Fuzzy analytic hierarchy process: A logarithmic fuzzy preference programming methodology

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Abstract

Fuzzy analytic hierarchy process (AHP) proves to be a very useful methodology for multiple criteria decision-making in fuzzy environments, which has found substantial applications in recent years. The vast majority of the applications use a crisp point estimate method such as the extent analysis or the fuzzy preference programming (FPP) based nonlinear method for fuzzy AHP priority derivation. The extent analysis has been revealed to be invalid and the weights derived by this method do not represent the relative importance of decision criteria or alternatives. The FPP-based nonlinear priority method also turns out to be subject to significant drawbacks, one of which is that it may produce multiple, even conflict priority vectors for a fuzzy pairwise comparison matrix, leading to entirely different conclusions. To address these drawbacks and provide a valid yet practical priority method for fuzzy AHP, this paper proposes a logarithmic fuzzy preference programming (LFPP) based methodology for fuzzy AHP priority derivation, which formulates the priorities of a fuzzy pairwise comparison matrix as a logarithmic nonlinear programming and derives crisp priorities from fuzzy pairwise comparison matrices. Numerical examples are tested to show the advantages of the proposed methodology and its potential applications in fuzzy AHP decision-making.

1. Introduction

As a practical yet popular methodology for dealing with fuzziness and uncertainty in multiple criteria decision-making (MCDM), fuzzy analytic hierarchy process (AHP) has found huge applications in recent years. Since fuzzy judgments are easier to provide than crisp judgments, it can be concluded that fuzzy AHP will find more applications in the near future. The use of fuzzy AHP for multiple criteria decision-making requires scientific approaches for deriving the weights from fuzzy pairwise comparison matrices. Existing approaches for fuzzy AHP weight derivation can be classified into two categories, one of which is to derive a set of fuzzy weights from a fuzzy pairwise comparison matrix, while the other is to derive a set of crisp weights from a fuzzy pairwise comparison matrix. The approaches for deriving fuzzy weights from fuzzy pairwise comparison matrices mainly include the geometric mean method [7], fuzzy logarithmic least-squares methods (LLSM) [2,62,66], Lambda–Max methods [19,63] and the linear goal programming (LGP) method [64]. The approaches for deriving crisp weights from fuzzy pairwise comparison matrices include the extent analysis [18] and the fuzzy preference programming (FPP) based nonlinear method [45].

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2. The FPP-based nonlinear priority method and its non-uniqueness in solutions

Suppose the decision maker (DM) provides fuzzy judgments instead of precise judgments for a pairwise comparison matrix. For example, it could be judged that criterion i is between \( l_i \) and \( u_i \) times as important as criterion j with \( m_i \) being the most likely times. Then, a fuzzy pairwise comparison matrix can be expressed as

\[
A = (\tilde{a}_{ij})_{n \times n} = \begin{bmatrix}
1 & (l_{12}, m_{12}, u_{12}) & \cdots & (l_{1n}, m_{1n}, u_{1n}) \\
(l_{21}, m_{21}, u_{21}) & 1 & \cdots & (l_{2n}, m_{2n}, u_{2n}) \\
\vdots & \vdots & \ddots & \vdots \\
(l_{n1}, m_{n1}, u_{n1}) & (l_{n2}, m_{n2}, u_{n2}) & \cdots & 1
\end{bmatrix},
\]

(1)

where \( l_i = 1/u_i \), \( m_i = 1/m_i \), \( u_i = 1/l_i \), and \( 0 < l_i \leq m_i \leq u_i \) for all \( i, j = 1, \ldots, n; j \neq i \). To find a crisp priority vector \( W = (w_1, \ldots, w_n)^T > 0 \) with \( \sum_{i=1}^{n} w_i = 1 \) for the fuzzy pairwise comparison matrix in (1), Mikhailov \([45]\) introduces the following membership function for each fuzzy judgment in \( \tilde{A} \):

\[
\mu_j(w_i/w_j) = \begin{cases}
\frac{w_i}{w_j} - \frac{l_j}{m_j}, & \frac{w_i}{w_j} \leq m_j, \\
\frac{w_i}{w_j} - \frac{m_j}{l_j}, & \frac{w_i}{w_j} \geq m_j,
\end{cases}
\]

(2)

where \( \mu_j(w_i/w_j) \) is the membership degree of \( w_i/w_j \) belonging to the fuzzy judgment \( \tilde{a}_{ij} = (l_{ij}, m_{ij}, u_{ij}) \). Let

\[
\lambda = \min \left\{ \mu_j(w_i/w_j) \mid i = 1, \ldots, n - 1; j = i + 1, \ldots, n \right\}.
\]

(3)

Then, \( \lambda \) is the minimum membership degree to which the crisp priority vector satisfies each fuzzy pairwise comparison. It is hoped that the priority vector should be able to maximize the DM’s satisfaction. For this hope, Mikhailov \([45]\) established the following FPP-based nonlinear priority model, which is an extension of the FPP priority method for crisp pairwise comparison matrix \([46]\) in fuzzy environments:

\[
\begin{align*}
\text{Maximize} & \quad \lambda \\
\text{Subject to} & \quad \mu_j(w_i/w_j) \geq \lambda, \quad i = 1, \ldots, n - 1; j = i + 1, \ldots, n, \\
& \quad \sum_{i=1}^{n} w_i = 1, \\
& \quad w_i \geq 0, \quad i = 1, \ldots, n,
\end{align*}
\]

(4)

which can be equivalently expressed as

\[
\begin{align*}
\text{Maximize} & \quad \lambda \\
\text{Subject to} & \quad -w_i + l_i w_j + \lambda (m_i - l_i) w_j \leq 0, \quad i = 1, \ldots, n - 1; j = i + 1, \ldots, n, \\
& \quad w_i - u_i w_j + \lambda (u_i - m_i) w_j \leq 0, \quad i = 1, \ldots, n - 1; j = i + 1, \ldots, n, \\
& \quad \sum_{i=1}^{n} w_i = 1, \\
& \quad w_i \geq 0, \quad i = 1, \ldots, n.
\end{align*}
\]

(5)

If the optimal objective value \( \lambda^* > 0 \), then the optimal solution \( w_1^*, \ldots, w_n^* \) satisfy \( l_i \leq w_i/w_j \leq u_i \); otherwise, there exists strong inconsistency among the fuzzy judgments and the optimal solutions only approximately satisfy the fuzzy pairwise comparison matrix.
For convenience, we refer to the method that uses the above model (5) for fuzzy AHP priority derivation as the FPP-based nonlinear priority method. With regard to this method, we have the following research findings:

1. Negative membership degree makes no sense.
2. Model (5) produces multiple optimal solutions when there exists strong inconsistency among the fuzzy judgments.
3. The priority vectors derived by using the upper or lower triangular elements of a fuzzy pairwise comparison matrix are not the same, even significantly different.

Consider the following fuzzy pairwise comparison matrix as an example:

\[
\tilde{B} = \begin{bmatrix}
1 & \left(\frac{1}{2}, \frac{2}{3}\right) & \left(\frac{1}{2}, \frac{2}{3}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) \\
\left(\frac{1}{2}, \frac{2}{3}\right) & 1 & (1, 1, 1) & \left(\frac{1}{2}, \frac{2}{3}\right) \\
\left(\frac{1}{2}, \frac{2}{3}\right) & (1, 1, 1) & 1 & \left(\frac{1}{2}, \frac{2}{3}\right) \\
\left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{3}{2}\right) & \left(\frac{1}{2}, \frac{3}{2}\right) & 1
\end{bmatrix}
\]

By using the upper triangular elements of \( \tilde{B} \), model (5) can be written as

Maximize \( \lambda \)

\[
\begin{align*}
-w_1 + 1.5w_2 + 0.5\lambda w_2 &\leq 0, \\
w_1 - 2.5w_2 + 0.5\lambda w_2 &\leq 0, \\
-w_1 + 1.5w_3 + 0.5\lambda w_3 &\leq 0, \\
w_1 - 2.5w_3 + 0.5\lambda w_3 &\leq 0, \\
-w_1 + 2w_4 / 3 + \lambda w_4 / 3 &\leq 0, \\
w_1 - 1.5w_4 + 0.5\lambda w_4 &\leq 0, \\
-w_2 + w_3 &\leq 0, \\
w_2 - w_3 &\leq 0, \\
-w_2 + 1.5w_4 + 0.5\lambda w_4 &\leq 0, \\
w_2 - 2.5w_4 + 0.5\lambda w_4 &\leq 0, \\
-w_3 + 0.4w_4 + 0.1\lambda w_4 &\leq 0, \\
w_3 - 2w_4 / 3 + \lambda w_4 / 6 &\leq 0, \\
w_1 + w_2 + w_3 + w_4 &\leq 1, \\
w_1, w_2, w_3, w_4 &\geq 0.
\end{align*}
\]

It is easy to find that \( W^\triangleright = (0.2178, 0.2489, 0.2489, 0.2844) \) and \( \tilde{W}^\triangleright = (0.4359, 0.1795, 0.1795, 0.2051) \) are both the optimal solutions to the above model with the same optimal value \( \tilde{\lambda}^\triangleright = -0.25 \) for \( \lambda \). Since \( \tilde{\lambda}^\triangleright = -0.25 < 0 \), it makes no sense as a membership degree. The two optimal solutions also produce conflict conclusions since the first one indicates that \( w_1 \) is the smallest, whereas the second one shows that \( w_1 \) is the biggest.

If we use the lower triangular elements of \( B \) for weight derivation, model (5) can then be written as

Maximize \( \lambda \)

\[
\begin{align*}
-w_2 + 0.4w_1 + 0.1\lambda w_1 &\leq 0, \\
w_2 - 2w_1 / 3 + \lambda w_1 / 6 &\leq 0, \\
-w_3 + 0.4w_1 + 0.1\lambda w_1 &\leq 0, \\
w_3 - 2w_1 / 3 + \lambda w_1 / 6 &\leq 0, \\
-w_3 + w_2 &\leq 0, \\
w_3 - w_2 &\leq 0, \\
-w_4 + 2w_1 / 3 + \lambda w_1 / 3 &\leq 0, \\
w_4 - 1.5w_1 + 0.5\lambda w_1 &\leq 0, \\
-w_4 + 0.4w_2 + 0.1\lambda w_2 &\leq 0, \\
w_4 - 2w_2 / 3 + \lambda w_2 / 6 &\leq 0, \\
-w_4 + 1.5w_3 + 0.5\lambda w_3 &\leq 0, \\
w_4 - 2.5w_3 + 0.5\lambda w_3 &\leq 0, \\
w_1 + w_2 + w_3 + w_4 &\leq 1, \\
w_1, w_2, w_3, w_4 &\geq 0.
\end{align*}
\]

It is not difficult to verify that this model produces many optimal solutions. The followings are just three of them:
It is seen that the normalization constraint \( P_i \) or as \( ~3. The LFPP-based nonlinear priority method

For the fuzzy pairwise comparison matrix in (1), we take its logarithm by the following approximate equation:

\[
\ln \tilde{a}_{ij} \approx \langle \ln l_{ij}, \ln m_{ij}, \ln u_{ij} \rangle, \quad i, j = 1, \ldots, n. \tag{6}
\]

That is, the logarithm of a triangular fuzzy judgment \( \tilde{a}_{ij} \) can still be seen as an approximate triangular fuzzy number, whose membership function can accordingly be defined as

\[
\mu_y \left( \ln \left( \frac{w_i}{w_j} \right) \right) = \begin{cases} 
\ln \left( \frac{\ln \left( \frac{w_i}{w_j} \right) - \ln l_{ij}}{\ln m_{ij} - \ln l_{ij}} \right), & \ln \left( \frac{w_i}{w_j} \right) \leq \ln m_{ij}, \\
\ln \left( \frac{\ln \left( \frac{w_i}{w_j} \right) - \ln u_{ij}}{\ln m_{ij} - \ln u_{ij}} \right), & \ln \left( \frac{w_i}{w_j} \right) \geq \ln m_{ij},
\end{cases}
\]

where \( \mu_y \left( \ln (w_i/w_j) \right) \) is the membership degree of \( \ln (w_i/w_j) \) belonging to the approximate triangular fuzzy judgment \( \ln \tilde{a}_{ij} = \langle \ln l_{ij}, \ln m_{ij}, \ln u_{ij} \rangle \). It is very natural that we hope to find a crisp priority vector to maximize the minimum membership degree \( \lambda = \min \{ \mu_y (\ln (w_i/w_j)) | i = 1, \ldots, n-1; j = i + 1, \ldots, n \} \). The resultant model can be constructed as

Maximize \( \lambda \)

Subject to

\[
\begin{align*}
\mu_y (\ln (w_i/w_j)) & \geq \lambda, i = 1, \ldots, n-1; j = i + 1, \ldots, n, \\
w_i & \geq 0, i = 1, \ldots, n.
\end{align*}
\]

or as

Maximize \( 1 - \lambda \)

Subject to

\[
\begin{align*}
\ln w_i - \ln w_j - \lambda \ln (m_{ij}/l_{ij}) & \geq \ln l_{ij}, i = 1, \ldots, n-1; j = i + 1, \ldots, n, \\
- \ln w_i + \ln w_j - \lambda \ln (u_{ij}/m_{ij}) & \geq - \ln u_{ij}, i = 1, \ldots, n-1; j = i + 1, \ldots, n, \\
w_i & \geq 0, i = 1, \ldots, n.
\end{align*}
\]

It is seen that the normalization constraint \( \sum_{i=1}^{n} w_i = 1 \) is not included in the above two equivalent models. This is because the models will become computationally complicated if the normalization constraint is included. As a matter of fact, normalization can be carried out after the priorities are obtained from model (9). Before normalization, without loss of generality, we can assume \( w_i \geq 1 \) for all \( i = 1, \ldots, n \) such that \( \ln w_i > 0 \) for \( i = 1, \ldots, n \). Note that the nonnegative assumption for \( \ln w_i \geq 0 \) \((i = 1, \ldots, n)\) is not essential. We make this assumption is just for convenience in solution.

Generally, there is no guarantee that model (9) can always produce a positive value for the membership degree \( \lambda \). The reason for producing a negative value for \( \lambda \) is that there are no weights that can meet all the fuzzy judgments in \( \tilde{A} \) within their support intervals. That is to say, not all the inequalities \( \ln w_i - \ln w_j - \lambda \ln (m_{ij}/l_{ij}) \geq \ln l_{ij} \) or \( - \ln w_i + \ln w_j - \lambda \ln (u_{ij}/m_{ij}) \geq - \ln u_{ij} \) can hold at the same time. To avoid \( \lambda \) from taking a negative value, we introduce nonnegative deviation variables \( \delta_{ij} \) and \( \eta_{ij} \) for \( i = 1, \ldots, n-1 \) and \( j = i + 1, \ldots, n \) such that they meet the following inequalities:

\[
\begin{align*}
\ln w_i - \ln w_j - \lambda \ln (m_{ij}/l_{ij}) + \delta_{ij} & \geq \ln l_{ij}, i = 1, \ldots, n-1; j = i + 1, \ldots, n, \\
- \ln w_i + \ln w_j - \lambda \ln (u_{ij}/m_{ij}) + \eta_{ij} & \geq - \ln u_{ij}, i = 1, \ldots, n-1; j = i + 1, \ldots, n.
\end{align*}
\]

It is the most desirable that the values of the deviation variables are the smaller the better. We thus propose the following LFPP-based nonlinear priority model for fuzzy AHP weight derivation:
Minimize \[ J = (1 - \lambda)^2 + M \sum_{i=1}^{n} \sum_{j=i+1}^{n} (\delta_{ij}^2 + \eta_{ij}^2) \]

Subject to
\[
\begin{align*}
x_i - x_j - \lambda \ln \frac{m_{ij}}{l_{ij}} + \delta_{ij} & \geq \ln l_{ij}, \quad i = 1, \ldots, n - 1; \quad j = i + 1, \ldots, n, \\
-x_i + x_j - \lambda \ln \frac{u_{ij}}{m_{ij}} + \eta_{ij} & \geq -\ln u_{ij}, \quad i = 1, \ldots, n - 1; \quad j = i + 1, \ldots, n,
\end{align*}
\]

where \(x_i = \ln w_i\) for \(i = 1, \ldots, n\) and \(M\) is a specified sufficiently large constant such as \(M = 10^3\). The main purpose of introducing a big constant \(M\) into the above model is to find the weights within the support intervals of fuzzy judgments without violations or with as little violations as possible. So, the first priority of model (10) is given to minimize the violations of the fuzzy judgments and the weights can then be optimized to maximize the DM’s satisfaction, namely, the value of \(\lambda\).

Theoretically, \(L_1\) or \(L_\infty\) norm can also be used instead of the \(L_2\) norm in the objective function of (10) to model the deviation variables and membership degree. If we do so, the model will be linear. However, the linear models based on these two norms may sometimes produce multiple optimal solutions. So, they are not considered.

Let \(x'_i\) (\(i = 1, \ldots, n\)) be the optimal solution to model (10). The normalized priorities for fuzzy pairwise comparison matrix \(\tilde{A} = (\tilde{a}_{ij})_{n \times n}\) can then be obtained as
\[
w_i' = \frac{\exp(x'_i)}{\sum_{j=1}^{n} \exp(x'_j)}, \quad i = 1, \ldots, n, \tag{11}
\]

where \(\exp()\) is the exponential function, namely, \(\exp(x'_i) = e^{x'_i}\) for \(i = 1, \ldots, n\). We refer to the method that utilizes model (10) for fuzzy AHP priority derivation as the LFPP methodology and the resultant priorities as the LFPP priorities.

With regard to the LFPP methodology, we have the following theorems.

**Theorem 1.** The priorities derived by the LFPP methodology from the upper triangular elements of a fuzzy pairwise comparison matrix are exactly the same as those derived from the lower triangular elements of the fuzzy pairwise comparison matrix.

**Proof.** Consider a pair of fuzzy judgments \(\tilde{a}_{ij} = (l_{ij}, m_{ij}, u_{ij})\) and \(\tilde{a}_{ji} = (1/u_{ij}, 1/m_{ij}, 1/l_{ij})\). The constraints of model (10) derived from \(\tilde{a}_{ij}\) can be written as
\[
\begin{align*}
x_j - x_i - \lambda \ln \frac{m_{ij}}{l_{ij}} + \delta_{ij} & \geq \ln l_{ij}, \\
-x_i + x_j - \lambda \ln \frac{u_{ij}}{m_{ij}} + \eta_{ij} & \geq -\ln u_{ij},
\end{align*}
\]

which can be equivalently expressed as
\[
\begin{align*}
x_i - x_j + \lambda \ln \frac{m_{ij}}{l_{ij}} - \delta_{ij} & \geq \ln l_{ij}, \\
x_i - x_j - \lambda \ln \frac{m_{ij}}{l_{ij}} + \eta_{ij} & \geq \ln u_{ij},
\end{align*}
\]

It is easy to see that these two inequalities are exactly the constraints of model (10) for \(\tilde{a}_{ji}\). That is to say, the constraints of model (10) for \(\tilde{a}_{ij}\) and \(\tilde{a}_{ji}\) are always the same. So, the use of the upper or lower triangular elements of a fuzzy pairwise comparison matrix for weight derivation will always give the same priorities when the LFPP methodology is applied. \(\square\)

**Theorem 2.** The LFPP methodology produces the unique normalized optimal priority vector for any fuzzy pairwise comparison matrix.

**Proof.** The objective function of model (10) is a strict convex function since its Hessian matrix is positively definite. The constraints of model (10) are all linear inequalities, which form a convex feasible region. Therefore, model (10) is a convex programming. From the theory of optimization, it is known that for a convex programming with a strict convex objective function, its local optimal solution is the only global optimal solution. So, the optimal solution to model (10) is unique. As a result, the normalized priority vector determined by model (10) is also unique. This completes the proof. \(\square\)

Generally speaking, it is desirable that a positive optimal value can be achieved for \(\lambda\). If its optimal value turns out to be \(\lambda^* = 0\), then there exists strong inconsistency among the fuzzy judgments unless \(\delta^* = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\delta_{ij}^2 + \eta_{ij}^2) = 0\). The bigger the value of \(\delta^*\), the stronger the inconsistency among the fuzzy judgments. So, the value of \(\delta^*\) can be treated as an inconsistency measure for fuzzy pairwise comparison matrices.

### 4. Numerical examples

In this section, we test three numerical examples using the proposed LFPP methodology to illustrate its advantages and potential applications in fuzzy AHP decision-making.
Example 1. Consider the $4 \times 4$ fuzzy pairwise comparison matrix $\tilde{B}$ in Section 2. Model (10) for this fuzzy pairwise comparison matrix can be written as

Minimize $J = (1 - \lambda)^2 + M \cdot \sum_{i=1}^{3} \sum_{j=1}^{4} (\delta_{ij}^2 + \eta_{ij}^2)$

Subject to $x_i - x_j - \lambda \ln(4/3) + \delta_{ij} \geq \ln(3/2)$, $\lambda = 5.46$.

Taking a sufficiently large number for $M$, say $M = 1000$, to solve this model with the Microsoft Excel Solver, we get the optimal solution as

$$x_1^* = 0.9775, \quad x_2^* = 0.6228, \quad x_3^* = 0.3693, \quad x_4^* = 0.5214, \quad \lambda = 0,$$

$$\delta_{12}^* = 0.051, \quad \delta_{13}^* = \delta_{14}^* = \delta_{23}^* = 0, \quad \delta_{24}^* = 0.304, \quad \delta_{34}^* = 0,$$

$$\eta_{12}^* = \eta_{13}^* = 0, \quad \eta_{14}^* = 0.051, \quad \eta_{23}^* = 0.253, \quad \eta_{24}^* = 0, \quad \eta_{34}^* = 0.253.$$

Based on which, we have normalized LFPP priorities as

$$w_1^* = \text{EXP}(x_1^*) / \sum_{i=1}^{4} \text{EXP}(x_i^*) = 0.3473, \quad w_2^* = \text{EXP}(x_2^*) / \sum_{i=1}^{4} \text{EXP}(x_i^*) = 0.2436,$$

$$w_3^* = \text{EXP}(x_3^*) / \sum_{i=1}^{4} \text{EXP}(x_i^*) = 0.1890, \quad \text{and} \quad w_4^* = \text{EXP}(x_4^*) / \sum_{i=1}^{4} \text{EXP}(x_i^*) = 0.2201.$$

Since $\delta^* = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\delta_{ij}^2 + \eta_{ij}^2) = 0.2271 \neq 0$, there exists strong inconsistency among the fuzzy judgments of $\tilde{B}$. It is more desirable that these fuzzy judgments can be rechecked to improve their qualities.

Example 2. Consider the following $4 \times 4$ fuzzy pairwise comparison matrix $\tilde{C}$:

$$\tilde{C} = \begin{bmatrix} 1 & (\frac{1}{2}, \frac{1}{2}, 1) & (\frac{1}{2}, \frac{1}{2}, \frac{3}{2}) & (\frac{1}{2}, 1, \frac{3}{2}) \\ (1, 2, 5) & 1 & (\frac{1}{2}, \frac{1}{2}, \frac{3}{2}) & (\frac{1}{2}, \frac{1}{2}, 1) \\ (\frac{1}{2}, 6) & (\frac{1}{2}, \frac{1}{2}, 2) & 1 & (\frac{1}{2}, \frac{1}{2}, \frac{3}{2}) \\ (\frac{1}{2}, 3, 7) & (\frac{1}{2}, \frac{1}{2}, \frac{3}{2}) & (\frac{1}{2}, \frac{1}{2}, \frac{3}{2}) & 1 \end{bmatrix}.$$  

By solving model (10) for this fuzzy pairwise comparison matrix, we obtain the optimal solution as

$$x_1^* = 0.0394, \quad x_2^* = 0.7325, \quad x_3^* = 0.9557, \quad x_4^* = 1.1380, \quad \lambda = 1,$$

$$\delta_{12}^* = \delta_{13}^* = \delta_{14}^* = \delta_{23}^* = \delta_{34}^* = 0 \quad \text{and} \quad \eta_{12}^* = \eta_{13}^* = \eta_{14}^* = \eta_{23}^* = \eta_{24}^* = \eta_{34}^* = 0,$$

based on which, we have normalized LFPP priorities as

$$w_1^* = \text{EXP}(x_1^*) / \sum_{i=1}^{4} \text{EXP}(x_i^*) = 0.1176, \quad w_2^* = \text{EXP}(x_2^*) / \sum_{i=1}^{4} \text{EXP}(x_i^*) = 0.2353,$$

$$w_3^* = \text{EXP}(x_3^*) / \sum_{i=1}^{4} \text{EXP}(x_i^*) = 0.2941 \quad \text{and} \quad w_4^* = \text{EXP}(x_4^*) / \sum_{i=1}^{4} \text{EXP}(x_i^*) = 0.3529.$$

Due to the fact that $\lambda = 1$, this set of priorities match the fuzzy pairwise comparison matrix $\tilde{C}$ perfectly well. In other words, the modal values of all the fuzzy judgments can be precisely fitted by this set of normalized priorities. Apparently, this is the most desirable situation, but may rarely happen in the real-world fuzzy AHP decision-making.
Selection of A Suitable Ship Registry Alternative

Economic factors
(C1)
- Capital and insurance costs (C11)
- Manning costs (C12)
- Bank finance (C13)
- Tax-related expenses (C14)

Social factors
(C2)
- Degree of control (C21)
- Labor quality and availability (C22)
- Safety standards and requirements (C23)
- Environmental issues (C24)

Political considerations
(C3)
- Level of bureaucracy (C31)
- Comparative advantages of country (C32)
- TNSR
- Malta
- Panama
- TISR

Fig. 1. Hierarchical structure for ship registry selection problem.

Fig. 2. The hierarchical structure considered by the extent analysis.

Table 1
Fuzzy pairwise comparison matrix of three selection criteria with respect to the decision goal and its priorities.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>LFPP priorities</th>
<th>EA priorities</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>(1, 1, 1)</td>
<td>(5/2, 3, 7/2)</td>
<td>(3/2, 2, 5/2)</td>
<td>0.5518</td>
<td>1</td>
</tr>
<tr>
<td>C2</td>
<td>(2/7, 1/3, 2/5)</td>
<td>(1, 1, 1)</td>
<td>(2/5, 1, 3/2)</td>
<td>0.2015</td>
<td>0</td>
</tr>
<tr>
<td>C3</td>
<td>(2/5, 1/2, 2/3)</td>
<td>(2/3, 1, 3/2)</td>
<td>(1, 1, 1)</td>
<td>0.2467</td>
<td>0</td>
</tr>
</tbody>
</table>

\( \lambda = 0.5023 \).

Table 2
Fuzzy pairwise comparison matrix of the four sub-criteria of economic factors (C1) and its normalized LFPP priorities.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>C11</th>
<th>C12</th>
<th>C13</th>
<th>C14</th>
<th>LFPP priorities</th>
<th>EA priorities</th>
</tr>
</thead>
<tbody>
<tr>
<td>C11</td>
<td>(1, 1, 1)</td>
<td>(3/2, 1, 3/2)</td>
<td>(1, 1, 1)</td>
<td>(2/5, 1/2, 2/3)</td>
<td>0.2329</td>
<td>0.1413</td>
</tr>
<tr>
<td>C12</td>
<td>(2/3, 1, 3/2)</td>
<td>(1, 1, 1)</td>
<td>(2/5, 1/2, 2/3)</td>
<td>(2/3, 1, 3/2)</td>
<td>0.1901</td>
<td>0.1797</td>
</tr>
<tr>
<td>C13</td>
<td>(1, 1, 1)</td>
<td>(3/2, 2, 5/2)</td>
<td>(1, 1, 1)</td>
<td>(2/5, 1/2, 2/3)</td>
<td>0.2450</td>
<td>0.2610</td>
</tr>
<tr>
<td>C14</td>
<td>(3/2, 2, 5/2)</td>
<td>(2/3, 1, 3/2)</td>
<td>(3/2, 2, 5/2)</td>
<td>(1, 1, 1)</td>
<td>0.3320</td>
<td>0.4179</td>
</tr>
</tbody>
</table>

\( \lambda^* = 0 \).
Example 3. Consider a ship registry selection problem investigated by Celik et al. [15]. The hierarchical structure for this selection problem is shown in Fig. 1, where $C_1$, $C_2$, and $C_3$ are three selection criteria, each involving some sub-criteria, and TNSR (Turkish National Ship Registry), Malta, Panama, and TISR (Turkish International Ship Registry) are four possible potential selection alternatives for Turkish ship owners.

Celik et al. [15] conducted a decision analysis using the extent analysis, which has been revealed to be invalid and may result in a wrong decision being made. In particular, this invalid priority method assigns a zero weight to each of the selection criteria $C_2$ and $C_3$. These two zero weights fundamentally change the hierarchical structure of the selection problem in Fig. 1. If the zero weights for $C_2$ and $C_3$ were true, then these two selection criteria should not have been considered and included in the hierarchical structure in Fig. 1 from the very beginning of decision analysis. The DM should use only one selection criterion $C_1$ for the hierarchical decision analysis, as shown in Fig. 2, which is the actual hierarchical structure considered by the extent analysis. Now that the DM considers multiple selection criteria for ship registry selection, none of them should be given a zero weight. In other words, assigning a zero weight to any selection criterion or sub-criterion in the hierarchical structure in Fig. 1 makes no sense. Therefore, the extent analysis should be rejected.

Here, we reinvestigate this selection problem using the proposed LFPP methodology to provide a correct application of the fuzzy AHP. Tables 1–7 show the fuzzy pairwise comparison matrices taken from Celik et al. [15] with a very slight change in the criteria for each sub-criterion.
for the fuzzy judgment $\hat{a}_{23}$ in Table 5 under the sub-criterion $C_{14}$. The priorities obtained by using the LFPP methodology are provided in the columns under the heading “LFPP priorities”. To illustrate the fact that the extent analysis can make a wrong decision, the priorities obtained by the extent analysis, which are marked as “EA priorities”, are provided in the last columns of these tables. The aggregated priorities are presented in Tables 8 and 9, from which it is seen that the LFPP methodology evaluates Malta as the best alternative, whereas the extent analysis draws a different conclusion which selects Panama as the best alternative. Without doubt, the decision conclusion made by the LFPP methodology takes account of all the selection criteria and sub-criteria, whereas the conclusion made by the extent analysis considers only the selection criterion $C_1$ without taking into account the other two criteria. From the point of view of multiple criteria decision-making, the conclusion made by the LFPP methodology is more convincing and more believable than that drawn by the extent analysis.

To verify the conclusion made by the LFPP methodology, we conduct an analysis using the modified logarithmic least-squares method (LLSM) proposed in [66], which derives normalized fuzzy weights for fuzzy pairwise comparison matrices. The final global fuzzy priorities for the four selection alternatives produced by the modified LLSM are pictured in Fig. 3, which also reveals that Malta is the best decision alternative. So, we have reason to reject the conclusion arrived at by the extent analysis.

It is worth pointing out that the FPP-based nonlinear priority method is not tested for this application example because it produces too many priority vectors for the fuzzy pairwise comparison matrix (i.e. the numerical example illustrated in Section 2) of the four selection alternatives with respect to the sub-criteria $C_{31}$ in Table 7, leading to the final conclusion lack of persuasiveness.

<table>
<thead>
<tr>
<th>Table 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuzzy pairwise comparison matrices of four selection alternatives with respect to the sub-criteria of $C_2$ and their normalized priorities.</td>
</tr>
<tr>
<td>TNSR</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td><strong>A: Comparisons of the four alternatives with respect to the sub-criterion $C_{21}$</strong></td>
</tr>
<tr>
<td>TNSR</td>
</tr>
<tr>
<td>Malta</td>
</tr>
<tr>
<td>Panama</td>
</tr>
<tr>
<td>TISR</td>
</tr>
<tr>
<td>$\hat{\lambda} = 0$</td>
</tr>
<tr>
<td><strong>B: Comparisons of the four alternatives with respect to the sub-criterion $C_{22}$</strong></td>
</tr>
<tr>
<td>TNSR</td>
</tr>
<tr>
<td>Malta</td>
</tr>
<tr>
<td>Panama</td>
</tr>
<tr>
<td>TISR</td>
</tr>
<tr>
<td>$\hat{\lambda} = 0$</td>
</tr>
<tr>
<td><strong>C: Comparisons of the four alternatives with respect to the sub-criterion $C_{23}$</strong></td>
</tr>
<tr>
<td>TNSR</td>
</tr>
<tr>
<td>Malta</td>
</tr>
<tr>
<td>Panama</td>
</tr>
<tr>
<td>TISR</td>
</tr>
<tr>
<td>$\hat{\lambda} = 0.1347$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuzzy pairwise comparison matrices of four selection alternatives with respect to the sub-criteria of $C_3$ and their normalized priorities.</td>
</tr>
<tr>
<td>TNSR</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td><strong>A: Comparisons of the four alternatives with respect to the sub-criterion $C_{31}$</strong></td>
</tr>
<tr>
<td>TNSR</td>
</tr>
<tr>
<td>Malta</td>
</tr>
<tr>
<td>Panama</td>
</tr>
<tr>
<td>TISR</td>
</tr>
<tr>
<td>$\hat{\lambda} = 0$</td>
</tr>
<tr>
<td><strong>B: Comparisons of the four alternatives with respect to the sub-criterion $C_{32}$</strong></td>
</tr>
<tr>
<td>TNSR</td>
</tr>
<tr>
<td>Malta</td>
</tr>
<tr>
<td>Panama</td>
</tr>
<tr>
<td>TISR</td>
</tr>
<tr>
<td>$\hat{\lambda} = 0$</td>
</tr>
<tr>
<td><strong>C: Comparisons of the four alternatives with respect to the sub-criterion $C_{33}$</strong></td>
</tr>
<tr>
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<tr>
<td>Malta</td>
</tr>
<tr>
<td>Panama</td>
</tr>
<tr>
<td>TISR</td>
</tr>
<tr>
<td>$\hat{\lambda} = 0.0952$</td>
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Table 8
Aggregation of the local priorities obtained by the LFPP methodology.

<table>
<thead>
<tr>
<th>C11</th>
<th>C12</th>
<th>C13</th>
<th>C14</th>
<th>Local priorities</th>
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</thead>
<tbody>
<tr>
<td>Weight</td>
<td>0.2329</td>
<td>0.1901</td>
<td>0.2450</td>
<td>0.3320</td>
</tr>
<tr>
<td>TNSR</td>
<td>0.1051</td>
<td>0.1244</td>
<td>0.2000</td>
<td>0.0968</td>
</tr>
<tr>
<td>Malta</td>
<td>0.4143</td>
<td>0.2736</td>
<td>0.2000</td>
<td>0.3385</td>
</tr>
<tr>
<td>Panama</td>
<td>0.3188</td>
<td>0.4264</td>
<td>0.2000</td>
<td>0.3388</td>
</tr>
<tr>
<td>TISR</td>
<td>0.1617</td>
<td>0.1756</td>
<td>0.4000</td>
<td>0.2258</td>
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</table>

<table>
<thead>
<tr>
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<th>C22</th>
<th>C23</th>
<th>Local priorities</th>
</tr>
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<td>0.2220</td>
</tr>
<tr>
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<td>0.4055</td>
</tr>
<tr>
<td>Malta</td>
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<td>0.2604</td>
</tr>
<tr>
<td>Panama</td>
<td>0.3139</td>
<td>0.2368</td>
<td>0.1671</td>
</tr>
<tr>
<td>TISR</td>
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<td>0.1683</td>
<td>0.1671</td>
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<table>
<thead>
<tr>
<th>C31</th>
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<th>C33</th>
<th>Local priorities</th>
</tr>
</thead>
<tbody>
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<td>0.2500</td>
<td>0.5000</td>
</tr>
<tr>
<td>TNSR</td>
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<td>0.1329</td>
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<td>Malta</td>
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<td>0.4704</td>
</tr>
<tr>
<td>Panama</td>
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<td>0.2059</td>
<td>0.1920</td>
</tr>
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<td>TISR</td>
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<td>0.1486</td>
<td>0.2048</td>
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<table>
<thead>
<tr>
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<th>C2</th>
<th>C3</th>
<th>Global priorities</th>
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<td>0.2467</td>
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<tr>
<td>TNSR</td>
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<td>0.3091</td>
<td>0.2141</td>
</tr>
<tr>
<td>Malta</td>
<td>0.3099</td>
<td>0.2704</td>
<td>0.3967</td>
</tr>
<tr>
<td>Panama</td>
<td>0.3168</td>
<td>0.2385</td>
<td>0.1947</td>
</tr>
<tr>
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<td>0.2440</td>
<td>0.1946</td>
<td>0.2193</td>
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</table>

Table 9
Aggregation of the local priorities obtained by the extent analysis

<table>
<thead>
<tr>
<th>C11</th>
<th>C12</th>
<th>C13</th>
<th>C14</th>
<th>Local priorities</th>
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</thead>
<tbody>
<tr>
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<td>0.1413</td>
<td>0.1797</td>
<td>0.2610</td>
<td>0.4179</td>
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<tr>
<td>TNSR</td>
<td>0</td>
<td>0</td>
<td>0.1645</td>
<td>0</td>
</tr>
<tr>
<td>Malta</td>
<td>0.5239</td>
<td>0.3482</td>
<td>0.1645</td>
<td>0.4076</td>
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<tr>
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<td>0.4761</td>
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<td>0.1847</td>
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</table>

<table>
<thead>
<tr>
<th>C21</th>
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<th>C23</th>
<th>Local priorities</th>
</tr>
</thead>
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</tr>
<tr>
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</tr>
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<td>0.2349</td>
<td>0.3850</td>
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<tr>
<td>Panama</td>
<td>0.4305</td>
<td>0.3136</td>
<td>0.0401</td>
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<td>0.2815</td>
<td>0.0316</td>
<td>0.0401</td>
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<table>
<thead>
<tr>
<th>C31</th>
<th>C32</th>
<th>C33</th>
<th>Local priorities</th>
</tr>
</thead>
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<tr>
<td>Weight</td>
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<td>0.1797</td>
<td>0.2610</td>
</tr>
<tr>
<td>TNSR</td>
<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>Malta</td>
<td>0.5239</td>
<td>0.3482</td>
<td>0.1645</td>
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<tr>
<td>Panama</td>
<td>0.4761</td>
<td>0.6518</td>
<td>0.1645</td>
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<tr>
<td>TISR</td>
<td>0</td>
<td>0</td>
<td>0.5065</td>
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</table>

<table>
<thead>
<tr>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>Global priorities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Priorities</td>
<td>0.5518</td>
<td>0.2015</td>
<td>0.2467</td>
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<tr>
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<td>Panama</td>
<td>0.3168</td>
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<td>0.1947</td>
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<td>TISR</td>
<td>0.2440</td>
<td>0.1946</td>
<td>0.2193</td>
</tr>
</tbody>
</table>
5. Conclusions

Fuzzy AHP has been playing an increasingly important role in multiple criteria decision-making under uncertainty and has found extensive applications in a wide variety of areas such as supplier selection, customer requirements assessment and the like. The use of fuzzy AHP for multiple criteria decision-making requires scientific weight derivation from fuzzy pairwise comparison matrices. Existing approaches for deriving fuzzy weights from fuzzy pairwise comparison matrices turn out to be too sophisticated and rare to be applied, while the approaches for deriving crisp weights from fuzzy pairwise comparison matrices prove to be either invalid or subject to significant drawbacks such as producing multiple even conflict priority vectors for a fuzzy pairwise comparison matrix, leading to distinct conclusions. To address these drawbacks and provide a valid yet practical priority method for fuzzy AHP, we have proposed in this paper a logarithmic fuzzy preference programming based methodology for fuzzy AHP priority derivation, which we refer to as the LFPP methodology. It formulates the priorities of a fuzzy pairwise comparison matrix as a logarithmic nonlinear programming and derives crisp priorities from fuzzy pairwise comparison matrices. Theoretical analysis has revealed that the LFPP methodology can produce a unique optimal priority vector for any fuzzy pairwise comparison matrix. It overcomes the significant drawbacks suffered by a so-called fuzzy preference programming based nonlinear priority method in the literature. Three numerical examples examined using the LFPP methodology have illustrated its advantages and potential applications. It is expected that the LFPP methodology can arouse more research interests and applications of the fuzzy AHP in the near future.

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References


