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A direct variational methods applied to Burgers' equation

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Abstract

In this study, a compact approximate method in limiting form for calculating the solution of Burgers' equation with appropriate boundary conditions is presented. The results obtained by present method are found to be in good agreement with those due to earlier authors and offers appreciable advantages for Burgers' like nonlinear problems.

Keywords: Direct variational method; Burgers' equation

AMS classification: 35Q53

1. Introduction

The study of the properties of the Burgers' equation has attracted considerable attention due to its application in the approximate theory of flow through shock wave propagating in a viscous fluid [5] and in the modelling of turbulence.

Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2} \tag{1}$$

is a quasilinear parabolic partial differential equation and for most of the fluid mechanics applications v is a small parameter. Burgers' equation and Navier–Stokes equation are similar due to the form of their nonlinear terms and the occurrence of higher order derivatives with small coefficients in both. Fortunately, Burgers' equation is one of the very few nonlinear partial differential equations which can be solved exactly for an arbitrary initial and boundary conditions

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[3, 5, 7]. Also, for $v = 0$, Burgers' equation reduces to the momentum equation from gas dynamics which is used as a test problem for several numerical schemes (see, for example, [2]).

But, the main difficulty arises in the numerical solution of Burgers' equation for small values of viscosity, i.e., for large Reynolds numbers. Miller [8] obtained some results by using a predictor–corrector method with those by an explicit method and the exact solution of Eq. (1) subject to boundary conditions $u(x,0) = \sin x$, $u(0,t) = u(1,t) = 0$. This study showed that for $0.01 \leqslant v \leqslant 1$ the explicit method, the predictor-corrector method and the exact solution are in good agreement. As v decreases from 10^{-2} and 10^{-4} to evaluate the exact solution is not practical because of the slow convergence of the Fourier series. Also, the explicit method produces unacceptable kinks and to overcome these very small space and time increments are required [2].

Recently, a new finite-element method which employs space–time elements and incorporates characteristics has been developed for the solution of Burgers' equation [11]. Again, Galerkin and Petrov–Galerkin finite-element methods involving a time dependent grid have been successfully used to obtain accurate numerical solutions [4]. Ali et al. [1] applied the finite-element approach using Galerkin method with quadratic spline interpolation functions and a constant grid of elements. On the other hand, [6, 10] revealed how complementary variational principles can be applied to nonlinear equations: the example chosen was Burgers' equation and a special test, the steady-state form of Burgers' equation, was solved numerically.

In this paper, direct variational method is used to solve Burgers' equation with the aim of generating an approximate solution with general boundary conditions in the form of the sequence $U_n(x,t)$ where $\lim_{n\to\infty} U_n = U$ which is the exact solution.

2. Direct variational methods

There is an extensive literature on direct variational methods. The comprehensive book is that of Rectorys [9] which contains numerous examples and references.

To make clear the idea of the method and refresh the minds, let us consider a simple example of an equation of the second order. Let Ω be a bounded region in the N-dimensional Euclidean space E_N with boundary Ω, let $x_1, \ldots, x_N$ be cartesian coordinates of the point $x \in E_N$. Denote $Q = \Omega \times (0, T)$. Let the Dirichlet problem for a parabolic equation be given

$$Au + \frac{\partial u}{\partial t} = f(x) \quad \text{in } Q, \tag{2}$$

$$u(x,0) = u_0(x), \tag{3}$$

$$u = 0 \quad \text{on } \Omega \times (0, T). \tag{4}$$

Denote

$$a(u,u) = \int_\Omega \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, \mathrm{d}x$$

and (V,u) the usual scalar product of real functions V and u in region Ω. Let us divide the interval $[0,T]$ into p subintervals of the same length h, and let us define the following functionals:

$$G_1(u) = a(u,u) + \frac{1}{h}(u,u) - 2(f,u) - \frac{2}{h}(u_0,u),$$

$$G_2(u) = a(u,u) + \frac{1}{h}(u,u) - 2(f,u) - \frac{2}{h}(u_1,u), \tag{5}$$

$$\vdots$$

$$G_p(u) = a(u,u) + \frac{1}{h}(u,u) - 2(f,u) - \frac{2}{h}(u_{p-1},u).$$

Under well-known assumptions on a $a(u,u)$ the functional $G_1(u)$ attains its minimum in a certain class of functions satisfying condition (4) and the minimizing function u_1 is the solution of elliptic problem

$$Au + \frac{1}{h}(u_1 - u_0) = f \quad \text{in } \Omega,\ u = 0 \text{ on } \Omega. \tag{6}$$

For every $t_i = i \cdot h$ $(i = 1, \ldots, p)$ being

$$\frac{1}{h}[u_i(x) - u_{i-1}(x)] \cong \frac{\partial u}{\partial t}(x,t_i)$$

each of the functions $u_i(x)$ can be taken as an approximation, in the hyperplane $t = t_i$ $(x \in \Omega)$, of the solution $u(x,t)$ of the problem (2)–(4). The approximation $u_1(x,t)$ can be defined in the whole Q, for example as a function continuous and piecewise linear in t for every fixed $x \in Q$, assuming the values $u_i(x)$ at the points $t = t_i$.

Thus,

$$u_i(x,t) = u_j(x) - \frac{t - t_j}{h}[u_{j+1}(x) - u_j(x)] \quad t_j \leqslant t \leqslant t_{j+1}. \tag{7}$$

Let us, in a similar way, construct $u_2(x,t)$ with the only difference that instead of dividing the interval $[0,T]$ into p subintervals of the length h as before, we divide it into $2p$ subintervals of the length $h_2 = h/2$. Going on in this way and dividing subsequently the interval $[0,T]$ into $4p, 8p, \ldots, 2^{n-1}p, \ldots$ subintervals, we construct a sequence of functions $u_n(x,t)$, defined in Q by the relations

$$u_n(x,t) = u_j^n(x) - \frac{t - t_j^n}{h_n}[u_{j+1}^n(x) - u_j^n(x)] \quad \text{for } t_j^n \leqslant t \leqslant t_{j+1}^n \tag{8}$$

where $h_n = h/2^{n-1}$, $t_j^n = j \cdot h_n$ $(j = 0, 1, \ldots, p \cdot 2^{n-1} - 1)$.

In this way we get the sequence $\{u_n(x,t)\}$ of approximate solutions of the problem (2)–(4).

3. Application to Burgers' equation

We are concerned with Burgers' equation

$$u_t + uu_x = vu_{xx} \tag{9}$$

for a real field, $u = u(x,t)$, and specially consider the initial value problem

$$u(x,0) = u_0(x) = \sin \pi x \tag{10a}$$

for periodic boundary conditions, on an interval of length L:

$$u(x + L, t) = u(x,t) = 0. \tag{10b}$$

Therefore, the problem involves the decay of a sinusoidal disturbance with time. Analytical solutions of these problems are often available through use of the Hoft–Cole [5,7] transformations. For the current problem, the Hoft–Cole transformation takes the form

$$u(x,t) = -2v(\theta_x/\theta). \tag{11}$$

Substitution of (11) into (9) results in the following linear equation for the function $\theta(x,t)$ in (11):

$$v\theta_{xx} = \theta_t. \tag{12}$$

Eq. (12) is recognised as the diffusion equation with v serving as viscosity of fluid, causes diffusion of any differences in velocity. The transformation to the linear diffusion equation of course renders the nonlinear Burgers' equation analytically solvable, and great variety of solutions have been developed following this technique. However, the boundary and initial conditions must be similarly transformed into the space $\theta(x,t)$ and this is where the limitations of the technique expose themselves.

Much of Burgers' work on the equation prior to the introduction of the Hoft–Cole transformation involved simple boundary conditions, typically homogeneous initial condition exhibiting spatial character in x. Later, using same transformation, the approach often followed in studying Burgers' equation has been to transform known heat conduction solutions, representing boundary and initial conditions of physical relevance to those problems, into solutions of boundary-value problems governed by Burgers' equation. In some cases these transformed solutions have obvious relevance to nonlinear physics governed by Burgers' equation and in others not [12].

But the above mentioned limitations may be overcome by the application of direct variational methods to these problems after the Hoft–Cole transformation. For the current problem, in this sense, we applied the direct variational method to generate limiting solution which can be generalised to any boundary-value problems governed by Burgers' equation.

4. Solution of equivalent diffusion problem by direct variational method

With suitable changes of nomenclature the Hoft–Cole transformations of (9), (10a) and (10b) are

$$v\theta_{xx} = \theta_t, \tag{13}$$

$$\theta_x(0,t) = \theta_x(1,t) = 0, \tag{14a}$$

$$\theta(x,0) = \exp\{-(2nv)^{-1}[1 - \cos(\pi x)]\} \tag{14b}$$

which is a parabolic initial-value problem.

In practice, the minima of functionals (5) are determined approximately, using some of the well-known direct methods, for example, the Ritz method. In order to simplify matter, we show how the direct variational methods already mentioned in Section 2 can be applied to the reduced version of Burgers' equation.

Now, there are many ways to choose base functions in variational calculation. A standard procedure is to choose suitable base functions containing r unknown parameters from the finite dimensional subspace, V say, of H. If we consider the problem defined by (13), (14a) and (14b), the suitable base functions can be chosen as

$$V_1(x) = \cos(\pi x), \quad V_2(x) = \cos(2\pi x), \quad \dots, \quad V_r(x) = \cos(r\pi x). \tag{15}$$

Let us construct a function

$$U_{r1}(x) = c_{11}V_1(x) + \cdots + c_{1r}V_r(x) \tag{16}$$

determining the unknown coefficients c_{1i} in a well-known way from the condition that $G_1(U_{r1})$ be minimal. Since $\cos(\pi x)$ and $\sin(\pi x)$ are orthogonal,

$$(((V_i, V_k))) = v\int_0^1 i\pi \sin(i\pi x)\cdot k\pi \sin(k\pi x)\,dx + h^{-1}\int_0^1 \cos(i\pi x)\cdot\cos(k\pi x)\,dx$$

where $h = T/p$ and $[0, T]$. Then,

$$(((V_i, V_k))) = \begin{cases} \dfrac{vi^2\pi^2}{2} + \dfrac{1}{2h}, & i = k, \\ 0, & i \neq k. \end{cases}$$

The right hand side of the system would be

$$\begin{aligned}(V_1, \theta_0) &= \int_0^1 [1 - (2\pi v)^{-1}(1 - \cos\pi x)]\cos\pi x\,dx \\ &= \int_0^1 \cos\pi x\,dx - (2\pi v)^{-1}\int_0^1 (\cos\pi x - \cos^2\pi x)\,dx \\ &= \tfrac{1}{4}\pi v.\end{aligned}$$

Consequently, Ritz system:

$$2^{-1}(v\pi^2 + h^{-1})c_{11} = \tfrac{1}{4}h\pi v$$

and

$$c_{11} = (4hv)^{-1}/[\tfrac{1}{2}(v\pi^2 + h^{-1})]. \tag{17}$$

Substituting (17) in (16)

$$U_{r1} = \tfrac{1}{2}(4h\pi v)^{-1}(v\pi^2 + h^{-1})^{-1}\cos\pi x \tag{18}$$

is obtained. Then, substitute U_{r1} into $G_2(u_{r1})$, similarly find

$$U_{r2} = (8h^2\pi v)^{-1}4(v\pi^2 + h^{-1})^{-2}\cos 2\pi x \tag{19}$$

so that $G_2(u_{r2})$ be minimal. Going on this way,

$$U_{rj}(x) = (2^{j+1}h^j\pi v)^{-1}2^j(v\pi^2 + h^{-1})^{-j}\cos 2j\pi x \quad (j = 1, 2, \ldots, p) \tag{20}$$

is easily obtained. Having these functions, we construct a function $\theta_1(x,t)$ substituting U_{rj} in to (8). Namely,

$$\theta_1(x,t) = U_{rj}(x) - \frac{t - t_0}{h}[U_{rj+1}(x) - U_{rj}(x)]. \tag{21}$$

Now, for $j = 0$

$$\begin{aligned}\theta_1(x,t) &= U_{r0}(x) - \frac{t - t_0}{h}[U_{r1}(x) - U_{r0}(x)] \\ &= 1 - (2\pi v)^{-1}(1 - \cos\pi x) + \frac{t - t_0}{h}\left[\frac{\cos\pi x}{4hv\pi(\frac{1}{2}\pi^2 v + \frac{1}{2}h)} - 1 + \frac{(1 - \cos\pi x)}{2v\pi}\right] \\ &= \frac{2\pi v - 1}{2\pi v}\left(1 - \frac{t - t_0}{h}\right) + \frac{(\pi^2 vh + 1) - \pi^2 v(t - t_0)}{2\pi v(\pi^2 vh + 1)}\cos\pi x;\end{aligned} \tag{22}$$

for $j = 1$

$$\begin{aligned}\theta_1(x,t) &= U_{r1}(x) - \frac{t - t_1}{h}[U_{r2}(x) - U_{r1}(x)] \\ &= \frac{(\pi^2 vh + 1) - \pi^2 v(t - t_1)}{2v\pi(\pi^2 vh + 1)^2}\cos\pi x;\end{aligned} \tag{23}$$

for $j = 2$

$$\begin{aligned}\theta_1(x,t) &= U_{r2}(x) - \frac{t - t_2}{h}[U_{r3}(x) - U_{r2}(x)] \\ &= \frac{(\pi^2 vh + 1) - \pi^2 v(t - t_2)}{2v\pi(\pi^2 vh + 1)^3}\cos\pi x.\end{aligned} \tag{24}$$

Generalising (22)–(24) we obtained the first step solution to the problem given by (13), (14a) and (14b) as

$$\theta_1(x,t) = \frac{2\pi v - 1}{2\pi v}\left(1 - \frac{t - t_0}{h}\right) + \frac{(\pi^2 vh + 1) - \pi^2 v(t - t_j)}{2\pi v(\pi^2 vh + 1)^{j+1}}\cos\pi x. \tag{25}$$

Then, dividing the interval $[0, T]$ into $2p, 4p, \ldots$ subintervals, we come in the same way to the functions $\theta_2(x,t), \theta_3(x,t), \ldots$ and generally to the function

$$\theta_n(x,t) = \frac{2\pi v - 1}{2\pi v}\left(1 - \frac{t - t_0}{h}\right)^n + \frac{(\pi^2 vh + 1) - \pi^2 v(t - t_j)}{2\pi v(\pi^2 vh + 1)^{j+n}}B^{n-1}\cos\pi x. \tag{26}$$

where $B = (n^2 vh + 1) - n^2 v(t - t_0)$, $(j = 0, 1, \ldots, p - 1)$ $(n = 1, 2, \ldots)$ which is of a similar form as the function (7). Hence, Eq. (26) generates successive approximations to solution for the problem given by (13), (14a) and (14b).

Finally, to achieve the solution to the current problem (9), (10a) and (10b) we substitute relevant values from (26) into (11) to obtain the approximate solution in the limiting form, i.e.,

$$U_n(x,t) = 2\pi v \frac{AB^{n-1}\sin \pi x}{C + AB^{n-1}\cos \pi x} \tag{27}$$

where

$$A = \frac{[(v\pi^2 h + 1) - v\pi^2(t - t_j)]}{[2v\pi(v\pi^2 h + 1)^{j+n}]},$$

$$B = (v\pi^2 h + 1) - v\pi^2(t - t_0),$$

$$C = \frac{(2v\pi - 1)}{2v\pi}\left(1 - \frac{t - t_0}{h}\right)^n.$$

Thus, $\lim_{n\to\infty} U_n(x,t) = u(x,t)$ would be the exact solution of the desired problem. I.e., this approach is easily generalised to yield approximate variational solution U_r containing r parameters whose fit to the exact solution increases with r and the approach is inductive. It also indicates that variational scheme is converging (see [9]). It should be noted also that all integrals in the calculation were carried out analytically, because of our choice of base functions, which makes the scheme more compact.

5. Numerical results and graphical interpretation

In order to evaluate the numerical solution, the mesh data and the problem parameters are taken to be $\delta x = 0.02$, $\delta t = 0.02$ and 0.01 and for various v values (i.e., Reynolds numbers).

For purpose of verification, the first step solution $U_1(x,t)$ is illustrated in Fig. 1 and 2 for $v = 1.0$ and 0.5 respectively. This simplification means that the solution can be calculated independently by hand. For $v = 1.0$, the solution for all t values is clearly symmetrical about $x = 0.5$ and the height of the peak in the centre and gradient at the centre point (the peak) is always zero, and the gradients at $x = 0$ and $x = 1$ takes symmetrical values for each time step. The program was then modified by setting $v = 0.5$ when the program was rerun the peak clearly moved from $x = 0.5$ towards $x = 1$ and as expected the gradient in the first semi-interval (0, 0.5) decreased quicker than that in the semi-interval (0.5, 1) due to the use of first step solution which is not a good approximation to the solution of current problem. Fig. 3 shows complete suffering for $v = 0.4$ for the first step solution which is also expected from the qualitative findings of the method. But the solution can easily be improved by using second or higher steps of the solution, when one requires a means of selecting number of the terms to be used in the solution series to obtain a "good approximation" for the given constant v value (i.e., Reynolds number). To do this, we modify the program in such a way that the difference in nt step solution and $(n-1)ht$ step should be less than the required tolerance value, i.e., $|U_n(x,t) - U_{n-1}(x,t)| < \varepsilon$ then we stop calculating higher steps of the solution in series and last step is taken as "good approximation" solution to the problem for given constant v value.

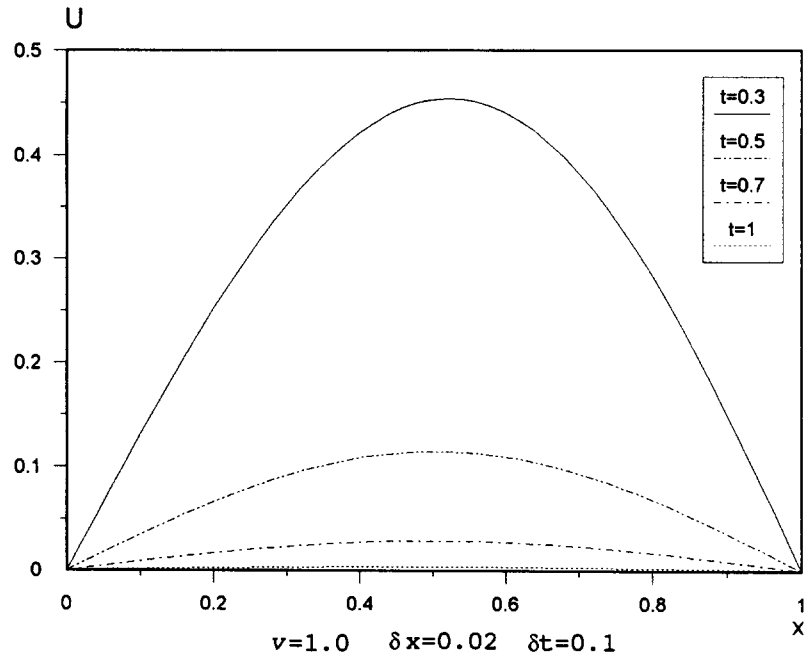


Fig. 1. Solution at different times for $v = 1.0$ for first step solution.

To see the efficiency and the correctness of series solution, we made the comparison with (exact) Fourier solution of [5] given by

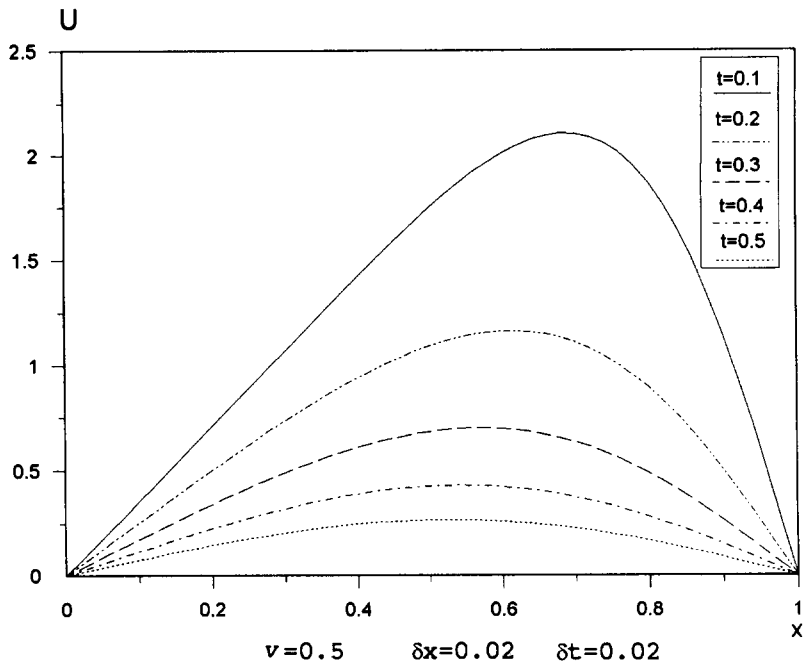
$$u(x,t) = 2\pi v \frac{\sum_{n=1}^{\infty} \exp\{-n^2\pi^2 vt\} A_n n \sin(n\pi x)}{\sum_{n=0}^{\infty} \exp\{-n^2\pi^2 vt\} A_n \cos(n\pi x)} \tag{28}$$

where

$$A_0 = \int_0^1 \exp\{-(2\pi v)^{-1}[1-\cos(\pi x)]\}\,dx, \tag{28a}$$

$$A_n = 2\int_0^1 \exp\{-(2\pi v)^{-1}[1-\cos(\pi x)]\}\cos(n\pi x)\,dx \tag{28b}$$

are Fourier coefficients (also see [8]). The results are shown in Table 1. The numerical results indicate that the present method compare very favourably with the results obtained from (exact) Fourier solution of [5]. In calculation we have used only six steps of our solution to get "good approximation" within the tolerance of 10^{-4} for $v = 0.01$. But the (exact) Fourier solution is obtained by setting the value $n = 28$ in (28) which is the value used by Miller in his calculations to get "good approximation" for $v = 0.01$. Even this coarse comparison shows that our solution is much more economical than (exact) Fourier series solution of [5]. Moreover, as Miller mentioned, calculation of Fourier coefficients is not possible analytically, therefore the choice of numerical method is very important. On the other hand, Miller notes, for smaller v almost all the area under the exponential term of (28a) and (28b) occurs near the abscissa $x = 0$, so the oscillatory effects of



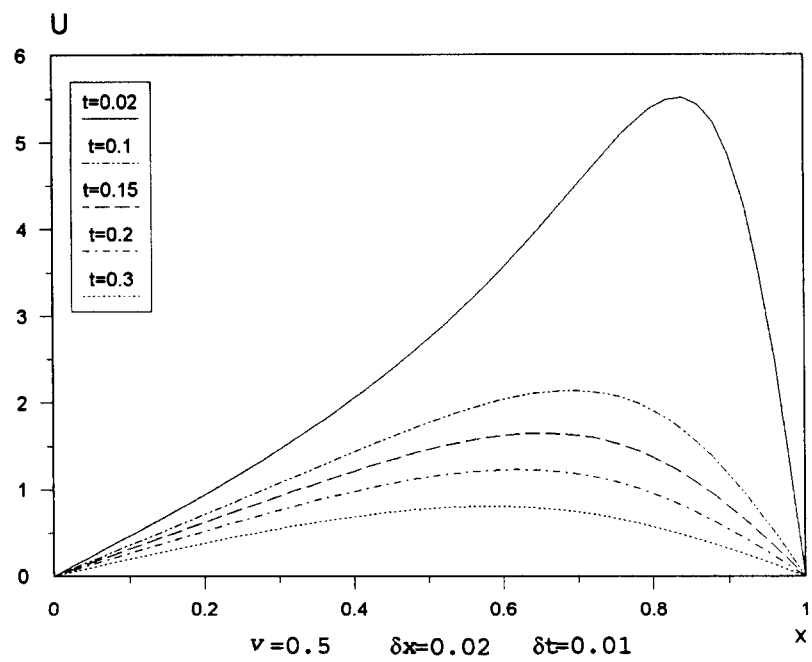
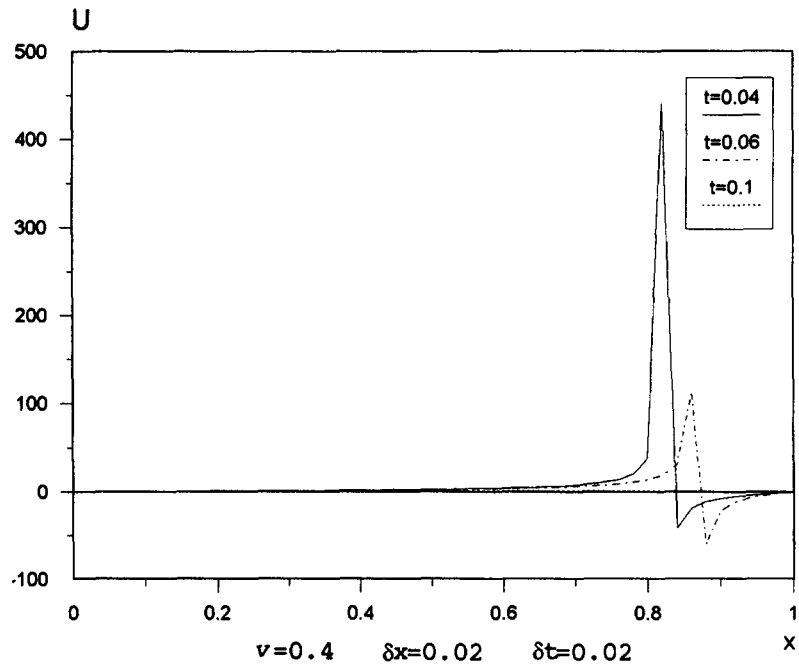


Fig. 2. Solution at different times for $\nu = 0.5$ for first step solution.



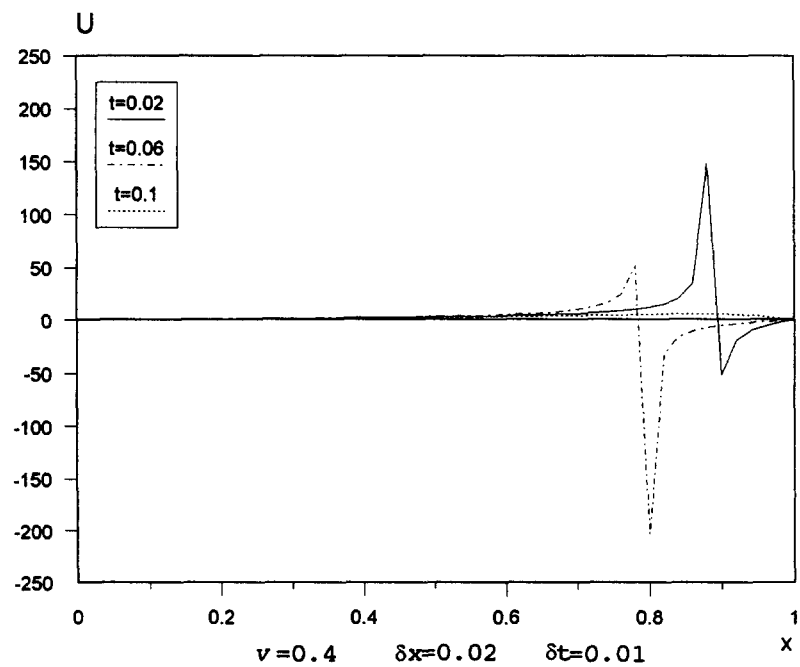


Fig. 3. Complete suffering for $\nu = 0.4$ for first step solution.

Table 1
Comparison of result at different times for $\nu = 0.01$

x	Cole (1951)	Present	Error
$t = 0.3$			
0.1	0.07593	0.07587	0.00006
0.2	0.15393	0.15390	0.00003
0.3	0.23607	0.23605	0.00002
0.4	0.32420	0.32421	−0.00001
0.5	0.41901	0.41900	0.00001
0.6	0.51704	0.51700	0.00004
0.7	0.60120	0.60117	0.00003
0.8	0.61652	0.61645	0.00007
0.9	0.44147	0.44125	0.00012
$t = 0.5$			
0.1	0.06817	0.06810	0.00007
0.2	0.13727	0.13722	0.00005
0.3	0.20795	0.20793	0.00002
0.4	0.27998	0.27999	−0.00001
0.5	0.35091	0.35091	0.0
0.6	0.41306	0.41303	0.00003
0.7	0.44734	0.44729	0.00005
0.8	0.41517	0.41510	0.00007
0.9	0.26610	0.26600	0.00010
$t = 0.8$			
0.1	0.05719	0.05715	0.00004
0.2	0.11411	0.11409	0.00002
0.3	0.16998	0.16997	0.00001
0.4	0.22295	0.22295	0.0
0.5	0.26892	0.26892	0.0
0.6	0.29992	0.29990	0.00002
0.7	0.30226	0.30221	0.00005
0.8	0.25746	0.25741	0.00005
0.9	0.15251	0.15245	0.00006
$t = 1.0$			
0.1	0.05045	0.05042	0.00003
0.2	0.10007	0.10004	0.00003
0.3	0.14759	0.14758	0.00001
0.4	0.19067	0.19067	0.0
0.5	0.22521	0.22521	0.0
0.6	0.24432	0.24431	0.00001
0.7	0.23810	0.23808	0.00002
0.8	0.19569	0.19566	0.00003
0.9	0.11257	0.11253	0.00004

$\cos(n\pi x)$ will not be felt until n is large. Thus, it would serve no purpose to include that portion of the interval $0 \leqslant x \leqslant 1$ for which the exponential factor is near zero. This handicap may be overcome by setting exponential factor to zero in that portion of the interval $0 \leqslant x \leqslant 1$ or using the asymptotic expansion of exponential factor such that this expansion may be integrated directly with $\cos(n\pi x)$. This also effects the accuracy and the efficiency of the (exact) Fourier method besides economy. Again, Miller notes, as ν become smaller, the initial values $u(x,0)$ were not calculated correctly, namely, the initial values were in error after $x = 0.5$ for $\nu = 0.01$ and negative values appear beyond $x = 0.6$. Further, as ν decreased to $\nu = 0.0001$ the initial values show errors at smaller x. This behaviour also resulted from the factor $\exp\{-n^2\pi^2\nu t\}$ in solution series (28) and Fourier coefficients (28a) and (28b). This behaviour may be overcome by implying more terms to be carried before truncation of the series or by taking more intervals for the numerical integration, which affects again the accuracy and the economy of the solution.

Finally, as a nature of Fourier solution, evaluation of the exact solution for $\nu < 0.01$ is not practical, because of the slow convergence of Fourier series (see again [8]).

Therefore, the values given in Table 1 are found to be adequate on the basis of the methods compared and no attempt is made to solve the problem for much lower values of ν. Also, we have not felt it worthwhile to make comparison with some other solutions, partly because most of them only gives graphical representations of the solutions without giving more information about it, partly make their comparisons with Cole's solution (or Miller's numerical solution) as we did.

On account of these, our simple approach, however, does not have this kind of inadequacy computationwise and the computational procedure has been found to be reasonably facile and this is verified by the speed of the computer calculations. The average computation time on the IBM compatible personal computers was in the vicinity of few seconds (unfortunately we cannot give the comparisons due to lack of the values of Miller's). All these show that our approach is much more economical as it stands. But, as a small drawback, for large times the accuracy and the stability of the numerical procedure must be questioned when ν is small. A comment concerning possible errors is in order. Thus the hypothesis of an error in computation must be rejected in our case.

6. Concluding remarks

An approximate method in limiting form for Burgers' equation with appropriate boundary conditions has been presented. The method can offer appreciable advantages for Burgers-like nonlinear problems involving different initial and boundary conditions. It is because the choice of the base functions are totally problem dependent. To use a boundary condition rather than the periodic one, it only makes the solution process more complicated compared to the present case. In special, the formulation of the problem in present form becomes computationally more economical partly because the incorporation of analytical calculation of the integrals makes it possible to achieve the high accuracy and numerical stability as demonstrated in the numerical example. Finally, since the method improves the solution iteratively requiring more steps and refined tolerance value for a given fixed ν values which makes it capable of solving Burgers' equation accurately for values of ν ranging from very small to large.

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