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Vanishing of beta function of non-commutative Φ_4^4 theory to all orders ☆

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Abstract

The simplest non-commutative renormalizable field theory, the ϕ^4 model on four-dimensional Moyal space with harmonic potential is asymptotically safe up to three loops, as shown by H. Grosse and R. Wulkenhaar, M. Disertori and V. Rivasseau. We extend this result to all orders.

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1. Introduction

Non-commutative (NC) quantum field theory (QFT) may be important for physics beyond the standard model and for understanding the quantum Hall effect [1]. It also occurs naturally as an effective regime of string theory [2,3].

The simplest NC field theory is the ϕ_4^4 model on the Moyal space. Its perturbative renormalizability at all orders has been proved by Grosse, Wulkenhaar and followers [4–7]. Grosse and Wulkenhaar solved the difficult problem of ultraviolet/infrared mixing by introducing a new harmonic potential term inspired by the Langmann–Szabo (LS) duality [8] between positions and momenta.

Other renormalizable models of the same kind, including the orientable fermionic Gross–Neveu model [9], have been recently also shown renormalizable at all orders and techniques such as the parametric representation have been extended to NCQFT [10]. It is now tempting to conjecture that commutative renormalizable theories in general have NC renormalizable extensions to Moyal spaces which imply new parameters. However the most interesting case, namely the one of gauge theories, still remains elusive.

Once perturbative renormalization is understood, the next problem is to compute the renormalization group (RG) flow. It is well known that the ordinary commutative ϕ_4^4 model is not asymptotically free in the ultraviolet regime. This problem, called the Landau ghost or triviality problem affects also quantum electrodynamics. It almost killed quantum field theory, which was resurrected by the discovery of ultraviolet asymptotic freedom in non-Abelian gauge theory [11].

An amazing discovery was made in [12]: the non-commutative ϕ_4^4 model does not exhibit any Landau ghost at one loop. It is not asymptotically free either. For any renormalized Grosse–Wulkenhaar harmonic potential parameter $\Omega_{\text{ren}} > 0$, the running Ω tends to the special LS dual point $\Omega_{\text{bare}} = 1$ in the ultraviolet. As a result the RG flow of the coupling constant is simply bounded.[1] This result was extended up to three loops in [13].

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[1] The Landau ghost can be recovered in the limit $\Omega_{\text{ren}} \to 0$.

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In this Letter we compute the flow at the special LS dual point $\Omega = 1$, and check that the beta function vanishes at all orders using a kind of Ward identity inspired by those of the Thirring or Luttinger models [14–16]. Note however that in contrast with these models, the model we treat has quadratic (mass) divergences.

The non-perturbative construction of the model should combine this result and a non-perturbative multiscale analysis [17,18]. Also we think the Ward identities discovered here might be important for the future study of more singular models such as Chern–Simons or Yang–Mills theories, and in particular for those which have been advocated in connection with the Quantum Hall effect [19–22].

In this Letter we give the complete argument of the vanishing of the beta function at all orders in the renormalized coupling, but we assume knowledge of renormalization and effective expansions as described e.g. in [18], and of the basic papers for renormalization of NC ϕ^4_4 in the matrix base [4–6].

2. Notations and main result

We adopt simpler notations than those of [12,13], and normalize so that $\theta = 1$, hence have no factor of π or θ.

The bare propagator in the matrix base at $\Omega = 1$ is

$$C_{mn;kl} = C_{mn}\delta_{ml}\delta_{nk}; \quad C_{mn} = \frac{1}{A+m+n}, \tag{2.1}$$

where $A = 2 + \mu^2/4$, $m, n \in \mathbb{N}^2$ (μ being the mass) and we used the notations

$$\delta_{ml} = \delta_{m_1 l_1}\delta_{m_2 l_2}, \quad m+n = m_1+m_2+n_1+n_2. \tag{2.2}$$

There are two version of this theory, the real and complex one. We focus on the complex case, the result for the real case follows easily [13].

The generating functional is:

$$\begin{aligned} Z(\eta,\bar\eta) &= \int d\phi\, d\bar\phi\, e^{-S(\bar\phi,\phi)+F(\bar\eta,\eta;\bar\phi,\phi)}, \\ F(\bar\eta,\eta;\bar\phi,\phi) &= \bar\phi\eta + \bar\eta\phi, \\ S(\bar\phi,\phi) &= \bar\phi X\phi + \phi X\bar\phi + A\bar\phi\phi + \frac{\lambda}{2}\phi\bar\phi\phi\bar\phi, \end{aligned} \tag{2.3}$$

where traces are implicit and the matrix X_{mn} stands for $m\delta_{mn}$. S is the action and F the external sources.

We denote $\Gamma^4(a,b,c,d)$ the amputated one particle irreducible four point function with external indices set to a,b,c,d. Furthermore we denote $\Sigma(a,b)$ the amputated one particle irreducible two point function with external indices set to a,b (also called the self-energy). The wave function renormalization is $1 - \partial\Sigma(0,0)$ where $\partial\Sigma(0,0) \equiv \partial_L\Sigma = \partial_R\Sigma = \Sigma(1,0) - \Sigma(0,0)$ is the derivative of the self-energy with respect to one of the two indices a or b [13]. Our main result is:

Theorem. *The equation*

$$\Gamma^4(0,0,0,0) = \lambda\big(1 - \partial\Sigma(0,0)\big)^2 \tag{2.4}$$

holds up to irrelevant terms[2] *to* all *orders of perturbation, either as a bare equation with fixed ultraviolet cutoff, or as an equation for the renormalized theory. In the latter case λ should still be understood as the bare constant, but reexpressed as a series in powers of* λ_{ren}.

3. Ward identities

We orient the propagators from a $\bar\phi$ to a ϕ. For a field $\bar\phi_{ab}$ we call the index a a *left index* and the index, b a *right index*. The first (second) index of a $\bar\phi$ *always* contracts with the second (first) index of a ϕ. Consequently for ϕ_{cd}, c is a *right index* and d is a *left index*.

Let $U = e^{\imath B}$ with B a small Hermitian matrix. We consider the "left" (as it acts only on the left indices) change of variables[3]:

$$\phi^U = \phi U; \quad \bar\phi^U = U^\dagger\bar\phi. \tag{3.1}$$

The variation of the action is, at first order:

$$\delta S = \phi U X U^\dagger\bar\phi - \phi X\bar\phi \approx \imath(\phi B X\bar\phi - \phi X B\bar\phi) = \imath B(X\bar\phi\phi - \bar\phi\phi X) \tag{3.2}$$

[2] Irrelevant terms include in particular all non-planar or planar with more than one broken face contributions.

[3] There is a similar "right" change of variables, acting only on the right indices.

and the variation of the external sources is:

$$\delta F = U^\dagger \bar\phi\eta - \bar\phi\eta + \bar\eta\phi U - \bar\eta\phi \approx -\imath B\bar\phi\eta + \imath\bar\eta\phi B = \imath B(-\bar\phi\eta + \bar\eta\phi). \tag{3.3}$$

We obviously have:

$$\frac{\delta \ln Z}{\delta B_{ba}} = 0 = \frac{1}{Z(\bar\eta,\eta)}\int d\bar\phi\, d\phi \left(-\frac{\delta S}{\delta B_{ba}} + \frac{\delta F}{\delta B_{ba}}\right)e^{-S+F}$$
$$= \frac{1}{Z(\bar\eta,\eta)}\int d\bar\phi\, d\phi\, e^{-S+F}\big(-[X\bar\phi\phi - \bar\phi\phi X]_{ab} + [-\bar\phi\eta + \bar\eta\phi]_{ab}\big). \tag{3.4}$$

We now take $\partial_\eta\partial_{\bar\eta}|_{\eta=\bar\eta=0}$ on the above expression. As we have at most two insertions we get only the connected components of the correlation functions.

$$0 = \big\langle \partial_\eta\partial_{\bar\eta}\big(-[X\bar\phi\phi - \bar\phi\phi X]_{ab} + [-\bar\phi\eta + \bar\eta\phi]_{ab}\big)e^{F(\bar\eta,\eta)}\big|_0\big\rangle_c, \tag{3.5}$$

which gives:

$$\left\langle \frac{\partial(\bar\eta\phi)_{ab}}{\partial\bar\eta}\frac{\partial(\bar\phi\eta)}{\partial\eta} - \frac{\partial(\bar\phi\eta)_{ab}}{\partial\eta}\frac{\partial(\bar\eta\phi)}{\partial\bar\eta} - [X\bar\phi\phi - \bar\phi\phi X]_{ab}\frac{\partial(\bar\eta\phi)}{\partial\bar\eta}\frac{\partial(\bar\phi\eta)}{\partial\eta}\right\rangle_c = 0. \tag{3.6}$$

Using the explicit form of X we get:

$$(a-b)\left\langle [\bar\phi\phi]_{ab}\frac{\partial(\bar\eta\phi)}{\partial\bar\eta}\frac{\partial(\bar\phi\eta)}{\partial\eta}\right\rangle_c = \left\langle \frac{\partial(\bar\eta\phi)_{ab}}{\partial\bar\eta}\frac{\partial(\bar\phi\eta)}{\partial\eta}\right\rangle_c - \left\langle \frac{\partial(\bar\phi\eta)_{ab}}{\partial\eta}\frac{\partial(\bar\eta\phi)}{\partial\bar\eta}\right\rangle_c,$$

and for $\bar\eta_{\beta\alpha}\eta_{\nu\mu}$ we get:

$$(a-b)\big\langle [\bar\phi\phi]_{ab}\phi_{\alpha\beta}\bar\phi_{\mu\nu}\big\rangle_c = \langle \delta_{a\beta}\phi_{\alpha b}\bar\phi_{\mu\nu}\rangle_c - \langle \delta_{b\mu}\bar\phi_{a\nu}\phi_{\alpha\beta}\rangle_c. \tag{3.7}$$

We now restrict to terms in the above expressions which are planar with a single external face, as all others are irrelevant. Such terms have $\alpha=\nu$, $a=\beta$ and $b=\mu$. The Ward identity for 2 point function reads:

$$(a-b)\big\langle [\bar\phi\phi]_{ab}\phi_{\nu a}\bar\phi_{b\nu}\big\rangle_c = \langle \phi_{\nu b}\bar\phi_{b\nu}\rangle_c - \langle \bar\phi_{a\nu}\phi_{\nu a}\rangle_c \tag{3.8}$$

(repeated indices are not summed).

Derivating further we get:

$$(a-b)\big\langle [\bar\phi\phi]_{ab}\partial_{\bar\eta_1}(\bar\eta\phi)\partial_{\eta_1}(\bar\phi\eta)\partial_{\bar\eta_2}(\bar\eta\phi)\partial_{\eta_2}(\bar\phi\eta)\big\rangle_c$$
$$= \big\langle \partial_{\bar\eta_1}(\bar\eta\phi)\partial_{\eta_1}(\bar\phi\eta)\big[\partial_{\bar\eta_2}(\bar\eta\phi)_{ab}\partial_{\eta_2}(\bar\phi\eta) - \partial_{\eta_2}(\bar\phi\eta)_{ab}\partial_{\bar\eta_2}(\bar\eta\phi)\big]\big\rangle_c + 1 \leftrightarrow 2. \tag{3.9}$$

Take $\bar\eta_{1\beta\alpha}$, $\eta_{1\nu\mu}$, $\bar\eta_{2\delta\gamma}$ and $\eta_{2\sigma\rho}$. We get:

$$(a-b)\big\langle [\bar\phi\phi]_{ab}\phi_{\alpha\beta}\bar\phi_{\mu\nu}\phi_{\gamma\delta}\bar\phi_{\rho\sigma}\big\rangle_c = \langle \phi_{\alpha\beta}\bar\phi_{\mu\nu}\delta_{a\delta}\phi_{\gamma b}\bar\phi_{\rho\sigma}\rangle_c - \langle \phi_{\alpha\beta}\bar\phi_{\mu\nu}\phi_{\gamma\delta}\bar\phi_{a\sigma}\delta_{b\rho}\rangle_c$$
$$+ \langle \phi_{\gamma\delta}\bar\phi_{\rho\sigma}\delta_{a\beta}\phi_{\alpha b}\bar\phi_{\mu\nu}\rangle_c - \langle \phi_{\gamma\delta}\bar\phi_{\rho\sigma}\phi_{\alpha\beta}\bar\phi_{a\nu}\delta_{b\mu}\rangle_c. \tag{3.10}$$

Again neglecting all terms which are not planar with a single external face leads to

$$(a-b)\big\langle \phi_{\alpha a}[\bar\phi\phi]_{ab}\bar\phi_{b\nu}\phi_{\nu\delta}\bar\phi_{\delta\alpha}\big\rangle_c = \langle \phi_{\alpha b}\bar\phi_{b\nu}\phi_{\nu\delta}\bar\phi_{\delta\alpha}\rangle_c - \langle \phi_{\alpha a}\bar\phi_{a\nu}\phi_{\nu\delta}\bar\phi_{\delta\alpha}\rangle_c.$$

Clearly there are similar identities for $2p$ point functions for any p.

The indices a and b are left indices, so that we have the Ward identity with an insertion on a left face[4] as represented in Fig. 1.

We conclude this section by several remarks on the real theory. If $\phi(x)$ is a real function then ϕ_{ab} is a Hermitian matrix. The action and the sources are:

$$S = \phi X\phi + \frac{\lambda}{4}\phi^4, \qquad F = \phi\eta. \tag{3.11}$$

We perform the change of variables (preserving the Hermitian character of ϕ):

$$\phi^U = U\phi U^\dagger \tag{3.12}$$

with constant U a unitary matrix. A straightforward computation shows that the Jacobian of this change of variables is 1 and the reader can check that the method above gives Ward identities identical with those of the complex model.

[4] There is a similar Ward identity obtained with the "right" transformation, consequently with the insertion on a right face.

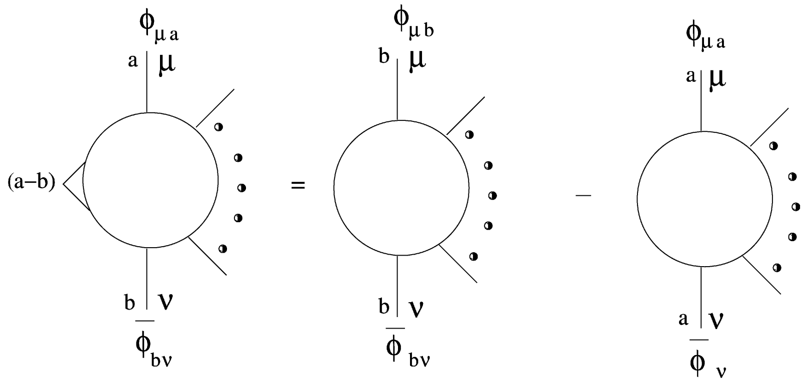

Fig. 1. The Ward identity for a $2p$ point function with insertion on the left face.

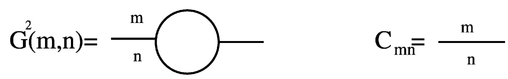

Fig. 2. The *dressed* and the bare propagators.

4. Proof of the theorem

We start this section by some definitions: we will denote $G^4(m,n,k,l)$ the connected four point function restricted to the planar one broken face case, where m, n, k, l are the indices of the external face in the correct cyclic order. The first index m always represents a left index.

Similarly, $G^2(m,n)$ is the connected planar one broken face two point function with m, n the indices on the external face (also called the *dressed* propagator, see Fig. 2). $G^2(m,n)$ and $\Sigma(m,n)$ are related by:

$$G^2(m,n) = \frac{C_{mn}}{1 - C_{mn}\Sigma(m,n)} = \frac{1}{C_{mn}^{-1} - \Sigma(m,n)}. \tag{4.1}$$

$G_{\text{ins}}(a,b;\dots)$ will denote the planar one broken face connected function with one insertion on the left border where the matrix index jumps from a to b. With this notations the Ward identity (3.8) writes:

$$(a-b)G^2_{\text{ins}}(a,b;\nu) = G^2(b,\nu) - G^2(a,\nu). \tag{4.2}$$

All the identities we use, either Ward identities or the Dyson equation of motion can be written either for the bare theory or for the theory with complete mass renormalization, which is the one considered in [13]. In the first case the parameter A in (2.1) is the bare one, A_{bare} and there is no mass subtraction. In the second case the parameter A in (2.1) is $A_{\text{ren}} = A_{\text{bare}} - \Sigma(0,0)$, and every two point 1PI subgraph is subtracted at 0 external indices.[5] Throughout this Letter ∂_L will denote the derivative with respect to a left index and ∂_R the one with respect to a right index. When the two derivatives are equal we will employ the generic notation ∂.

Let us prove first the theorem in the mass-renormalized case, then in the next subsection in the bare case. Indeed the mass renormalized theory used is free from any quadratic divergences, and remaining logarithmic subdivergences in the ultra violet cutoff can be removed easily by going, for instance, to the "useful" renormalized effective series, as explained in [13].

We analyze a four point connected function $G^4(0,m,0,m)$ with index $m \neq 0$ on the right borders. This explicit break of left–right symmetry is adapted to our problem.

Consider a $\bar{\phi}$ external line and the first vertex hooked to it. Turning right on the m border at this vertex we meet a new line (the slashed line in Fig. 3). The slashed line either separates the graph into two disconnected components ($G^4_{(1)}$ and $G^4_{(2)}$ in Fig. 3) or not ($G^4_{(3)}$ in Fig. 3). Furthermore, if the slashed line separates the graph into two disconnected components the first vertex may either belong to a four point component ($G^4_{(1)}$ in Fig. 3) or to a two point component ($G^4_{(2)}$ in Fig. 3).

We stress that this is a *classification* of graphs: the different components depicted in Fig. 3 take into account all the combinatoric factors. Furthermore, the setting of the external indices to 0 on the left borders and m on the right borders distinguishes the $G^4_{(1)}$ and $G^4_{(2)}$ from their counterparts "pointing upwards": indeed, the latter are classified in $G^4_{(3)}$!

We have thus the Dyson equation:

$$G^4(0,m,0,m) = G^4_{(1)}(0,m,0,m) + G^4_{(2)}(0,m,0,m) + G^4_{(3)}(0,m,0,m). \tag{4.3}$$

[5] These mass subtractions need not be rearranged into forests since 1PI 2 point subgraphs never overlap non-trivially.

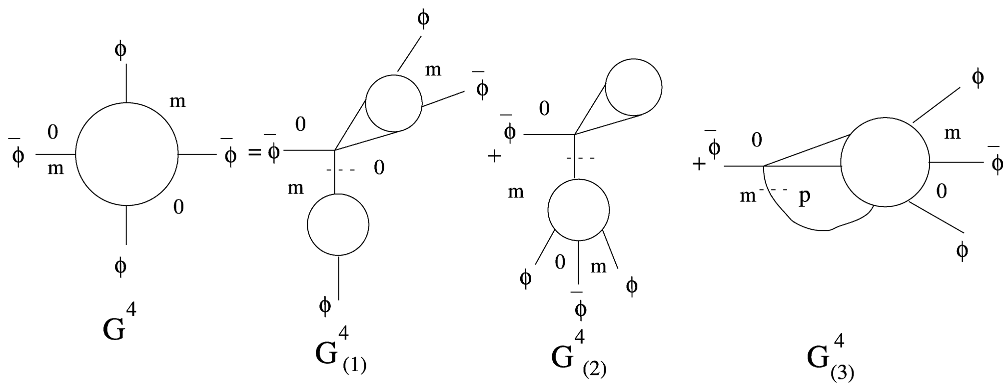

Fig. 3. The Dyson equation.

The second term, $G^4_{(2)}$, is zero. Indeed the mass renormalized two point insertion is zero, as it has the external left index set to zero. Note that this is an insertion exclusively on the left border. The simplest case of such an insertion is a (left) tadpole. We will (naturally) call a general insertion touching only the left border a "generalized left tadpole" and denote it by T^L.

We will prove that $G^4_{(1)} + G^4_{(3)}$ yields $\Gamma^4 = \lambda(1 - \partial\Sigma)^2$ after amputation of the four external propagators.

We start with $G^4_{(1)}$. It is of the form:

$$G^4_{(1)}(0,m,0,m) = \lambda C_{0m} G^2(0,m) G^2_{\text{ins}}(0,0;m). \tag{4.4}$$

By the Ward identity we have:

$$G^2_{\text{ins}}(0,0;m) = \lim_{\zeta\to 0} G^2_{\text{ins}}(\zeta,0;m) = \lim_{\zeta\to 0} \frac{G^2(0,m) - G^2(\zeta,m)}{\zeta} = -\partial_L G^2(0,m). \tag{4.5}$$

Using the explicit form of the bare propagator we have $\partial_L C^{-1}_{ab} = \partial_R C^{-1}_{ab} = \partial C^{-1}_{ab} = 1$. Reexpressing $G^2(0,m)$ by Eq. (4.1) we conclude that:

$$\begin{aligned} G^4_{(1)}(0,m,0,m) &= \lambda C_{0m} \frac{C_{0m} C^2_{0m}[1 - \partial_L \Sigma(0,m)]}{[1 - C_{0m}\Sigma(0,m)](1 - C_{0m}\Sigma(0,m))^2} \\ &= \lambda\big[G^2(0,m)\big]^4 \frac{C_{0m}}{G^2(0,m)}\big[1 - \partial_L\Sigma(0,m)\big]. \end{aligned} \tag{4.6}$$

The self energy is (again up to irrelevant terms [5]):

$$\Sigma(m,n) = \Sigma(0,0) + (m+n)\partial\Sigma(0,0). \tag{4.7}$$

Therefore up to irrelevant terms ($C^{-1}_{0m} = m + A_{\text{ren}}$) we have:

$$G^2(0,m) = \frac{1}{m + A_{\text{bare}} - \Sigma(0,m)} = \frac{1}{m[1 - \partial\Sigma(0,0)] + A_{\text{ren}}}, \tag{4.8}$$

and

$$\frac{C_{0m}}{G^2(0,m)} = 1 - \partial\Sigma(0,0) + \frac{A_{\text{ren}}}{m + A_{\text{ren}}}\partial\Sigma(0,0). \tag{4.9}$$

Inserting Eq. (4.9) into Eq. (4.6) holds:

$$G^4_{(1)}(0,m,0,m) = \lambda\big[G^2(0,m)\big]^4 \left(1 - \partial\Sigma(0,0) + \frac{A_{\text{ren}}}{m + A_{\text{ren}}}\partial\Sigma(0,0)\right)\big[1 - \partial_L\Sigma(0,m)\big]. \tag{4.10}$$

For the $G^4_{(3)}(0,m,0,m)$ one starts by "opening" the face which is "first on the right". The summed index of this face is called p (see Fig. 3). For bare Green functions this reads:

$$G^{4,\text{bare}}_{(3)}(0,m,0,m) = C_{0m} \sum_p G^{4,\text{bare}}_{\text{ins}}(p,0;m,0,m). \tag{4.11}$$

When passing to mass renormalized Green functions one must be cautious. It is possible that the face p belonged to a 1PI two point insertion in $G^4_{(3)}$ (see the left-hand side in Fig. 4). Upon opening the face p this 2 point insertion disappears (see right-hand side of Fig. 4)! When renormalizing, the counterterm corresponding to this kind of two point insertion will be substracted on the

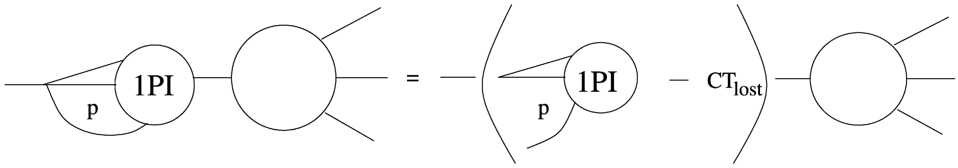

Fig. 4. Two point insertion and opening of the loop with index p.

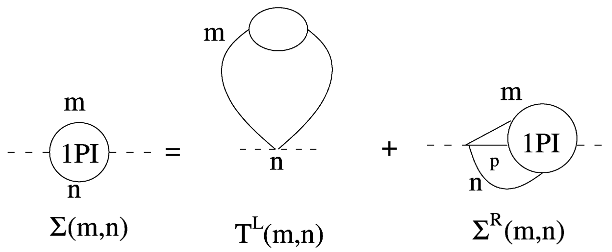

Fig. 5. The self energy.

left-hand side of Eq. (4.11), but not on the right-hand side. In the equation for $G^4_{(3)}(0,m,0,m)$ one must therefore *add its missing counterterm*, so that:

$$G^4_{(3)}(0,m,0,m) = C_{0m}\sum_p G^4_{\text{ins}}(0,p;m,0,m) - C_{0m}(CT_{\text{lost}})G^4(0,m,0,m). \tag{4.12}$$

It is clear that not all 1PI 2 point insertions on the left-hand side of Fig. 4 will be "lost" on the right-hand side. If the insertion is a "generalized left tadpole" it is not "lost" by opening the face p (imagine a tadpole pointing upwards in Fig. 4: clearly it will not be opened by opening the line). We will call the 2 point 1PI insertions "lost" on the right-hand side $\Sigma^R(m,n)$. Denoting the generalized left tadpole T^L we can write (see Fig. 5):

$$\Sigma(m,n) = T^L(m,n) + \Sigma^R(m,n). \tag{4.13}$$

Note that as $T^L(m,n)$ is an insertion exclusively on the left border, it does not depend upon the right index n. We therefore have $\partial\Sigma(m,n) = \partial_R\Sigma(m,n) = \partial_R\Sigma^R(m,n)$.

The missing mass counterterm writes:

$$CT_{\text{lost}} = \Sigma^R(0,0) = \Sigma(0,0) - T^L. \tag{4.14}$$

In order to evaluate $\Sigma^R(0,0)$ we proceed by opening its face p and using the Ward identity (3.8), to obtain:

$$\Sigma^R(0,0) = \frac{1}{G^2(0,0)}\sum_p G^2_{\text{ins}}(0,p;0) = \frac{1}{G^2(0,0)}\sum_p \frac{1}{p}\left[G^2(0,0) - G^2(p,0)\right] = \sum_p \frac{1}{p}\left(1 - \frac{G^2(p,0)}{G^2(0,0)}\right). \tag{4.15}$$

Using Eqs. (4.12) and (4.15) we have:

$$G^4_{(3)}(0,m,0,m) = C_{0m}\sum_p G^4_{\text{ins}}(0,p;m,0,m) - C_{0m}G^4(0,m,0,m)\sum_p \frac{1}{p}\left(1 - \frac{G^2(p,0)}{G^2(0,0)}\right). \tag{4.16}$$

But by the Ward identity (3.11):

$$C_{0m}\sum_p G^4_{\text{ins}}(0,p;m,0,m) = C_{0m}\sum_p \frac{1}{p}\left(G^4(0,m,0,m) - G^4(p,m,0,m)\right). \tag{4.17}$$

The second term in Eq. (4.17), having at least three denominators linear in p, is irrelevant.[6] Substituting Eq. (4.17) in Eq. (4.16) we have:

$$G^4_{(3)}(0,m,0,m) = C_{0m}\frac{G^4(0,m,0,m)}{G^2(0,0)}\sum_p \frac{G^2(p,0)}{p}. \tag{4.18}$$

[6] Any perturbation order of $G^4(p,m,0,m)$ is a polynomial in $\ln(p)$ divided by p^2. Therefore the sums over p above are always convergent.

To conclude we must evaluate the sum in Eq. (4.18). Using Eq. (4.8) we have:

$$\sum_p \frac{G^2(p,0)}{p} = \sum_p \frac{G^2(p,0)}{p}\left(\frac{1}{G^2(0,1)} - \frac{1}{G^2(0,0)}\right)\frac{1}{1-\partial\Sigma(0,0)}. \tag{4.19}$$

In order to interpret the two terms in the above equation we start by performing the same manipulations as in Eq. (4.15) for $\Sigma^R(0,1)$. We get:

$$\Sigma^R(0,1) = \sum_p \frac{1}{p}\left(1 - \frac{G^2(p,1)}{G^2(0,1)}\right) = \sum_p \frac{1}{p}\left(1 - \frac{G^2(p,0)}{G^2(0,1)}\right), \tag{4.20}$$

where in the second equality the we have neglected an irrelevant term.

Substituting Eqs. (4.15) and (4.20) into Eq. (4.19) we get:

$$\sum_p \frac{G^2(p,0)}{p} = \frac{\Sigma^R(0,0) - \Sigma^R(0,1)}{1-\partial\Sigma(0,0)} = -\frac{\partial_R\Sigma^R(0,0)}{1-\partial\Sigma(0,0)} = -\frac{\partial\Sigma(0,0)}{1-\partial\Sigma(0,0)} \tag{4.21}$$

as $\partial_R\Sigma^R = \partial\Sigma$. Hence:

$$\begin{aligned} G^4_{(3)}(0,m,0,m) &= -C_{0m}G^4(0,m,0,m)\frac{1}{G^2(0,0)}\frac{\partial\Sigma(0,0)}{1-\partial\Sigma(0,0)} \\ &= -G^4(0,m,0,m)\frac{A_{\text{ren}}\partial\Sigma(0,0)}{(m+A_{\text{ren}})[1-\partial\Sigma(0,0)]}. \end{aligned} \tag{4.22}$$

Using (4.10) and (4.22), Eq. (4.3) rewrites as:

$$\begin{aligned} &G^4(0,m,0,m)\left(1 + \frac{A_{\text{ren}}\partial\Sigma(0,0)}{(m+A_{\text{ren}})[1-\partial\Sigma(0,0)]}\right) \\ &\quad = \lambda\big(G^2(0,m)\big)^4\left(1-\partial\Sigma(0,0) + \frac{A_{\text{ren}}}{m+A_{\text{ren}}}\partial\Sigma(0,0)\right)\big[1-\partial_L\Sigma(0,m)\big]. \end{aligned} \tag{4.23}$$

We multiply (4.23) by $[1-\partial\Sigma(0,0)]$ and amputate four times. As the differences $\Gamma^4(0,m,0,m,) - \Gamma^4(0,0,0,0)$ and $\partial_L\Sigma(0,m) - \partial_L\Sigma(0,0)$ are irrelevant we get:

$$\Gamma^4(0,0,0,0) = \lambda\big(1-\partial\Sigma(0,0)\big)^2. \qquad \square \tag{4.24}$$

4.1. Bare identity

Let us explain now why the main theorem is also true as an identity between bare functions, without any renormalization, but with ultraviolet cutoff.

Using the same Ward identities, all the equations go through with only few differences:

- We should no longer add the lost mass counterterm in (4.12).
- The term $G^4_{(2)}$ is no longer zero.
- Eq. (4.9) and all propagators now involve the bare A parameter.

But these effects compensate. Indeed the bare $G^4_{(2)}$ term is the left generalized tadpole $\Sigma - \Sigma^R$, hence

$$G^4_{(2)}(0,m,0,m) = C_{0,m}\big(\Sigma(0,m) - \Sigma^R(0,m)\big)G^4(0,m,0,m). \tag{4.25}$$

Eq. (4.9) becomes up to irrelevant terms

$$\frac{C^{\text{bare}}_{0m}}{G^{2,\text{bare}}(0,m)} = 1 - \partial_L\Sigma(0,0) + \frac{A_{\text{bare}}}{m+A_{\text{bare}}}\partial_L\Sigma(0,0) - \frac{1}{m+A_{\text{bare}}}\Sigma(0,0). \tag{4.26}$$

The first term proportional to $\Sigma(0,m)$ in (4.25) combines with the new term in (4.26), and the second term proportional to $\Sigma^R(0,m)$ in (4.25) is exactly the former "lost counterterm" contribution in (4.12). This proves (2.4) in the bare case.

5. Conclusion

Since the main result of this Letter is proved up to irrelevant terms which converge at least like a power of the ultraviolet cutoff, as this ultraviolet cutoff is lifted towards infinity, we not only get that the beta function vanishes in the ultraviolet regime, but that it vanishes fast enough so that the total flow of the coupling constant is bounded. The reader might worry whether this conclusion is still true for the full model which has $\Omega_{\text{ren}} \neq 1$, hence no exact conservation of matrix indices along faces. The answer is yes, because the flow of Ω towards its ultra violet limit $\Omega_{\text{bare}} = 1$ is very fast (see e.g. [13, Section II.2]).

The vanishing of the beta function is a step towards a full non-perturbative construction of this model without any cutoff, just like e.g. the one of the Luttinger model [15,23]. But NC ϕ^4_4 would be the first such *four-dimensional* model, and the only one with non-logarithmic divergences. Tantalizingly, quantum field theory might actually behave better and more interestingly on non-commutative than on commutative spaces.

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