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Critical exponents for semilinear dissipative wave equations in $\mathbf{R}^N$

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Abstract

We shall present new critical exponents $1+2m/N$ with $m \in [1,2]$ to the Cauchy problem $u_{tt} - \Delta u + u_t = |u|^{p-1}u$ with the initial data $[u_0, u_1] \in (H^1(\mathbf{R}^N) \cap L^m(\mathbf{R}^N)) \times (L^2(\mathbf{R}^N) \cap L^m(\mathbf{R}^N))$; that is, the small data global existence property can be derived to the Cauchy problem above with power $1+2m/N < p < +\infty$ $(N = 1, 2)$. Furthermore, the small data global nonexistence of solutions will be discussed in the case when $1 < p < 1+2m/N$ $(N \geqslant 1)$. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction

In this paper we are concerned with the following Cauchy problem in $\mathbf{R}^N$:

$$u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) = |u(t,x)|^{p-1}u(t,x),$$
$$(t,x) \in (0,\infty) \times \mathbf{R}^N, \tag{1.1}$$

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$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbf{R}^N. \tag{1.2}$$

In the sequel, $\|\cdot\|_q$ and $\|\cdot\|_{H^1}$ stand for the usual $L^q(\mathbf{R}^N)$-norm and $H^1(\mathbf{R}^N)$-norm, respectively. For simplicity of notation, in particular, we write $\|\cdot\|$ instead of $\|\cdot\|_2$.

For the Cauchy problem (1.1)–(1.2) in $\mathbf{R}^N$ $(N \geqslant 1)$ with the usual nonlinearity

$$1 + \frac{4}{N} \leqslant p < \frac{N+2}{N-2}, \tag{1.3}$$

for the small initial data without $L^1 \times L^1$ assumption Nakao and Ono [1] have already derived the global existence of small weak solutions $u(t,x)$ and the decay estimates

$$\|u(t,\cdot)\|^2 \leqslant C, \qquad \|u_t(t,\cdot)\|^2 + \|\nabla u(t,\cdot)\|^2 \leqslant C(1+t)^{-1}. \tag{1.4}$$

Their argument is based on the so-called (modified) potential well method combined with the energy method whose idea has originally come from [2] and [3]. In some sense, in the L^2-framework we may be able to say that the critical exponent is equal to $1 + 4/N$.

On the other hand, very recently Ikehata et al. [4] have just proved that the small data global existence property to the problem (1.1)–(1.2) occurs in the case when $1 + 2/N < p < +\infty$ $(N = 1, 2)$ to the problem (1.1)–(1.2) with the initial data $[u_0, u_1] \in (H^1(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)) \times (L^2(\mathbf{R}^N) \cap L^1(\mathbf{R}^N))$. In this case, $1+2/N$ is the critical exponent and in fact, it is called the Fujita exponent in the semilinear parabolic equation case.

So, a question naturally rises whether the critical exponent is equal to $1 + 2m/N$ or not if we choose the class of the initial data to be in $(H^1(\mathbf{R}^N) \cap L^m(\mathbf{R}^N)) \times (L^2(\mathbf{R}^N) \cap L^m(\mathbf{R}^N))$ with $m \in [1,2]$.

Note that in [5] the same critical exponent $1 + 2/N$ for all $N \geqslant 1$ has been already found to the problem (1.1)–(1.2) with the nonlinearity $|u|^{p-1}u$ replaced by $|u|^p$ in the framework of "compactly supported" initial data. However, we cannot find the question above from the argument in [5].

Before introducing our results we shall present a new extended critical exponent,

$$1 + \frac{2m}{N} < p < +\infty \quad (N = 1, 2), \tag{1.5}$$

where $m \in [1,2]$. Our first result reads as follows.

Theorem 1.1. *Let* $N = 1, 2$. *Suppose that* (1.5) *is satisfied. Then there exists a real number* $\epsilon_0 > 0$ *such that if the initial data* $(u_0, u_1) \in (H^1(\mathbf{R}^N) \cap L^m(\mathbf{R}^N)) \times (L^2(\mathbf{R}^N) \cap L^m(\mathbf{R}^N))$ *with* $m \in [1,2]$ *further satisfy*

$$I_{0,u} = \|u_0\|_m + \|u_0\|_{H^1} + \|u_1\|_m + \|u_1\| \leqslant \epsilon_0,$$

the problem (1.1)–(1.2) *admits a global solution* $u \in C([0,\infty); H^1(\mathbf{R}^N)) \cap C^1([0,\infty); L^2(\mathbf{R}^N))$ *satisfying the decay property*

$$\|u(t,\cdot)\|^2 \leqslant C I_{0,u}^2 (1+t)^{-N(1/m-1/2)},$$
$$\|u_t(t,\cdot)\|^2 + \|\nabla u(t,\cdot)\|^2 \leqslant C I_{0,u}^2 (1+t)^{-1-N(1/m-1/2)}$$

with some generous constant $C > 0$.

Remark 1.1. This result implies that the decay condition of the initial data as $|x| \to +\infty$ reflects on the critical exponent, and so that on the small data global existence property. We shall present a conjecture that for higher dimensional case $N \geqslant 3$ (cf. [5]), $1 + 2m/N < p < (N+2)/[N-2]^+$ also becomes the region for which the small data global existence property occurs with the initial data as in Theorem 1.1. Note that these results are closely related with the diffusion structure of Eq. (1.1) (see [6] and [7]).

In the case when $N \geqslant 3$ Theorem 1.1 can be read as follows.

Theorem 1.2. *Let* $3 \leqslant N \leqslant 6$ *and suppose that* (m,p) *satisfies*

$$\frac{\sqrt{N^2+16N}-N}{4} \leqslant m < \min\left\{2, \frac{N}{N-2}\right\},$$
$$1 + \frac{2m}{N} < p \leqslant \frac{N}{N-2}.$$

Then one has the same conclusion as in Theorem 1.1.

Next let us discuss the counterpart of the condition (1.5); that is, under the assumption

$$1 < p < 1 + \frac{2m}{N} \quad (N \geqslant 1) \tag{1.6}$$

we shall derive the blowup property to the Cauchy problem (1.1)–(1.2).

In [8] and [5] the global nonexistence property to the problem (1.1)–(1.2) with nonlinearity $|u|^{p-1}u$ replaced by $|u|^p$ have already been discussed in the case when $1 < p \leqslant 1 + 2/N$ or $1 < p < 1 + 2/N$, so that the present question concerning blowup of solutions under (1.6) is quite natural. Our third result reads as follows.

Theorem 1.3. *Let* $N \geqslant 1$ *and assume* (1.6) *with* $1 \leqslant m \leqslant 2$. *Then, for any* $\epsilon > 0$ *there exists* $(u_0, u_1) \in (H^1(\mathbf{R}^N) \cap L^m(\mathbf{R}^N)) \times (L^2(\mathbf{R}^N) \cap L^m(\mathbf{R}^N))$ *satisfying*

$$I_{0,u} \equiv \|u_0\|_{H^1} + \|u_0\|_m + \|u_1\| + \|u_1\|_m \leqslant \epsilon$$

such that the associated solution $u(t,x)$ *to the problem* (1.1)–(1.2) *does not exist globally in time.*

By the way, in the occasion of the proof of Theorems 1.1 and 1.2, we shall proceed our argument based on the following well-known result (cf. [9] and [1]):

Proposition 1.1. *Suppose* $1 < p \leqslant N/[N-2]^+$ *with* $N \geqslant 1$. *For each* $(u_0, u_1) \in H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$, *there exists a maximal existence time* $T_{\max} > 0$ *such that the problem* (1.1)–(1.2) *has a unique solution* $u \in C([0, T_{\max}); H^1(\mathbf{R}^N)) \cap C^1([0, T_{\max}); L^2(\mathbf{R}^N))$. *If* $T_{\max} < +\infty$, *then*

$$\lim_{t \uparrow T_{\max}} \left[\|u(t,\cdot)\| + \|\nabla u(t,\cdot)\| + \|u(t,\cdot)\| \right] = +\infty.$$

2. Proof of Theorems 1.1, 1.2 and 1.3

In this section first let us prove Theorems 1.1 and 1.2. For this aim we first prepare the following lemma, that is the Gagliardo–Nirenberg inequality.

Lemma 2.1. *Let* $1 \leqslant r < q \leqslant 2N/[N-2]^+$, $2 \leqslant q$ *and* $N \geqslant 1$. *Then the inequality*

$$\|v\|_q \leqslant K_0 \|\nabla v\|^{\theta} \|v\|_r^{1-\theta}, \quad v \in H^1\big(\mathbf{R}^N\big),$$

holds with some constant $K_0 > 0$ *and*

$$\theta = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{r} - \frac{1}{2} + \frac{1}{N}\right)^{-1}$$

provided that $0 < \theta \leqslant 1$.

Now we shall consider the linear wave equation

$$v_{tt}(t,x) - \Delta v(t,x) + v_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}^N, \tag{2.1}$$

$$v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \quad x \in \mathbf{R}^N. \tag{2.2}$$

For this linear problem, in [10] and [11] the so-called $L^p - L^q$ estimates of solutions have been already derived. Thus by using these $L^p - L^q$ estimates and the so-called Duhamel principle we shall handle the semilinear problem (1.1)–(1.2).

Proposition 2.1 [10, Proposition 3.2]. *Let* $N \geqslant 1$. *Then for each* $(v_0, v_1) \in (H^1(\mathbf{R}^N) \cap L^m(\mathbf{R}^N)) \times (L^2(\mathbf{R}^N) \cap L^m(\mathbf{R}^N))$ *with* $m \in [1,2]$, *the solution* v *of* (2.1)–(2.2) *satisfies*

$$\|v(t,\cdot)\|^2 \leqslant C\big(\|v_0\| + \|v_0\|_m + \|v_1\| + \|v_1\|_m\big)^2 (1+t)^{-N(1/m-1/2)}.$$

Proposition 2.2 [11, Lemma 1]. *Let* $N \geqslant 1$. *Then for each* $(v_0, v_1) \in C_0^\infty(\mathbf{R}^N) \times C_0^\infty(\mathbf{R}^N)$, *the solution* v *of* (2.1)–(2.2) *satisfies*

$$\|v_t(t,\cdot)\|^2 + \|\nabla v(t,\cdot)\|^2 \leqslant C\big(\|v_0\|_{H^1} + \|v_0\|_m + \|v_1\| + \|v_1\|_m\big)^2 (1+t)^{-1-N(1/m-1/2)}.$$

Of course, in [10] and [11] the more precise estimates have been derived. Anyway, based on Propositions 2.1 and 2.2 let us derive the following total energy decay estimates to the weak solution of the linear problem (2.1)–(2.2).

Proposition 2.3. *Let* $N \geqslant 1$. *Then for each* $(v_0, v_1) \in (H^1(\mathbf{R}^N) \cap L^m(\mathbf{R}^N)) \times (L^2(\mathbf{R}^N) \cap L^m(\mathbf{R}^N))$ *with* $m \in [1,2]$, *the weak solution* $v \in C([0,+\infty); H^1(\mathbf{R}^N)) \cap C^1([0,+\infty); L^2(\mathbf{R}^N))$ *to the problem* (2.1)–(2.2) *has the decay estimates*

$$\|v(t,\cdot)\|^2 \leqslant C\big(\|v_0\| + \|v_0\|_m + \|v_1\| + \|v_1\|_m\big)^2 (1+t)^{-N(1/m-1/2)},$$

$$\|v_t(t,\cdot)\|^2 + \|\nabla v(t,\cdot)\|^2 \leqslant C\big(\|v_0\|_{H^1} + \|v_0\|_m + \|v_1\| + \|v_1\|_m\big)^2 (1+t)^{-1-N(1/m-1/2)}.$$

Proof. Since $v_0 \in H^1(\mathbf{R}^N) \cap L^m(\mathbf{R}^N)$ and $v_1 \in L^2(\mathbf{R}^N) \cap L^m(\mathbf{R}^N)$ can be approximated by smooth functions $\{\phi_n\} \subset C_0^\infty(\mathbf{R}^N)$ and $\{\psi_n\} \subset C_0^\infty(\mathbf{R}^N)$ satisfying

$$\|\phi_n - v_0\|_{H^1} + \|\phi_n - v_0\|_m \to 0 \quad (n \to +\infty),$$

$$\|\psi_n - v_1\| + \|\psi_n - v_1\|_m \to 0 \quad (n \to +\infty),$$

the desired statement is rather standard. □

Under these preparations we can prove Theorem 1.1. The proof will be done along the same way as in [12].

Proof of Theorem 1.1. First define a semigroup $S(t): H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N) \to H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$ by

$$S(t): \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \mapsto \begin{bmatrix} v(t,\cdot) \\ v_t(t,\cdot) \end{bmatrix},$$

where $v(t,\cdot) \in C([0,+\infty); H^1(\mathbf{R}^N)) \cap C^1([0,+\infty); L^2(\mathbf{R}^N))$ is a unique solution to the "linear" problem (2.1)–(2.2).

The following well-known inequalities are useful in order to derive some decay rate (see [13]).

Lemma 2.2. *If* $\beta > 1$ *and* $\eta \leqslant \beta$, *then there exists a constant* $C_{\beta,\eta} > 0$ *depending only on* β *and* η *such that*

$$\int_0^t (1+t-s)^{-\eta}(1+s)^{-\beta}\,ds \leqslant C_{\beta,\eta}(1+t)^{-\eta}$$

for all $t \geqslant 0$.

In the following paragraph we set $I_{0,u} = I_0$ for simplicity. Now we shall derive the decay property of a nonlinear problem (1.1)–(1.2). By a standard semigroup theory, the nonlinear problem (1.1)–(1.2) is rewritten as the integral equation

$$U(t) = S(t)U_0 + \int_0^t S(t-s)F(s)\,ds, \tag{2.3}$$

where

$$U(t) = \begin{bmatrix} u(t,\cdot) \\ u_t(t,\cdot) \end{bmatrix}, \qquad U_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \qquad F(s) = \begin{bmatrix} 0 \\ f(u(s,\cdot)) \end{bmatrix}$$

with $f(u)(x) = |u(x)|^{p-1}u(x)$.

We proceed our argument based on the way of [14]. In order to show global existence, it suffices to obtain the a priori estimates for $\|u_t(t,\cdot)\| + \|\nabla u(t,\cdot)\|$ and $\|u(t,\cdot)\|$ in the interval of existence $[0, T_{\max})$ (see Proposition 1.1). As a result of Proposition 2.3, first one has

Lemma 2.3. *Under the assumptions as in Theorem* 1.1*, we have*

$$\|S(t)U_0\|_E \leqslant C I_0 (1+t)^{-1/2-(N/2)(1/m-1/2)}$$

on $[0, T_{\max})$*, where we set*

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_E = \|v\| + \|\nabla u\|.$$

Furthermore, if

$$I(s) = \big\|f(u(s,\cdot))\big\| + \big\|f(u(s,\cdot))\big\|_m < +\infty \tag{2.4}$$

for each $s \in [0,t]$ with $t \in [0, T_{\max})$, then from Proposition 2.3 we have

$$\|S(t-s)F(s)\|_E \leqslant C I(s)(1+t-s)^{-1/2-(N/2)(1/m-1/2)}. \tag{2.5}$$

Thus from (2.3) one can estimate $U(t)$ as follows:

$$\begin{aligned} \|U(t)\|_E &\leqslant C I_0 (1+t)^{-1/2-(N/2)(1/m-1/2)} \\ &\quad + C\int_0^t (1+t-s)^{-1/2-(N/2)(1/m-1/2)} I(s)\,ds. \end{aligned} \tag{2.6}$$

Take $K > 0$ so large and choose $T \in (0, T_m)$ so small such as

$$\|U(t)\|_E \leqslant KI_0(1+t)^{-1/2-(N/2)(1/m-1/2)} \quad \text{on } [0,T), \tag{2.7}$$

$$\|u(t)\| \leqslant KI_0(1+t)^{-(N/2)(1/m-1/2)} \quad \text{on } [0,T). \tag{2.8}$$

By Lemma 2.1 and the assumption (1.5) we have

$$\bigl\|f(u(s,\cdot))\bigr\|_m = \|u(t,\cdot)\|_{mp}^p \leqslant K_0\|u(s,\cdot)\|^{p(1-\theta_1)}\|\nabla u(s,\cdot)\|^{p\theta_1}$$

with $\theta_1 = N(mp-2)/2mp \in (0,1]$. Note that (1.5) implies

$$mp > 2 \tag{2.9}$$

for each $m \in [1,2]$. Similarly one has

$$\bigl\|f(u(s,\cdot))\bigr\| \leqslant K_0\|u(s,\cdot)\|^{p(1-\theta_2)}\|\nabla u(s,\cdot)\|^{p\theta_2}$$

with $\theta_2 = N(p-1)/2p \in (0,1]$. Therefore, as long as (2.7) and (2.8) hold one gets

$$\begin{aligned}
\bigl\|f(u(s,\cdot))\bigr\|_m &\leqslant K_0\bigl\{KI_0(1+s)^{-(N/2)(1/m-1/2)}\bigr\}^{p(1-\theta_1)} \\
&\quad \times \bigl\{KI_0(1+s)^{-1/2-(N/2)(1/m-1/2)}\bigr\}^{p\theta_1} \\
&= K_0K^pI_0^p(1+s)^{-p(\theta_1/2+(N/2)(1/m-1/2))}, \\
\bigl\|f(u(s,\cdot))\bigr\| &\leqslant K_0\bigl\{KI_0(1+s)^{-(N/2)(1/m-1/2)}\bigr\}^{p(1-\theta_2)} \\
&\quad \times \bigl\{KI_0(1+s)^{-1/2-(N/2)(1/m-1/2)}\bigr\}^{p\theta_2} \\
&= K_0K^pI_0^p(1+s)^{-p(\theta_2/2+(N/2)(1/m-1/2))}.
\end{aligned}$$

Summarizing these calculations we have the following lemma which shows the validity of the condition (2.4).

Lemma 2.4. *As long as* (2.7) *and* (2.8) *hold on* $[0,T)$ *we have*

$$\begin{aligned}
\bigl\|f(u(t,\cdot))\bigr\|_m &\leqslant K_0K^pI_0^p(1+t)^{-p(N/2)(1/m-1/2)-N(mp-2)/4m}, \\
\bigl\|f(u(t,\cdot))\bigr\| &\leqslant K_0K^pI_0^p(1+t)^{-p(N/2)(1/m-1/2)-N(p-1)/4m}.
\end{aligned}$$

By applying Lemmas 2.3 and 2.4 to (2.3) we see that

$$\begin{aligned}
\|U(t)\|_E &\leqslant CI_0(1+t)^{-1/2-(N/2)(1/m-1/2)} \\
&\quad + C\int_0^t (1+t-s)^{-1/2-(N/2)(1/m-1/2)}K_0K^pI_0^p \\
&\quad \times \bigl\{(1+s)^{-\gamma_1} + (1+s)^{-\gamma_2}\bigr\}\,ds \\
&\leqslant CI_0(1+t)^{-1/2-(N/2)(1/m-1/2)} \\
&\quad + CK_0K^pI_0^p\int_0^t (1+t-s)^{-1/2-(N/2)(1/m-1/2)}(1+s)^{-\gamma_1}\,ds,
\end{aligned}$$

where

$$\gamma_1 = \frac{N}{2}p\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{N(mp-2)}{4m},$$
$$\gamma_2 = \frac{N}{2}p\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{N(p-1)}{4}. \tag{2.10}$$

Note that $\gamma_1 > 1$ because of the assumptions (1.5) and $\gamma_1 \leqslant \gamma_2$. Thus from Lemma 2.2 and again (1.5) it follows that

$$\|U(t)\|_E \leqslant CI_0(1+t)^{-1/2-(N/2)(1/m-1/2)} + CK_0K^pI_0^p(1+t)^{-1/2-(N/2)(1/m-1/2)}$$

with some constant $C > 0$. Setting

$$Q_0(I_0, K) = C + CK_0K^pI_0^{p-1},$$

we get the following lemma.

Lemma 2.5. *As long as* (2.7) *and* (2.8) *hold on* $[0, T)$ *we get*

$$\|U(t)\|_E \leqslant I_0Q_0(I_0, K)(1+t)^{-1/2-(N/2)(1/m-1/2)}.$$

Next let us derive the L^2-estimates for the local solution $u(t, x)$ to the problem (1.1)–(1.2). Indeed, we have from (2.3) and Proposition 2.3 that

$$\|u(t,\cdot)\| \leqslant CI_0(1+t)^{-(N/2)(1/m-1/2)} + C\int_0^t (1+t-s)^{-(N/2)(1/m-1/2)} I(s)\,ds.$$

Therefore, it follows from Lemma 2.4 and the similar argument to Lemma 2.5 that

$$\begin{aligned}\|u(t,\cdot)\| &\leqslant CI_0(1+t)^{-(N/2)(1/m-1/2)} \\ &\quad + C\int_0^t (1+t-s)^{-(N/2)(1/m-1/2)}K_0K^pI_0^p \\ &\quad \times \left[(1+s)^{-N(mp-2)/4m-p(N/2)(1/m-1/2)} \right. \\ &\quad \left. + (1+s)^{-N(p-1)/4-p(N/2)(1/m-1/2)}\right]ds \\ &\leqslant CI_0(1+t)^{-(N/2)(1/m-1/2)} \\ &\quad + CK_0K^pI_0^p\int_0^t (1+t-s)^{-(N/2)(1/m-1/2)} \\ &\quad \times (1+s)^{-N(mp-2)/4m-p(N/2)(1/m-1/2)}\,ds\end{aligned}$$

with some generous constant $C > 0$, where $N(mp-2)/4m + p(N/2)(1/m - 1/2) > 1$ because of (1.5). This together with Lemma 2.2 implies

$$\|u(t,\cdot)\| \leqslant CI_0(1+t)^{-(N/2)(1/m-1/2)} + CK_0K^pI_0^p(1+t)^{-(N/2)(1/m-1/2)}.$$

Thus we have

Lemma 2.6. *As long as* (2.7) *and* (2.8) *hold on* $[0,T)$ *it follows that*

$$\|u(t,\cdot)\| \leqslant I_0Q_0(I_0,K)(1+t)^{-(N/2)(1/m-1/2)}.$$

Take $K > C$ so large and take I_0 so small such as

$$CK_0K^pI_0^{p-1} < K - C. \tag{2.11}$$

For such $K > 0$ and I_0 we have

$$Q_0(I_0,K) < K.$$

Therefore, by combining this with Lemmas 2.5 and 2.6 we see that

$$\|U(t)\|_E < KI_0(1+t)^{-1/2-(N/2)(1/m-1/2)}, \tag{2.12}$$

$$\|u(t,\cdot)\| < KI_0(1+t)^{-(N/2)(1/m-1/2)} \tag{2.13}$$

on $[0,T)$. Thus (2.7), (2.8) and (2.12), (2.13) show that under the assumption (2.11), the local solution $u(t,\cdot)$ exists globally in time and these estimates hold in fact for all $t \geqslant 0$. Taking

$$\epsilon_0 = \left(\frac{K-C}{CK_0K^p}\right)^{1/(p-1)},$$

the proof of Theorem 1.1 is now finished. □

Proof of Theorem 1.2. Under the assumptions as in Theorem 1.2, we have $mp > 2$, $N(mp-2)/2mp \in (0,1]$, $\gamma_1 > 1$ and $\gamma_1 \leqslant \gamma_2$ (see (2.9) and (2.10)). So the same argument as in Theorem 1.1 can be proceeded and we have the desired conclusion. □

Finally let us prove Theorem 1.3. For this purpose we set

$$E(u,v) = J(u) + \frac{1}{2}\|v\|^2, \qquad J(u) = \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}.$$

In the occasion of the proof of Theorem 1.3 the following blowup result due to Levine [15] is crucial in our argument (see also [16]).

Lemma 2.7 ([15]). *If* $(u_0,u_1) \in H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$ *satisfies* $E(u_0,u_1) < 0$, *then the associated solution to the problem* (1.1)–(1.2) *does not exist globally in time.*

Proof of Theorem 1.3. Let $\phi \in (H^1(\mathbf{R}^N) \cap L^m(\mathbf{R}^N))\backslash\{0\}$ and $\alpha > 0$ such that $N/m < \alpha < 2/(p-1)$ be arbitrarily fixed. This is guaranteed by an assumption (1.6). For $\lambda > 0$, set

$$\phi_\lambda(x) = \lambda^\alpha \phi(\lambda x).$$

Then one has

$$E(\phi_\lambda, 0) = J(\phi_\lambda) < 0 \tag{2.14}$$

for sufficiently small $\lambda > 0$. Indeed, first we have

$$J(\phi_\lambda) = \frac{1}{2}\lambda^{2\alpha+2-N}\|\nabla\phi\|^2 - \frac{1}{p+1}\lambda^{\alpha(p+1)-N}\|\phi\|_{p+1}^{p+1}.$$

Since $\alpha < 2/(p-1)$ implies $2\alpha + 2 - N > \alpha(p+1) - N$, the desired (2.14) can be derived if we take $\lambda > 0$ small enough. Therefore, the solution $u_\lambda(t,x)$ to the problem (1.1)–(1.2) with $(\phi_\lambda, 0)$ as the initial data blows up in a finite time because of Lemma 2.7. On the other hand, we see that

$$\begin{aligned}&\|\phi_\lambda\|_m + \|\phi_\lambda\| + \|\nabla\phi_\lambda\| \\ &\quad = \lambda^{\alpha-N/m}\|\phi\|_m + \lambda^{\alpha-N/2}\|\phi\| + \lambda^{\alpha+1-N/2}\|\nabla\phi\|.\end{aligned}$$

Since $m \leqslant 2$ and $N/m < \alpha$ imply $0 < \alpha - N/m \leqslant \alpha - N/2 < \alpha - N/2 + 1$, for each $\epsilon > 0$ if we take $\lambda > 0$ further sufficiently small, one has

$$\|\phi_\lambda\|_m + \|\phi_\lambda\| + \|\nabla\phi_\lambda\| \leqslant \epsilon,$$

which implies the desired statement. □

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