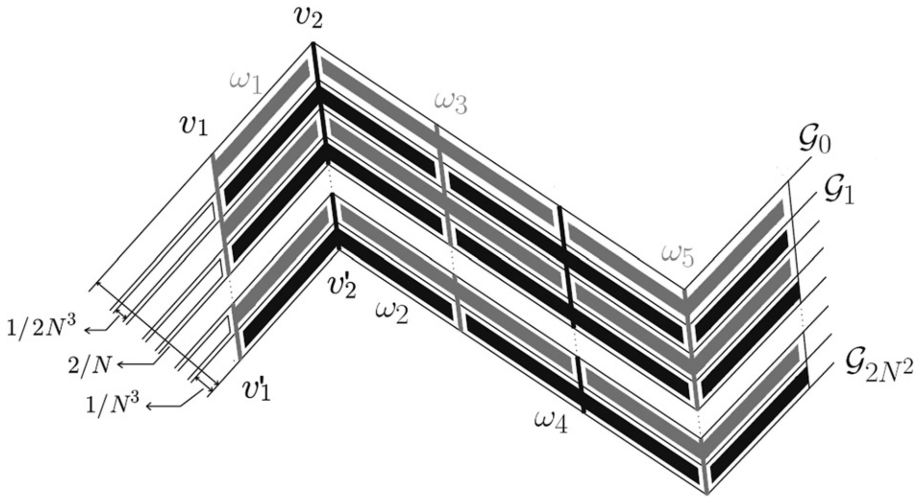

Fig. 1. The labyrinth.

- $\mathcal{A} = \bigcup_{i=0}^{N^2-1} \overline{(\operatorname{Int}\mathcal{G}_{2i}) \setminus (\operatorname{Int}\mathcal{G}_{2i+1})}$ and $\widetilde{\mathcal{A}} = \bigcup_{i=1}^{N^2} \overline{(\operatorname{Int}\mathcal{G}_{2i-1}) \setminus (\operatorname{Int}\mathcal{G}_{2i})}$.
- $\mathcal{R} = \bigcup_{i=0}^{2N^2} \mathcal{G}_i$.
- $\mathcal{B} = \bigcup_{i=1}^{N} L_{2i}$ and $\widetilde{\mathcal{B}} = \bigcup_{i=0}^{N-1} L_{2i+1}$.
- $\mathcal{L} = \mathcal{B} \cap \mathcal{A}$, $\widetilde{\mathcal{L}} = \widetilde{\mathcal{B}} \cap \widetilde{\mathcal{A}}$ and $H = \mathcal{R} \cup \mathcal{L} \cup \widetilde{\mathcal{L}}$.
- $\Omega_N = \{z \in (\operatorname{Int}\mathcal{G}_0) \setminus (\operatorname{Int}\mathcal{G}_{2N^2}) \mid \operatorname{dist}_{(\mathbb{C},\langle\cdot,\cdot\rangle)}(z, H) \geqslant 1/(4N^3)\}$.
- ω_i is the union of the segment L_i and those connected components of Ω_N that have nonempty intersection with L_i, for $i = 1, \ldots, 2N$.
- $\varpi_i = \{z \in \mathbb{C} \mid \operatorname{dist}_{(\mathbb{C},\langle\cdot,\cdot\rangle)}(z, \omega_i) < \delta(N)\}$, is chosen so that the sets $\overline{\varpi_i}$, $i = 1, \ldots, 2N$, are pairwise disjoint.

After constructing the labyrinth (see Fig. 1), we are going to list some of its properties.

Claim 1. *If N is large enough, for any $i = 1, \ldots, 2N$, one has*

A. $\operatorname{diam}_{(\mathbb{C},\langle\cdot,\cdot\rangle)}(\varpi_i) < \mathtt{const}/N$.
B. $\operatorname{diam}_{\mathbb{H}^2_+}(\mathcal{N}(X(\varpi_i))) < 1/\sqrt{N}$, *where* $\operatorname{diam}_{\mathbb{H}^2_+}$ *is the intrinsic diameter in* $\mathbb{H}^2_+$. *Here,* $\mathcal{N}$ *is the map defined in Section 2.1.*
C. *Denote by* (g, Φ_3) *the Weierstrass data of the immersion* X. *Then, there exists a subset* $I_0 \subset \{1, \ldots, 2N\}$ *such that*
   - $|g(z)| \neq 1$ $\forall z \in \varpi_j$, $\forall j \in I_0$.
   - $g(z) \neq \infty$ $\forall z \in \varpi_j$, $\forall j \in J_0 = \{1, \ldots, 2N\} \setminus I_0$.
D. *Let* $\lambda^2\langle\cdot,\cdot\rangle$ *be a conformal metric in* $\overline{\operatorname{Int} P}$. *Assume there exists* $c \in \mathbb{R}^+$ *so that*

$$\lambda \geqslant \begin{cases} c & \text{in } \operatorname{Int} P, \\ cN^4 & \text{in } \Omega_N. \end{cases}$$

Then, for any curve α *in* $\overline{\operatorname{Int} P}$ *connecting* P^{ζ_0} *and* P, *one has* $\operatorname{length}(\alpha, \lambda\langle\cdot,\cdot\rangle) > \mathtt{const}\, cN$, *where* $\mathtt{const}$ *does not depend on* c.

Proof. Checking item A in the above claim is straightforward. Item B is a consequence of item A and the fact that $\mathcal{N}$ is a differentiable map. For a sufficiently large N, item C holds since item A and because of g is a meromorphic function. In order to prove item D, we denote by α_j as the piece of α connecting $P^{j/N}$ and $P^{(j+1)/N}$, for $j = 0, \ldots, N^2 - 1$. Then, either the Euclidean length of α_j is greater than $\mathtt{const}/N$ or the length of $\alpha_j \cap \Omega_N$ is greater than $1/2N^3$. This fact and our assumption about λ imply item D. $\square$

At this point, we construct a sequence $F_0 = X, F_1, \ldots, F_{2N}$ of conformal maximal immersions (with boundary and, possibly, lightlike singularities) defined in $\overline{\operatorname{Int} P}$.

Claim 2. *We will construct the sequence in order to satisfy the following list of statements, for $i = 1, \ldots, 2N$.*

(a1$_i$) $F_i(z) = \operatorname{Re}(\int_0^z \phi^i(u)\,du) + V$. *Here,* $V \in \mathbb{R}^3$ *is a fixed vector. It does not depend on* i.
(a2$_i$) $\|\phi^i - \phi^{i-1}\|_0 \leqslant 1/N^2$ *in* $\overline{\operatorname{Int} P} \setminus \varpi_i$.
(a3$_i$) $\|\phi^i\|_0 \geqslant N^{7/2}$ *in* ω_i.

(a4$_i$) $\|\phi^i\|_0 \geqslant \mathtt{const}/\sqrt{N}$ *in* ϖ_i.
(a5$_i$) *Assume* (g_i, ϕ_3^i) *are the Weierstrass data of* F_i. *Then, the following two assertions hold.*
 (a5.1$_i$) $|g_i(z)| \neq 1$, $\forall z \in \varpi_j$, $\forall j \in I_0$, $j > i$. *Hence, the Gauss map* G_i *of the immersion* F_i *is well defined in* ϖ_j *for those* j. *Moreover,* $\mathrm{dist}_{\mathbb{H}^2}(G_i(z), G_{i-1}(z)) < 1/N^2$, *for any* $z \in \varpi_j$ *and for any* $j \in I_0$, $j > i$, *where by* $\mathrm{dist}_{\mathbb{H}^2}$ *we mean the intrinsic distance in* $\mathbb{H}^2$.
 (a5.2$_i$) $g_i(z) \neq \infty$, $\forall z \in \varpi_j$, $\forall j \in J_0$, $j > i$. *Furthermore, one has* $|g_i(z) - g_{i-1}(z)| < 1/N^2$, *for any* $z \in \varpi_j$, *for any of those* j.
(a6$_i$) *There exists* $S_i = \{e_1, e_2, e_3\}$ *an orthonormal frame in* $\mathbb{L}^3$, *such that*
 (a6.1$_i$) $\mathrm{dist}_{\mathbb{H}^2_+}(e_3, \mathcal{N}(X(z))) < \mathtt{const}/\sqrt{N}$, *for any* $z \in \overline{\varpi_i}$.
 (a6.2$_i$) $(F_i(z))_{(3,S_i)} = (F_{i-1}(z))_{(3,S_i)}$, *for all* z *in* $\overline{\mathrm{Int}\,P}$.
(a7$_i$) $\|F_i - F_{i-1}\|_0 < \mathtt{const}/N^2$ *in* $(\mathrm{Int}\,P) \setminus \varpi_i$.

Proof. The sequence $F_0, F_1, \ldots, F_{2N}$ is constructed in a recursive way. Assume that we already have $F_0, F_1, \ldots, F_{j-1}$ satisfying the assertions (a1$_i$), ..., (a7$_i$), $i = 1, \ldots, j-1$. Before constructing F_j, we need to check the following claim.

Claim 3. *For a large enough* N, *the following statements hold.*

(b1) $\|\phi^{j-1}\|_0 \leqslant \mathtt{const}$ *in* $(\mathrm{Int}\,P) \setminus (\bigcup_{k=1}^{j-1} \varpi_k)$.
(b2) $\|\phi^{j-1}\|_0 \geqslant \mathtt{const}$ *in* $(\mathrm{Int}\,P) \setminus (\bigcup_{k=1}^{j-1} \varpi_k)$.
(b3) *The diameter in* $\mathbb{R}^3$ *of* $F_{j-1}(\varpi_j)$ *is less than* $1/\sqrt{N}$.
(b4) *Assume* $j \in I_0$. *Then,*
 (b4.1) *The diameter in* $\mathbb{H}^2$ *of* $G_{j-1}(\varpi_j)$ *is less than* $1/\sqrt{N}$. *In particular, there exists* $p \in G_{j-1}(\varpi_j)$ *such that* $\mathrm{dist}_{\mathbb{H}^2}(p, G_{j-1}(z)) < 1/\sqrt{N}$, *for any* $z \in \varpi_j$.
 On the other hand, suppose $j \in J_0$.
 (b4.2) *Consider the set*

$$\Gamma := \left\{ \frac{G_{j-1}(z)}{\|G_{j-1}(z)\|_0} \;\middle|\; z \in \varpi_j,\ \left|g_{j-1}(z)\right| \neq 1 \right\}.$$

 Denote by Γ^+ *(resp.* Γ^-*) as the part of* Γ *corresponding to* $\mathbb{H}^2_+$ *(resp.* $\mathbb{H}^2_-$*). Then, there exists* $p \in \Gamma^+$ *so that* $\mathrm{dist}_{\mathbb{S}^2}(\pm p, q) < 1/\sqrt{N}$, *for all* $q \in \Gamma^\pm$.
(b5) *There exists an orthonormal frame* $S_j = \{e_1, e_2, e_3\}$ *in* $\mathbb{L}^3$, *where* $e_3 \in \mathbb{H}^2_+$ *and the following assertions hold.*
 (b5.1) $\mathrm{dist}_{\mathbb{H}^2_+}(e_3, \mathcal{N}(X(z))) \leqslant \mathtt{const}/\sqrt{N}$, *for all* $z \in \varpi_j$.
 (b5.2) $\mathrm{dist}_{\mathbb{H}^2_+}(e_3, \pm q) \geqslant \mathtt{const}/\sqrt{N}$ *and* $\mathrm{dist}_{\mathbb{H}^2_-}(-e_3, \pm q) \geqslant \mathtt{const}/\sqrt{N}$, *for any* q *in the set* $\{G_{j-1}(z) \mid z \in \varpi_j,\ |g_{j-1}(z)| \neq 1\}$. *We mean that we only have to compute the distance if both points are in the same connected component of* $\mathbb{H}^2$.

Proof. To deduce (b1) and (b2) we have to use just (a2$_k$), $k = 1, \ldots, j-1$. Item (b3) is a consequence of (b1) and Claim 1.A. In order to prove (b4) we distinguish cases. If $j \in I_0$, taking into account Claim 1.A and Claim 1.C we obtain that the diameter of $G_0(\varpi_j)$ is bounded by $\mathtt{const}/N$. Then, we can apply (a5.1$_k$), $k = 1, \ldots, j-1$, to conclude (b4.1). On the other hand, if $j \in J_0$, we use again Claim 1.A and Claim 1.C to deduce that $\mathrm{diam}_{\mathbb{C}}(g_0(\varpi_j)) < \mathtt{const}/N$. Therefore, (a5.2$_k$), $k = 1, \ldots, j-1$, imply that $\mathrm{diam}_{\mathbb{C}}(g_{j-1}(\varpi_j)) < \mathtt{const}/N$. This fact guarantees (b4.2) for a large enough N. We also have taken into account that if $|g_{i-1}(z)| < 1 < |g_{i-1}(z')|$ and $g_{i-1}(z) \approx g_{i-1}(z')$, then $G_{j-1}(z) \approx -G_{j-1}(z')$.

The proof of (b5) is slightly more complicated. First, assume that $j \in I_0$. Without loss of generality we can assume that $G_{j-1}(\varpi_j) \subset \mathbb{H}^2_+$, otherwise we would work with $-G_{j-1}(\varpi_j)$. Consider p given by property (b4.1), then to obtain (b5.2), it suffices to take e_3 in $C = \{q \in \mathbb{H}^2_+ \mid \mathrm{dist}_{\mathbb{H}^2_+}(p, q) > 2/\sqrt{N}\}$. Moreover, in order to satisfy (b5.1), the vector e_3 must be chosen as follows.

- If $C \cap \mathcal{N}(X(\varpi_j)) \neq \emptyset$, then we take e_3 in that set. Therefore (b5.1) holds because of Claim 1.B.
- If $C \cap \mathcal{N}(X(\varpi_j)) = \emptyset$, then we take $e_3 \in C$ such that $\mathrm{dist}_{\mathbb{H}^2_+}(e_3, q') < 2/\sqrt{N}$ for some $q' \in \mathcal{N}(X(\varpi_j))$. This choice is possible since (b4.1). Again Claim 1.B. guarantees (b5.1).

Assume now that $j \in J_0$. We define the sets

$$\Lambda_\pm := \left\{ \frac{q}{\|q\|_0} \;\middle|\; q \in \mathbb{H}^2_\pm \right\} \subset \mathbb{S}^2, \qquad \Xi := \left\{ \frac{\mathcal{N}(X(z))}{\|\mathcal{N}(X(z))\|_0} \;\middle|\; z \in \varpi_j \right\} \subset \Lambda_+.$$

In order to prove assertion (b5) in this case, we are going to use the following statement. There exists $e_3 \in \mathbb{H}^2_+$ so that the vector $\widehat{e}_3 = e_3/\|e_3\|_0$ satisfies

(i) $\mathrm{dist}_{\Lambda_+}(\widehat{e}_3, q) \leqslant \mathtt{const}/\sqrt{N}$, for all $q \in \Xi$.

(ii) $\mathrm{dist}_{\Lambda_+}(\widehat{e}_3, \pm q) \geqslant \mathtt{const}/\sqrt{N}$ and $\mathrm{dist}_{\Lambda_-}(-\widehat{e}_3, \pm q) \geqslant \mathtt{const}/\sqrt{N}$ for any $q \in \Gamma$. Again, we mean that we only have to compute the distance if both points are in Λ_+ or both in Λ_-.

Indeed, the proof consists of the same arguments as above but using (b4.2) instead of (b4.1). Then, (b5.1) is a consequence of (i) and the fact that $\|\mathcal{N}(X(\varpi_j))\|_0$ is bounded (not depending on N). Moreover, (ii) implies (b5.2). Hence, e_3 proves property (b5) in this case. □

Now, we can continue with the proof of Claim 2. Let (g^{j-1}, ϕ_3^{j-1}) be the Weierstrass data of the immersion F_{j-1} in the basis S_j given by (b5). For any $\alpha > 0$, consider $h_\alpha : \overline{\mathrm{Int}\,P} \to \mathbb{C}$ a holomorphic function without zeros and satisfying

- $|h_\alpha - 1| < 1/\alpha$ in $\overline{\mathrm{Int}\,P} \setminus \varpi_j$.
- $|h_\alpha - \alpha| < 1/\alpha$ in ω_j.

This family of functions is given by Runge's Theorem. Using h_α as a López–Ros parameter, we define F_j in the coordinate system S_j as $g^j = g^{j-1}/h_\alpha$ and $\phi_3^j = \phi_3^{j-1}$. Taking into account that $h_\alpha \to 1$ (resp. $h_\alpha \to \infty$) uniformly in $\overline{\mathrm{Int}\,P} \setminus \varpi_j$ (resp. in ω_j), as $\alpha \to \infty$, it is clear that properties (a1$_j$), (a2$_j$), (a3$_j$), (a5$_j$) and (a7$_j$) hold for a large enough (in terms of N) value of the parameter α. Moreover, using (b5.1) we obtain (a6.1$_j$) and to get (a6.2$_j$) we use that $\phi_3^{j-1} = \phi_3^j$ in the frame S_j. Finally, we are going to prove (a4$_j$). Consider $z \in \varpi_j$ with $|g^{j-1}(z)| \neq 1$. Using the stereographic projection for $\mathbb{H}^2$ from the point $e_3 \in \mathbb{H}^2_+$, from property (b5.2) one has

$$\frac{\sinh(\frac{\mathtt{const}}{\sqrt{N}})}{\cosh(\frac{\mathtt{const}}{\sqrt{N}}) + 1} \leqslant \left|g^{j-1}(z)\right| \leqslant \frac{\sinh(\frac{\mathtt{const}}{\sqrt{N}})}{\cosh(\frac{\mathtt{const}}{\sqrt{N}}) - 1}.$$

On the other hand, if $|g^{j-1}(z)| = 1$, then the above inequalities trivially hold, so they occur for any $z \in \varpi_j$. Therefore,

$$\|\phi^j\|_0 \geqslant |\phi_3^j| = |\phi_3^{j-1}| \geqslant \sqrt{2}\|\phi^{j-1}\|_0 \frac{|g^{j-1}|}{1+|g^{j-1}|^2} \geqslant \mathtt{const}\cdot\tanh\left(\frac{\mathtt{const}}{\sqrt{N}}\right) \geqslant \frac{\mathtt{const}}{\sqrt{N}} \quad \text{in } \varpi_j,$$

where we have used (a6.2$_j$) and (b2). This fact proves (a4$_j$) and concludes the proof of Claim 2. □

Remark 5. Notice that in the definition of F_i in property (a1$_i$), we need the addition of the fixed vector V. Otherwise, it would be $F_i(0) = (0,0,0)$. In particular, $X(0) = (0,0,0) \notin \mathbb{B}(r)$, which is absurd.

Remark 6. Let $S_i = \{e_1, e_2, e_3\}$ be the $\mathbb{L}^3$-orthonormal basis given by property (a6$_i$). Consider $\widetilde{S}_i = \{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3\}$ an $\mathbb{R}^3$-orthonormal basis such that $\{e_1, e_2\}$ and $\{\widetilde{e}_1, \widetilde{e}_2\}$ define the same plane, and e_3 and $\widetilde{e}_3$ lie in the same halfspace determined by that plane, i.e., $\widetilde{e}_3 = -\mathcal{J}(e_3)/\|e_3\|_0$, where $\mathcal{J}(e_3^1, e_3^2, e_3^3) = (e_3^1, e_3^2, -e_3^3)$. Then, one has

- $\mathrm{dist}_{\mathbb{S}^2}(\widetilde{e}_3, \mathcal{N}_0(X(z))) < \mathtt{const}/\sqrt{N}$, for any $z \in \varpi_i$, where $\mathcal{N}_0$ is the map that was defined in Section 2.1.
- $(F_i(z))_{(3,\widetilde{S}_i)} = (F_{i-1}(z))_{(3,\widetilde{S}_i)}$.

Now, we establish some properties of the final immersion F_{2N}.

Claim 4. *If N is large enough, then F_{2N} satisfies*

(c1) $2s < \mathrm{dist}_{(\overline{\mathrm{Int}\,P}, d\sigma_{F_{2N}})}(P, P^\epsilon)$, *where by $d\sigma_{F_{2N}}$ we represent the lift metric of the immersion F_{2N}.*
(c2) $\|F_{2N} - X\|_0 < \mathtt{const}/N$, *in* $\overline{\mathrm{Int}\,P} \setminus (\bigcup_{i=1}^{2N} \varpi_i)$.
(c3) *There exists a polygon Q such that*
 (c3.1) $\overline{\mathrm{Int}\,P^\epsilon} \subset \mathrm{Int}\,Q \subset \overline{\mathrm{Int}\,Q} \subset \mathrm{Int}\,P$.
 (c3.2) $s < \mathrm{dist}_{(\overline{\mathrm{Int}\,P}, d\sigma_{F_{2N}})}(z, P^\epsilon) < 2s$, *for any $z \in Q$.*
 (c3.3) $F_{2N}(\overline{\mathrm{Int}\,Q}) \subset \mathbb{B}(R)$, *where $R = \sqrt{r^2 - 4s^2} - \epsilon$.*

Proof. Properties (b2), (a2$_i$), (a3$_i$) and (a4$_i$), $i = 1, \ldots, 2N$, guarantee that the conformal coefficient $\lambda^0_{F_{2N}}$ of the lift metric of F_{2N} satisfies

$$\lambda^0_{F_{2N}} = \frac{\|\phi^{2N}\|_0}{\sqrt{2}} \geqslant \begin{cases} \frac{\mathtt{const}}{\sqrt{N}} & \text{in } \mathrm{Int}\,P \\ \frac{\mathtt{const}}{\sqrt{N}} N^4 & \text{in } \Omega_N. \end{cases}$$

Therefore, Claim 1.D imply that

$$\mathrm{dist}_{(\overline{\mathrm{Int}\,P}, d\sigma_{F_{2N}})}\left(P, P^\epsilon\right) \geqslant \mathrm{dist}_{(\overline{\mathrm{Int}\,P}, d\sigma_{F_{2N}})}\left(P, P^{\zeta_0}\right) > \frac{\mathtt{const}}{\sqrt{N}} N = \mathtt{const}\sqrt{N} > 2s,$$

for a large enough N. We have proved (c1). Property (c2) trivially holds from $(\mathrm{a2}_i)$, $i = 1, \dots, 2N$.

In order to construct the polygon Q of the assertion (c3), we consider the set

$$\mathcal{K} = \{z \in (\operatorname{Int} P) \setminus (\operatorname{Int} P^\epsilon) \mid s < \operatorname{dist}_{(\overline{\operatorname{Int} P}, d\sigma_{F_{2N}})}(z, P^\epsilon) < 2s\}.$$

From (c1), $\mathcal{K}$ is a nonempty open subset of $(\operatorname{Int} P) \setminus (\operatorname{Int} P^\epsilon)$, and P and P^ϵ are contained in different connected components of $\mathbb{C} \setminus \mathcal{K}$. Therefore, we can choose a polygon Q on $\mathcal{K}$ satisfying (c3.1) and (c3.2).

The proof of (c3.3) is more complicated. Consider $z \in \overline{\operatorname{Int} Q}$. First, we assume that $z \in (\operatorname{Int} P) \setminus (\cup_{i=1}^{2N} \varpi_i)$. Then, we can use properties $(\mathrm{a2}_i)$, $i = 1, \dots, 2N$, to conclude that $\|F_{2N}(z) - X(z)\|_0 < \mathtt{const}/N$. Moreover, from the hypotheses of Lemma 1, we have $X(z) \in \mathrm{B}(r)$. Hence, $F_{2N}(z) \in \mathrm{B}(R)$, if N is large enough.

On the other hand, suppose that there exists $i \in \{1, \dots, 2N\}$ with $z \in \varpi_i$. Choose a curve $\gamma : [0,1] \to \operatorname{Int} P$ satisfying $\gamma(0) \in P^\epsilon$, $\gamma(1) = z$ and $\operatorname{length}(\gamma, d\sigma_{F_{2N}}) < 2s$. This choice is possible since (c3.2). Label

$$t_0 = \sup\{t \in [0,1] \mid \gamma(t) \in \partial\varpi_i\}, \quad z_0 = \gamma(t_0).$$

Notice that this supremum exists because $\varpi_i \subset (\operatorname{Int} P) \setminus \overline{\operatorname{Int} P^\epsilon}$ (for a large enough N). Now, consider the basis $\widetilde{S}_i$ introduced in Remark 6, then we have

$$\left\| \left(F_{2N}(z) - X(z)\right)_{(*,\widetilde{S}_i)} \right\| \leqslant 2s + \frac{\mathtt{const}}{\sqrt{N}}, \tag{9}$$

$$\left| \left(F_{2N}(z) - X(z)\right)_{(3,\widetilde{S}_i)} \right| < \frac{\mathtt{const}}{N}. \tag{10}$$

Indeed,

$$\begin{aligned} \left\| \left(F_{2N}(z) - X(z)\right)_{(*,\widetilde{S}_i)} \right\| &\leqslant \left\| F_{2N}(z) - F_{2N}(z_0) \right\|_0 + \left\| F_{2N}(z_0) - F_{i-1}(z_0) \right\|_0 + \left\| F_{i-1}(z_0) - F_{i-1}(z) \right\|_0 + \left\| F_{i-1}(z) - X(z) \right\|_0 \\ &\leqslant \operatorname{length}(\gamma, d\sigma_{F_{2N}}) + \frac{\mathtt{const}}{N} + \frac{1}{\sqrt{N}} + \frac{\mathtt{const}}{N} < 2s + \frac{\mathtt{const}}{\sqrt{N}}, \end{aligned}$$

where we have used $(\mathrm{a7}_j)$, $j = 1, \dots, 2N$, and (b3). On the other hand, taking Remark 6 and $(\mathrm{a7}_j)$, $j = 1, \dots, 2N$, into account, we conclude

$$\begin{aligned} \left| \left(F_{2N}(z) - X(z)\right)_{(3,\widetilde{S}_i)} \right| &\leqslant \left\| F_{2N}(z) - F_i(z) \right\|_0 + \left| \left(F_i(z) - F_{i-1}(z)\right)_{(3,\widetilde{S}_i)} \right| + \left\| F_{i-1}(z) - X(z) \right\|_0 \\ &< \frac{\mathtt{const}}{N} + \frac{\mathtt{const}}{N} = \frac{\mathtt{const}}{N}. \end{aligned}$$

At this point, consider the following statement. Its proof is elemental, we leave the details to the reader.

Claim 5. *Let $0 < x < t$. Consider $p \in \mathrm{B}(t)$ and $v \in \mathbb{R}^3$ with $\langle \mathcal{N}_0(p), v \rangle_0 = 0$ and $\|v\|_0 = x$. Then, $p + v \in \mathrm{B}(\sqrt{t^2 - x^2})$.*

Now, Remark 6, Eqs. (5), (9) and (10), and the above claim guarantee that $F_{2N}(z) \in \mathrm{B}(R)$, if N was chosen large enough (see Fig. 2). This proves (c3.3) and finishes the proof of Claim 4. $\square$

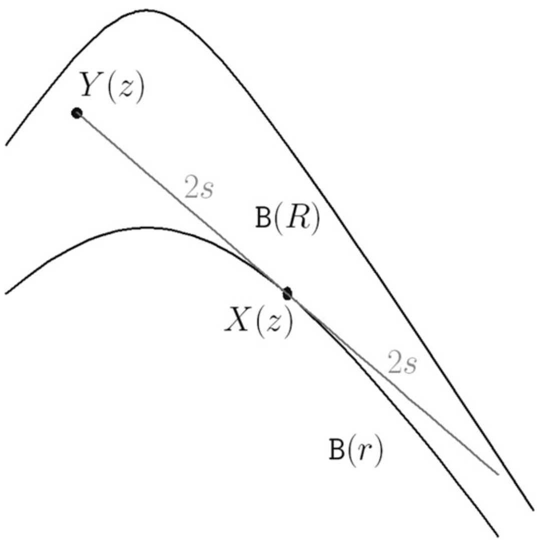

Fig. 2. The effect of the deformation.

From Claim 4 it is straightforward to check that (for N large enough) $Y = F_{2N} : \overline{\operatorname{Int} Q} \to \mathbb{L}^3$ satisfies the conclusion of Lemma 1.

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