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Some complexity results about threshold graphs

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Abstract

The problem of determining whether a graph G contains a threshold subgraph containing at least h edges is shown to be NP-complete if h is part of the input as the problems of minimum threshold completion, weighted 2-threshold partition and weighted 2-threshold covering. We also prove that the k-cyclic scheduling problem is NP-complete for all fixed k, a result used to show that deciding whether a threshold r-hypergraph contains a Hamiltonian cycle is NP-complete.

Key words: Threshold graph; Complexity; Cyclic scheduling

1. Introduction

The concept of threshold graph was introduced by Chvátal and Hammer [1] in 1977. While some complexity results are known for problems related with threshold graphs such as recognition [1] or Hamiltonicity [4] of threshold graphs, covering of a graph with three threshold graphs [10], the problem of determining whether a graph can be covered by two threshold graphs has an unknown complexity status [3, 5]. In this paper we prove that, given a graph $G = (V, E)$, deciding if G contains a threshold subgraph with at least h edges is NP-complete if h is part of the input. This implies directly that the problem of the minimum completion of a graph to obtain a threshold graph is NP-complete and we also show that the problems of finding a partition or a covering of weight at least W of a weighted graph by two threshold graphs are NP-complete even if the weights are restricted to be among three different integers.

We also look at the cyclic scheduling problem: Given n integers, a bound B and a positive integer A, is there a cyclic ordering of the n numbers such that the sum of any A consecutive ones in the ordering is at most B. It was shown in [6] that this problem is in P if $A \leqslant 2$ and we prove that it is NP-complete for all fixed $A \geqslant 3$. It turns out that there is a hypergraph formulation for this problem transforming it to the decision problem for the Hamilitonicity of threshold r-hypergraphs ($r \geqslant 3$ fixed) which is thus also NP-complete.

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2. Definitions and notations

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs on disjoint vertex sets. We write $G_1 + G_2$ for the union of G_1, G_2 and the complete bipartite graph on V_1 and V_2. For $v \in V_1$, we note by $N_{G_1}(v)$ the set of nodes adjacent to v in G_1.

A graph $G = (V, E)$ is a *threshold graph* [1] if and only if G does not contain $2K_2$, P_4 or C_4 as induced subgraphs. Alternative equivalent definitions are known, as:

(a) There is a partition of V into disjoints sets K, I and an ordering $\{u_1, u_2, \dots, u_n\}$ of the nodes in I such that K induces a clique in G, I is a stable set of vertices and $N_G(u_1) \subseteq N_G(u_2) \subseteq \cdots \subseteq N_G(u_n)$,

(b) there exist a real valued function $y: V \to \mathbb{R}$ and a real Y such that $S \subseteq V$ is a stable set if and only if $y(S) \leqslant Y$, where $y(S) := \sum(y(v) \mid v \in S)$.

A partition of V satisfying definition (a) will be called a (K, I) *partition of* G.

The three definitions above of threshold graphs can be generalized for hypergraphs, but they are no longer equivalent. In the following, we will use a generalization of the third definition [7]: A hypergraph $H = (V, E)$ is a *threshold r-hypergraph* if there exist an integer valued function $y: V \to Z$ and two integers r and Y such that $S \subseteq V$ is a hyperedge if and only if $|S| = r$ and $y(S) > Y$.

A *maximum threshold subgraph of* G is a threshold subgraph of G containing the largest number of edges of G among all threshold subgraphs of G.

A *weighted graph* $G = (V, E, w)$ is a graph where each edge $e \in E$ has an integer weight $w(e)$. For $A \subseteq E$, we write $w(A) := \sum(w(e) \mid e \in A)$.

We consider the following problems:

THRESHOLD SUBGRAPH.

Instance: A graph $G = (V, E)$ and a positive integer $h \leqslant |E|$.

Problem: Is there a subgraph $T = (V, E')$ of G with $|E'| \geqslant h$ such that T is a threshold graph?

THRESHOLD COMPLETION.

Instance: A graph $G = (V, E)$ and a positive integer h.

Problem: Is there a threshold graph $G' = (V, E')$ which can be obtained by adding at most h edges to G?

WEIGHTED 2-THRESHOLD-PARTITION.

Instance: A doubly weighted graph $G = (V, E, w_1, w_2)$ and an integer W.

Problem: Can G be partitioned into two threshold subgraphs T_1 and T_2 such that $w_1(T_1) + w_2(T_2) \geqslant W$?

WEIGHTED 2-THRESHOLD-COVERING.

Instance: A doubly weighted graph $G = (V, E, w_1, w_2)$ and an integer W.

Problem: Can G be covered by two threshold subgraphs T_1 and T_2 such that $w_1(T_1) + w_2(T_2) \geqslant W$?

HAMILTONIAN THRESHOLD r-HYPERGRAPH.

Instance: A threshold r-hypergraph $H = (V, E)$ given by the set V, the function $y: V \to Z$ and the real Y.

Problem: Does H contain a Hamiltonian cycle?

What we call a Hamiltonian cycle in an r-hypergraph H on n nodes is a cyclic ordering of the n nodes such that each r consecutive nodes form a hyperedge of H.

MONOTONE 3-PARTITION (MON3-PART).

Instance: A set U of $3k$ elements, a positive bound T and a size $t(u) \in Z$ for each $u \in U$ with $0 < t(u) < T$ and $\sum_{u \in U} t(u) = kT$.

Problem: Can U be partitioned into k ordered disjoint sets of cardinality three, $V_1, \ldots, V_k$, such that for any triple $V_i = (a_i, b_i, c_i)$, $i = 1, \ldots, k$, $t(a_i) + t(b_i) + t(c_i) = T$, $t(a_i) \leqslant t(a_{i+1})$ and $t(a_i) + t(b_i) \leqslant t(a_{i+1}) + t(b_{i+1})$?

K-CYCLIC SCHEDULING.

Instance: A finite set C and a length $l(c) \in Z^+$ for each $c \in C$, a positive integer B.

Problem: Is there a cyclic ordering of the elements of C such that the sum of the lengths of any K consecutive elements is at most B?

CLIQUE.

Instance: A graph $G = (V, E)$ and a positive integer $k \leqslant |V|$.

Problem: Does G contain a clique of size k or more?

ONE-IN-THREE 3-SAT.

Instance: A set U of variables and a collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$ and does not contain any negated variable.

Problem: Is there a truth assignment for U such that each clause in C has exactly one true variable?

NUMERICAL MATCHING WITH TARGET SUMS (NMTS).

Instance: Disjoint sets A and B each containing m elements, a size $s(a) \in Z^+$ for each element $a \in A \cup B$, and a target vector $\langle C_1, \ldots, C_m \rangle$ with positive integer entries.

Problem: Can $A \cup B$ be partitioned into m disjoint sets $D_1, \ldots, D_m$, each containing exactly one element from each of A and B, such that, for $i = 1, \ldots, m$, $\sum_{a \in D_i} s(a) = C_i$?

The three last problems are well-known NP-complete problems [2, problems [GT19], [LO4] and [SP17]] and we will show that all the other problems are also NP-complete.

3. Polynomial transformation between CLIQUE and THRESHOLD SUBGRAPH

In this section, we write $G_N = (J_N, \emptyset)$ the graph on N nodes containing no edges.

Lemma 1. *Let $G = (V, E)$ be a graph and $N > |V| + 1$ be an integer. Consider the graph $G' = G + G_N$ and let $T = (V \cup J_N, E^*)$ be a maximum threshold subgraph of G'. There exists a (K, I) partition of T such that $K \subseteq V$.*

Proof. Suppose that the lemma is not true. As T is a threshold graph, we can find a (K, I) partition of T with the good properties except that K is not contained in V, i.e., $K \cap J_N = \{j\}$ (note that K cannot contain more than one node of J_N, as two nodes of J_N are not joined by an edge in G') and one may assume that K is chosen as large as possible. The neighborhood of j in T has to contain at least one node of $V - K$, otherwise $N_T(j)$ is contained in K and the sets $K' := K - \{j\}$ and $I' := I \cup \{j\}$ will form a (K, I) partition of T satisfying the lemma, a contradiction. Let u be a neighbor of j in $V - K$. If $N_T(u) = K$, then redefine $K := K \cup \{u\}$ and $I := I - \{u\}$ and this contradicts the choice of K as large as possible. Thus, $N_T(u)$ is strictly contained in K. Now, for each v in $J_N - \{j\}$, we have $N_T(v) \subseteq N_T(u) - \{j\}$, since j is not in $N_T(v)$. It follows that $N_T(v)$ is strictly contained in $K - \{j\}$.

Consider now the subgraph H induced by V in T. Since an induced subgraph of a threshold graph is threshold, H is threshold. Let now S be the threshold graph obtained by adding to H all edges between J_N and $K - \{j\}$. Then, as $N > |V| + 1$, S has more edges than T since $|N_T(j)| \leqslant |V|$, $|N_S(j)| \geqslant 1$ and for each v in $J_N - \{j\}$, $N_S(v) = K - \{j\}$ whereas $N_T(v)$ is strictly contained in $K - \{j\}$. But this contradicts the choice of T. □

Lemma 2. *Let $G = (V, E)$ be a graph and $N > |V|^2$ be an integer. Consider the graph $G' := G + G_N$ and let $T = (V \cup J_N, E^*)$ be a maximum threshold subgraph of G'. There exists a (K, I) partition of T such that K induces in T a maximum clique of G.*

Proof. Lemma 1 implies that we can find a (K, I) partition of T such that $K \subseteq V$ and for all $p \in J_N$, $N_T(p) = K$. The number of edges of T is then: $|E^*| = N|K| + M < N(|K| + 1)$, where M is the the number of edges of T contained in E.

Suppose that, in contradiction to the lemma, there exists a clique in G induced by $K_1 \subseteq V$ such that $|K_1| > |K|$ and construct the threshold graph containing the induced clique on the nodes in K_1 and the edges joining each $j \in J_N$ with each $k \in K_1$. This threshold subgraph of G' has a number of edges $N|K_1| + \frac{1}{2}|K_1|(|K_1| - 1)$. But, as T is a maximum threshold subgraph of G' and $|K_1| > |K|$, we have:

$$|E^*| < N(|K| + 1) \leqslant N|K_1| < N|K_1| + \tfrac{1}{2}|K_1|(|K_1| - 1),$$

a contradiction with the maximality of T. □

Consider now the following algorithm:

Input: A graph $G = (V, E)$.

Output: The cardinality θ of the maximum clique of G.

Step 1. Let $N := |V|^2 + 1$, construct the graph $G + G_N$ and compute the number of edges d of the maximum threshold graph contained in it.

Step 2. Construct the graph $G + G_{N+1}$ and compute the number of edges e of the maximum threshold graph contained in it.

Step 3. Set $\theta := e - d$.

The proof this algorithm works is the following: As shown above, the threshold graphs with respectively d and e edges both contain a maximum clique K of G and, letting r be the number of edges contained in $E^* \cap E$ but not in K, we have:

$$d = N|K| + \tfrac{1}{2}|K|(|K| - 1) + r_1,$$

$$e = (N + 1)|K| + \tfrac{1}{2}|K|(|K| - 1) + r_2.$$

The maximality of the two threshold graphs implies $r_1 = r_2$, so

$$d - e = |K| = \theta.$$

This implies that if we can solve THRESHOLD SUBGRAPH in polynomial time for all h, then we can solve CLIQUE in polynomial time also, and as recognizing threshold graphs is in P [1], the proof of the NP-completeness of THRESHOLD SUBGRAPH is completed.

This result implies directly that THRESHOLD COMPLETION is NP-complete: Finding the smallest threshold graph containing a given graph G is equivalent to finding the maximum threshold subgraph contained in the complement of the graph G, as the complement of a threshold graph is a threshold graph [1].

Note that the NP-completeness of THRESHOLD-SUBGRAPH implies that WEIGHTED 2-THRESHOLD-PARTITION and WEIGHTED 2-THRESHOLD-COVERING are NP-complete: Let $G = (V, E)$ be a graph and h a positive integer forming an instance of THRESHOLD SUBGRAPH. We construct an instance of WEIGHTED 2-THRESHOLD-PARTITION (respectively 2-THRESHOLD-COVERING) as follows: Let $W := h$ and $G' = (V, E', w_1, w_2)$ be the complete doubly weighted graph on the nodes in V with, for all $e \in E'$,

$$w_1(e) := \begin{cases} 1, & \text{if } e \in E, \\ -|E|, & \text{otherwise,} \end{cases}$$

$$w_2(e) := 0.$$

As the complement of a threshold graph is a threshold graph [1] WEIGHTED 2-THRESHOLD-PARTITION (respectively 2-THRESHOLD-COVERING) has an answer "yes" if and only if THRESHOLD SUBGRAPH has an answer "yes".

Note that it is also possible to show that if we restrict the instances of WEIGHTED 2-THRESHOLD-PARTITION and WEIGHTED 2-THRESHOLD-COVERING such that all the weights w_1 of the edges are among three nonnegative integers and w_2 is always 0, these two problems are also NP-complete (the proof uses a reduction from ONE-IN-THREE-3SAT).

4. Polynomial transformation between NMTS and MON3-PART

The following theorem which is the key result for the remaining proofs of this paper was found by Troyon [8] and is given here with his kind permission. We have to mention that Wöginger was the first in investigating the complexity status of monotone partition problems and has found a proof of the NP-completeness of MONOTONE 4-PARTITION [9].

Theorem 3 [8]. *NUMERICAL MATCHING WITH TARGET SUMS is polynomially transformable to MONOTONE 3-PARTITION.*

Proof. Let A, B, C and s be an instance of NUMERICAL MATCHING WITH TARGET SUMS. For short, we will write only "x" for $s(x)$ or $t(x)$ in the following. We note the elements of A (respectively B) by $a_1, \dots, a_m$ (respectively $b_1, \dots, b_m$) and suppose that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_m$.

Construct the following instance of MONOTONE 3-PARTITION:

Let $F := \max\{C_i \mid i = 1, \dots, m\}$. We may suppose w.l.o.g. that $n \geqslant 3$ and $F > 0$. Define $T := 7$ and

$$
\begin{aligned}
x_i &:= 8(ma_i + im^2F + i) + 1, && i = 1, \dots, m,\\
y_i &:= 8mb_i + 2, && i = 1, \dots, m,\\
z_{ij} &:= 8(-mC_j - im^2F - i) + 4, && i = 1, \dots, m, j = 1, \dots, m,\\
h_{ij} &:= 8(mC_j + im^2F + mF + i) - 4, && i = 1, \dots, m, j = 1, \dots, m,\\
p_i &:= -8mF + 7, && i = 1, \dots, m^2 - m,\\
q_i &:= 8(-mC_i + m^4F) + 7, && i = 1, \dots, m,\\
r_i &:= 8(-im^2F - i - mF - m^4F) + 4, && i = 1, \dots, m,
\end{aligned}
$$

and let these $3m^2 + 3m$ numbers form the set U. We will prove that these two instances always produce the same answer for their respective problems. Notice that p_i is a constant and hence in the following we will denote it only by p.

(a) Suppose that NMTS has an answer "yes" with solution $a_i + b_{j(i)} = C_{k(i)}$, $i = 1, \dots, m$ and consider the following triples for the corresponding MON3-PART problem:

$$
\begin{aligned}
&(x_i, y_{j(i)}, z_{i,k(i)}) && \text{for } i = 1, \dots, m,\\
&(r_i, q_{k(i)}, h_{i,k(i)}) && \text{for } i = 1, \dots, m,\\
&(z_{u,v}, h_{u,v}, p) && \text{for } (u, v) \neq (i, k(i)) \text{ for all } i = 1, \dots, m.
\end{aligned}
$$

To express that two triples (a, b, c) and (e, f, g) satisfy the monotonicity constraints, we write $(a, b, c) \leqslant (e, f, g)$.

First, we show that $(r_i, q_{k(i)}, h_{i,k(i)}) \leqslant (z_{u,v}, h_{u,v}, p)$ for all possible choices for i, u, v:

(i) Obviously, we have $r_i \leqslant z_{u,v}$, for all i, u, v and

(ii) $r_i + q_{k(i)} = 8(-(im^2 + m)F - i - mC_{k(i)}) + 11 \leqslant 8mF = z_{u,v} + h_{u,v}$ for all i, u and v.

Now, we show that $(z_{u,v}, h_{u,v}, p) \leqslant (x_i, y_{j(i)}, z_{i,k(i)})$ for all possible choices for i, u, v:

(i) It is clear that $z_{u,v} \leqslant x_i$ for all i, u and v and

(ii) $z_{u,v} + h_{u,v} = 8mF \leqslant x_i + y_{j(i)}$ for all i, u and v.

Thus we only have to find an ordering for each three types of triples to prove that MON3-PART has answer "yes".

Consider the ordering according to decreasing i for the triples $(r_i, q_{k(i)}, h_{i,k(i)})$. Obviously, we have $(r_{i+1}, q_{k(i+1)}, h_{i+1,k(i+1)}) \leqslant (r_i, q_{k(i)}, h_{i,k(i)})$ for $i = 1, \dots, m-1$.

Consider the ordering of the triples $(z_{u,v}, h_{u,v}, p)$ obtained according to decreasing u and the triples with the same u are ordered according to decreasing C_v. As

(i) $z_{u,v} \leqslant z_{u',v'}$ if $u > u'$ and

(ii) $z_{u,v} \leqslant z_{u,v'}$ if $C_v \geqslant C_{v'}$,

the produced ordering is monotone.

Last, order the triples $(x_i, y_{j(i)}, z_{i,k(i)})$ according to increasing i. As we have assumed that $a_i \leqslant a_{i+1}$, for $i = 1, \dots, m-1$, this ordering is also monotone.

Hence, MON3-PART has answer "yes".

(b) Suppose that MON3-PART has answer "yes" with solution $(v_{i,1}, v_{i,2}, v_{i,3})$ for $i = 1, \dots, m$. The equation $v_{i,1} + v_{i,2} + v_{i,3} \equiv T$ (modulo 8), i.e., $v_{i,1} + v_{i,2} + v_{i,3} \equiv 7$ (modulo 8), shows that only the following triples (with unordered entries) may appear:

(x, y, z), (x, y, h), (x, y, r), (x, p, p), (x, p, q), (x, q, q), (z, z, p), (z, z, q), (z, h, p), (z, h, q), (z, p, r), (z, q, r), (h, h, p), (h, h, q), (h, p, r), (h, q, r), (p, r, r), or (q, r, r).

But as the sum of the entries of each of the triples (x, y, h), (x, p, p), (x, p, q), (x, q, q), (z, z, q), (z, h, q), (h, h, p), (h, h, q) is strictly greater than 7 and that of each of the triples (x, y, r), (z, z, p), (z, p, r), (z, q, r), (h, p, r), (p, r, r), and (q, r, r) is smaller than 0, the only triples which may appear in the solution of MON3-PART are the triples (x, y, z), (z, h, p) or (h, q, r).

This implies:

(i) Exactly m triples of type (x, y, z) appear in the solution. For one such triple $(x_i, y_j, z_{e,k})$ we have $x_i + y_j + z_{e,k} \equiv 7$ (modulo $8m$), i.e., $(8i + 1) + 2 + (-8e + 4) = 7 + 8(i - e) \equiv 7$ (modulo $8m$) which implies $i = e$ and hence $a_i + b_j = C_k$.

(ii) Exactly m triples of type (h, q, r) appear in the solution. By the same argument as above one can show that for one such triple $(h_{i,j}, q_k, r_e)$, one has $i = e$ and since $h_{ij} + q_k + r_i = 8m(C_j - C_k) + 7 = T$, we have $C_j = C_k$. Hence $h_{i,j} = h_{i,k}$ and we may suppose w.l.o.g. that $j = k$, i.e., for each $v = 1, \dots, m$, exactly one $h_{u,v}$ appears in the triples (h, q, r).

(iii) Exactly $m^2 - m$ triples of type (z, h, p) appear in the solution. With the same arguments one can show that for one such triple $(z_{i,j}, h_{k,e}, p)$ one has $i = k$ and $C_j = C_e$. Once again, as $z_{i,j} = z_{i,e}$, we may suppose w.l.o.g. that $j = e$.

Now, (ii) and (iii) imply that for each $v = 1, \dots, m$, exactly one $z_{u,v}$ appears in the triples of type (x, y, z), i.e. all targets appears in these triples.

Hence NMTS has an answer "yes". □

Note that the above constructed instance of MON3-PART has $t(u) < 0$ for some u, but by adding to all the numbers $\max|t(u)| + 1$ and using the bound $T := T + 3(\max|t(u)| + 1)$, the constructed instance satisfies the given condition on the size function.

5. Polynomial transformation between MON3-PART and 3-CYCLIC SCHEDULING

Let U, T and the size function t be an instance of MON3-PART. Construct the following instance of 3-CYCLIC SCHEDULING: Define $C := U \cup \{a, b, c, d, e\}$, $l(u) := t(u)$ for all $u \in U$, $l(a)$, $l(b)$, $l(c)$, $l(d) := 0$, $l(e) := T$, $B := T$.

It is obvious that any solution $V_1, \ldots, V_k$ to MON3-PART allows the construction of a solution to the corresponding instance of 3-CYCLIC SCHEDULING in the following way: Take the elements corresponding to $V_k, \ldots, V_1$ in that order and then take a, b, e, c, d.

It is also clear that any solution to 3-CYCLIC SCHEDULING induces a solution for MON3-PART: the element e in the solution of 3-CYCLIC SCHEDULING has to be preceded and followed by two elements with length 0, i.e., by $\{a, b, c, d\}$. Now, partition the elements of $C - \{a, b, c, d, e\}$ in k sets of three consecutive elements in the cyclic ordering. The corresponding triples form a solution to MON3-PART.

6. Polynomial transformation between MON3-PART and 4-CYCLIC SCHEDULING

Let U, T and the size function t be an instance of MON3-PART (recall that $|U| = 3k$). Construct the following instance of 4-CYCLIC SCHEDULING: Let D be a set of $k - 1$ elements, $C := U \cup D \cup \{a, b, c, d, e\}$, $l(u) := t(u)$ for all $u \in U$, $l(d_i) := 3T$ for all $d_i \in D$, $l(a)$, $l(b)$, $l(c)$, $l(d)$, $l(e) := T$, $B := 4T$.

It is obvious that any solution $V_1, \ldots, V_k$ to MON3-PART allows the construction of a solution to the corresponding instance of 4-CYCLIC SCHEDULING in the following way:

$$a_k, b_k, c_k, d_{k-1}, a_{k-1}, b_{k-1}, c_{k-1}, d_{k-2}, \ldots, a_2, b_2, c_2, d_1, a_1, b_1, c_1, a, b, c, d, e,$$

where $V_i = (a_i, b_i, c_i)$ for $i = 1, \ldots, k$ and $D = \{d_1, \ldots, d_{k-1}\}$.

Now suppose that the constructed instance of 4-CYCLIC SCHEDULING has a solution. Then any $d_i \in D$ has to be preceded and followed by three elements with length $< T$, as the sum of the three elements preceding or following d_i is $\leqslant T$ and no element has length 0. This implies that the solution looks like:

$$a_k, b_k, c_k, d_{k-1}, a_{k-1}, b_{k-1}, c_{k-1}, d_{k-2}, \ldots, a_2, b_2, c_2, d_1, a_1, b_1, c_1, a, b, c, d, e,$$

with the same notation as above, and the triples $V_i = (a_i, b_i, c_i)$ for $i = 1, \ldots, k$ form a solution to MON3-PART.

7. Polynomial transformation between K-CYCLIC SCHEDULING and $(K + 1)$-CYCLIC SCHEDULING

This transformation will show that K-CYCLIC SCHEDULING is NP-complete for all fixed $K \geqslant 5$. As shown by the transformation between MON3-PART and

4-CYCLIC SCHEDULING, 4-CYCLIC SCHEDULING is NP-complete even if we restrict the instances such that $|C| \equiv 0 \pmod 4$. We will give a polynomial transformation of instances of K-CYCLIC SCHEDULING with $|C| \equiv 0 \pmod K$ to instances of $(K+1)$-CYCLIC SCHEDULING with $|C'| \equiv 0 \pmod{(K+1)}$.

Let C, l, B be an instance of K-CYCLIC SCHEDULING with $|C| \equiv 0 \pmod K$. We construct the following instance of $(K+1)$-CYCLIC SCHEDULING: Let D' be a set of $|C|/K$ elements, define $C' := C \cup D'$, $l'(c) := l(c)$ for all $c \in C$, $l'(d') := 2B$ for all $d' \in D'$ and $B' := 3B$ and let C', l', B' be the instance of $(K+1)$-CYCLIC SCHEDULING.

Obviously, if K-CYCLIC SCHEDULING has a solution then one can obtain a solution to $(K+1)$-CYCLIC SCHEDULING in the following way: Partition C in $|C|/K$ K-uples of consecutive elements in the solution of K-CYCLIC SCHEDULING and add one element of D' between each of these K-uples.

Now, if the constructed instance of $(K+1)$-CYCLIC SCHEDULING has a solution, then the K elements preceding and following any $d' \in D$ have to be in $C' - D'$, which implies that exactly K elements of $C' - D'$ are between any two elements of D' in the solution. Thus, the solution obtained by deleting all elements of D' is a solution to K-CYCLIC SCHEDULING.

8. Polynomial transformation between K-CYCLIC SCHEDULING and HAMILTONIAN THRESHOLD K-HYPERGRAPH

This transformation is straightforward: Let C, l and B be an instance of K-CYCLIC SCHEDULING. The corresponding instance of HAMILTONIAN THRESHOLD K-HYPERGRAPH is the following: Each element of C has a corresponding node in the hypergraph $H = (V, E)$, the function y is defined by $y(v) := -l(v)$ for all $v \in V$ and $Y := -B - 1$. Then, a hyperedge e exists if and only if the sum of the length of the K corresponding elements of C is at most B, which implies that K-CYCLIC SCHEDULING has an answer "yes" if and only if HAMILTONIAN THRESHOLD K-HYPERGRAPH has an answer "yes".

Note that 2-CYCLIC SCHEDULING and HAMILTONIAN THRESHOLD 2-HYPERGRAPH are solvable in polynomial time [4, 6] whereas MON-4-PART is NP-complete [9] as probably MON-K-PART for all fixed $K \geqslant 5$.

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