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Note

The Football Pool Problem for 7 and 8 Matches

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A method to bet on a football lottery with N games in such a way as to guarantee second prize, with the least number of forecasts (bets) is described. It is shown that for $N = 7$, second prize is guaranteed with 225 bets; for $N = 8$, second prize is achieved with 567 bets.

1. INTRODUCTION

We intend to forecast the outcome (win, lose, or draw) of N football matches in such a way that no matter what the outcome of the matches, one of the forecasts will have no less than $N - 1$ correct results. In various countries such a lottery exists, with $N = 12$, 13, or 14 (most commonly, $N = 13$).

The least number $A(N)$ of forecast to guarantee second prize is unkown, for the most values of N. The existing results are, $A(3) = 5$, $A(4) = 9$, $A(5) = 27$, $A(13) = 3^{10}$, as indicated in [1]. For $N = 9$, a "good" upper bound is known, $A(9) \leqslant 2 \times 3^6$, as shown in [1]. The proof that $A(5) = 27$ takes 10 pages, as presented in [2].

In this note, we shall construct upper bounds for $A(7)$ and $A(8)$. From what is known thus far, the trivial upper bounds are $A(7) \leqslant A(5) \times 3^2$ and $A(8) \leqslant A(5) \times 3^3$. The results to be presented in this paper have been announced by the authors in [3]. A sharper bound for $A(7)$ will be obtained, however.

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2. Notation and Definitions

Denote by R_3^N the set of vectors with N components chosen from the ring of integers mod 3. The Hamming distance $d(x, y)$ of two vectors of R_3^N is the number of places in which the components of **x** and **y** differ.

For any element $b \in R_3^N$, we define $N(b)$ as the set formed by all $x \in R_3^N$ such that $d(b, x) = 1$. The elements of $N(b)$ are the "neighbors" of b. We say that a set $Q \subset R_3^N$ is covered by a set $S \subset R_3^N$, if for any $z \in Q$, there is a (bet) $b \in S$ such that $d(b, z) \leqslant 1$. In this case, we also say that S is a covering of Q. The aim of this note is to find a covering set of R_3^N, for $N = 7$ and 8, with the least number of elements. We recall that any $b \in R_3^N$ has $2N$ neighbors and, therefore, $A(N) \geqslant 3^N/(2N+1)$. Rounding off, we have that $A(7) \geqslant 146$ and $A(8) \geqslant 386$. For $M = N - 4$, R_3^N can be partitioned into 3^M subsets $S_{k_1, k_2, \dots, k_M}$ with $k_i \in \{0, 1, 2\}$. The components of this partition are of the form, $S_{k_1, \dots, k_M} = x \in \{R_3^N : x = [k_1, k_2, \dots, k_M, x_1, x_2, x_3, x_4]\}$. Each set $S_{k_1, \dots, k_M}$ contains 81 elements of R_3^N. Thus, for example,

$$S_{101} = \{x \in R_3^7 : x = [1, 0, 1, x_1, x_2, x_3, x_4]\}$$

and

$$S_{1121} = \{x \in R_3^8 : x = [1, 1, 2, 1, x_1, x_2, x_3, x_4]\}.$$

The "neighbor sets" of $S_{k_1, \dots, k_M}$, denoted by $N(S_{k_1, \dots, k_M})$, are all sets $S_{p, \dots, p_M}$ such that the ordered M tuples $(k_1, \dots, k_M)$ and $(p_1, \dots, p_M)$ differ, precisely, in one position. For example, $N(S_{111}) = \{S_{101}, S_{110}, S_{011}, S_{112}, S_{121}, S_{211}\}$. Analogously, we define $N(N(S_{k_1, \dots, k_M}))$ as the set formed by all neighbor sets of the sets belonging to $N(S_{k_1, \dots, k_M})$.

TABLE I

	00	01	02	10	11	12	20	21	22
00	●								
01					●				
02									●
10						●			
11							●		
12		●							
20								●	
21			●						
22				●					

3. The Perfect Group R_3^4

Here R_3^4 is a group of order 81 and, by Lagrange's theorem, has a subgroup of order 9. For example, the nine elements of the set B^1,

$$B^1 = \{0000, 0111, 0222, 1012, 1120, 1201, 2021, 2102, 2210\}$$

displayed in Table I, form such a subgroup.

In this note, we shall use a partition of R_3^4, consisting of the following 9 disjoint subsets:

$$\begin{aligned}
B^1 &= \{0000, 0111, 0222, 1012, 1120, 1201, 2021, 2102, 2210\}\\
B^2 &= \{0001, 0112, 0220, 1010, 1121, 1202, 2022, 2100, 2211\}\\
B^3 &= \{0002, 0110, 0221, 1011, 1122, 1200, 2020, 2101, 2212\}\\
B^4 &= \{0010, 0121, 0202, 1022, 1100, 1211, 2001, 2112, 2220\}\\
B^5 &= \{0011, 0122, 0200, 1020, 1101, 1212, 2002, 2110, 2221\}\\
B^6 &= \{0012, 0120, 0201, 1021, 1102, 1210, 2000, 2111, 2222\}\\
B^7 &= \{0020, 0101, 0212, 1002, 1110, 1221, 2011, 2122, 2200\}\\
B^8 &= \{0021, 0102, 0210, 1000, 1111, 1222, 2012, 2120, 2201\}\\
B^9 &= \{0022, 0100, 0211, 1001, 1112, 1220, 2010, 2121, 2202\}.
\end{aligned}$$

It is easily verified that, for $N=4$ football matches, second prize is guaranteed if we choose any B^i as in our forecasts. In other words, any set B^i is a covering for R_3^4. Furthermore, this is an optimal solution since $A(4) \geqslant 9$.

4. Choice of Bets

We shall place our bets in some (but not all) of the sets $S_{k_1,\ldots,k_M}$ of the partition of R_3^N.

All bets will be of the form, $(k_1,\ldots,k_M,\ x_1,x_2,x_3,x_4)$, where (x_1,x_2,x_3,x_4) runs through a set B^j, for some j.

For example, in R_3^7, the insertion of B^1 in S_{101} yields the following nine bets:

101 0000	101 1012	101 2021
101 0111	101 1120	101 2102
101 0222	101 1201	101 2210.

We notice that, because of this choice, the total number of bets will be a multiple of nine.

4.1. *The Strategy for Choosing the Bets*

The strategy adopted is to maximize the number of partition components of R_3^N in which no bets are placed and to obtain their "coverage" by the least number of bets placed in neighbor sets.

For $N=7$, it is possible to choose 6 partition components of R_3^7 in which no bets are placed; 20 partition components will then have 9 bets each. Coverage (solution) is obtained by placing $5 \times 9 = 45$ bets in the remaining partition component.

For $N=8$, we shall be able to choose 24 partition components of R_3^8 in which no bets are placed; 54 partition components will then have 9 bets each. Coverage is obtained by placing 27 bets in each of the 3 remaining partition components.

Two sufficient conditions for coverage are given by Lemmas 1 and 2.

LEMMA 1. *For any j, the nine bets* $(k_1,..., k_M, x_1, x_2, x_3, x_4)$, $(x_1, x_2, x_3, x_4) \in B^j$, *yield a covering of* $S_{k_1,...,k_M}$.

Proof. This is so because B^j covers R_3^4.

LEMMA 2. *Here* $S_{k_1,...,k_M}$ *is covered if, for each* $i = 1,..., 9$ *there is a neighbor set in which* B^i *is inserted.*

Proof. Let $(k_1,..., k_M, x_1, x_2, x_3, x_4) \in S_{k_1,...,k_M}$. By construction $B^1,..., B^9$ form a partition of R_3^4; hence for some i, $(x_1, x_2, x_3, x_4) \in B^i$, and by assumption B^i is inserted in some neighbor set $S_{p_1,...,p_M}$ of $S_{k_1,...,k_M}$. Therefore, there is a bet $(p_1,..., p_M, x_1, x_2, x_3, x_4 \in S_{p_1,...,p_M}$ such that

$$d[(k_1,..., k_M, x_1, x_2, x_3, x_4), (p_1,..., p_M, x_1, x_2, x_3, x_4)] = 1.$$

5. THE FOOTBALL POOL WITH 7 MATCHES

In this section, we construct a covering of R_3^7, as follows:

(1) Insert B^5, B^6, B^7, B^8, and B^9 in S_{111}.

(2) The sets belonging to $N(S_{111})$ remain without bets.

(3) Insert B^1 in S_{001}, S_{122}, and S_{210}.

Insert B^2 in S_{010}, S_{221}, and S_{102} (indices as given but shifted cyclicly).

(4) Insert B^3 in S_{100}, S_{212}, and S_{021} (another cyclic shift).

Insert B^4 in S_{012}, S_{120}, and S_{201} (indices form a cyclic set).

(5) Insert an arbitrary B^k in each of the sets S_{xyz}, where all indices are 0 or 2.

Now we shall prove that the bets constructed form a covering of R_3^7.

THEOREM 1. *In a football pool with 7 games it is possible to guarantee second prize with 225 bets.*

Proof. By Lemma 1, all partition sets with one or more insertions B^i are covered.

Since our insertions were made using cyclic shifts it is sufficient to show that S_{011} and S_{211} are covered. Then

S_{001} is a neighbor set of S_{011} and contains B^1,
S_{010} is a neighbor set of S_{011} and contains B^2,
S_{021} is a neighbor set of S_{011} and contains B^3,
S_{012} is a neighbor set of S_{011} and contains B^4,
S_{111} is a neighbor set of S_{011} and contains B^5, B^6, B^7, B^8 and B^9.

Thus, S_{011} is covered. The proof for S_{211} is similar.

We conclude that al 27 partition sets of R_3^7 are covered. There are 20 partition sets with one B^i and S_{111} has five insertions of type B^i. Thus, the total number of bets is $9 \times (20+5) = 225$.

6. THE FOOTBALL POOL WITH 8 MATCHES

We construct a covering of R_3^8, as follows:

(a) The sets belonging to $N(S_{xxxx})$, where $x = 0, 1,$ or 2 remain without bets.

(b) The sets $S_{xxxx} (x = 0, 1, 2)$ have B^7, B^8 and B^9 inserted.

(c) Let x, y, z denote different elements from $\{0, 1, 2\}$:

- (i) Insert B^1 in S_{xyxy} (6 choices).
- (ii) Insert B^2 in S_{xyxz} and S_{zxxx} (12 choices).
- (iii) Insert B^3 in S_{xyyx} (6 choices).
- (iv) Insert B^4 in S_{yxxz} and S_{xyzx} (12 choices).
- (v) Insert B^5 in S_{xxyy} (6 choices).
- (vi) Insert B^6 in S_{xxyz} and S_{yzxx} (12 choices).

THEOREM 2. *In a football pool with 8 matches it is possible to guarantee second prize with 567 bets.*

Proof. Because of the symmetry of the construction it is sufficient to show that S_{0001} is covered. For $N(S_{0001})$ we have

$$S_{0101} \text{ has } B^1,\ S_{0201} \text{ has } B^2,\ S_{1001} \text{ has } B^3,\ S_{2001} \text{ has } B^4,$$
$$S_{0011} \text{ has } B^5,\ S_{0021} \text{ has } B^6,\ S_{0000} \text{ has } B^7, B^8, B^9.$$

Thus, the sets (bets) inserted above form a covering of R_3^8 with $9 \times (3 \times 3 + 54) = 567$ bets.

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