Journal of Combinatorial Theory, Series B 100 (2010) 497–509

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

On the vertex-stabiliser in arc-transitive digraphs

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ARTICLE INFO

Article history:
Received 4 June 2009
Available online 2 July 2010

Keywords:
Digraphs
Graphs
Groups
Arc-transitive
Vertex-stabiliser
Prime out-valence
Alter-exponent
Alternet

ABSTRACT

We discuss a possible approach to the study of finite arc-transitive digraphs and prove an upper bound on the order of a vertex-stabiliser in locally cyclic arc-transitive digraphs of prime out-valence.

1. Introduction

By a *digraph* $\vec{\Gamma}$, we shall mean an ordered pair (V, A) where $V = \mathrm{V}(\vec{\Gamma})$ is a non-empty set and $A = \mathrm{A}(\vec{\Gamma}) \subseteq V \times V$ is an arbitrary binary relation on V. We shall refer to V and A in a graph theoretical fashion as the *vertex-set* and the *arc-set*, and call their elements *vertices* and *arcs* of $\vec{\Gamma}$, respectively. A digraph $\vec{\Gamma}$ is called a *graph* when the relation $\mathrm{A}(\vec{\Gamma})$ is symmetric and is said to be *asymmetric* provided that the relation $\mathrm{A}(\vec{\Gamma})$ is asymmetric. We will assume throughout the paper that digraphs are finite, that is, that V is a finite set.

An *s-arc* in a digraph $\vec{\Gamma}$ is a sequence $\alpha = (v_0, \ldots, v_s)$ of $s+1$ vertices of $\vec{\Gamma}$ such that each three consecutive vertices in the sequence are pairwise distinct and such that (v_{i-1}, v_i) is an arc of $\vec{\Gamma}$ for every $i \in \{1, \ldots, s\}$.

An automorphism of $\vec{\Gamma}$ is a permutation of $\mathrm{V}(\vec{\Gamma})$ which preserves the relation $\mathrm{A}(\vec{\Gamma})$. Let G be a subgroup of the full automorphism group $\mathrm{Aut}(\vec{\Gamma})$ of $\vec{\Gamma}$. We say that $\vec{\Gamma}$ is G-vertex-transitive or G-arc-transitive provided that G acts transitively on $\mathrm{V}(\vec{\Gamma})$ or $\mathrm{A}(\vec{\Gamma})$, respectively. Similarly, we say that

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0095-8956/$ – see front matter
doi:10.1016/j.jctb.2010.03.002

$\vec{\Gamma}$ is (G,s)-*arc-transitive* if G acts transitively on the set of s-arcs of $\vec{\Gamma}$. When $G = \mathrm{Aut}(\vec{\Gamma})$, the prefix G in the above notation is usually omitted.

One of the oldest and most studied questions in algebraic graph theory is about the order and structure of the vertex-stabiliser G_v in a finite connected G-arc-transitive graph. It is somewhat surprising that, under some additional assumptions, the order of the vertex-stabiliser can often be bounded by a function of the valence. For example, a beautiful result of Tutte [13] shows that $|G_v|$ is bounded by 48 in the case of G-arc-transitive graphs of valence 3, and, by a result of Weiss [17], it follows that a similar bound exists for arc-transitive graphs of any prime valence. This was further generalised in a long series of papers by Trofimov (see [14] for a summary), in which he proved the existence of a function f with the property that $|G_v| \leqslant f(d)$ for any connected $(G,2)$-arc-transitive graph of valence d. Existence of such a function was conjectured by Weiss [16] for any locally primitive graph (a G-arc-transitive graph Γ for which G_v acts primitively on the neighbourhood $\Gamma(v)$).

The situation is quite different in the case of asymmetric G-arc-transitive digraphs, where the order of the vertex-stabiliser G_v cannot be bounded by a function of the out-valence even when it is assumed that the permutation group $G_v^{\vec{\Gamma}^+(v)}$ induced by the action of G_v on the out-neighbourhood $\vec{\Gamma}^+(v)$ is regular.

To see this, consider the family of the *wreath digraphs* $\vec{W}(n,k)$, defined as the lexicographic product $\vec{C}_n[\overline{K_k}]$, of the directed cycle $\vec{C}_n$ on $n \geqslant 3$ vertices with the empty graph on $k \geqslant 2$ vertices (i.e. the digraphs with vertex-set $\mathbb{Z}_n \times \mathbb{Z}_k$ and arcs of the form $((i,j),(i+1,j'))$ for $i \in \mathbb{Z}_n$ and $j, j' \in \mathbb{Z}_k$). Note that the full automorphism group of $\vec{W}(n,k)$ is isomorphic to the wreath product $\mathbb{Z}_n \wr S_k$ and contains an arc-transitive subgroup G isomorphic to $\mathbb{Z}_n \wr \mathbb{Z}_k$ with vertex-stabiliser $G_v \cong \mathbb{Z}_k^{n-1}$. Even though the group $G_v^{\vec{\Gamma}^+(v)}$ is cyclic of order k (and thus regular), the vertex-stabiliser G_v has order k^{n-1}, and is therefore not only arbitrary large, but even grows exponentially with the number of vertices of the digraphs $\vec{W}(n,k)$.

This example suggests that, in the context of arc-transitive digraphs, the correct question is not whether the order of the vertex-stabiliser can be bounded by a constant, but, rather, whether it can be bounded by a reasonably tame function (polynomial, say) of the number of vertices. The wreath digraphs show that such a bound does not exist in the class of all arc-transitive digraphs (even when restricted to prime out-valence and cyclic local action). However, as the theorem below states, if we exclude a particular family of "problematic" digraphs (denoted by $\mathrm{Pl}^r(\vec{W}(n,p))$ and defined precisely in Section 2.2) the existence of such a bound can indeed be proved.

Theorem 1.1. *Let $\vec{\Gamma}$ be a finite connected asymmetric G-arc-transitive digraph of prime out-valence p, let v be a vertex of $\vec{\Gamma}$ and suppose that $G_v^{\vec{\Gamma}^+(v)}$ is a cyclic group of order p. Then one of the following holds.*

(i) $\vec{\Gamma} \cong \mathrm{Pl}^r(\vec{W}(n,p))$ *for some $n \geqslant 3$ and $r \geqslant 0$, or*
(ii) $|G_v| \leqslant |\mathrm{V}(\vec{\Gamma})|^3$.

Let us mention at this point that the assumption on $\vec{\Gamma}$ being finite connected and G-arc-transitive implies that it is vertex-transitive and strongly connected (see for example [19, Section 4.4] for the definition of strong connectivity of digraphs, and [10, Lemma 2] for the proof of the latter fact).

Note that for each graph $\mathrm{Pl}^r(\vec{W}(n,p))$, there exists a group of automorphisms G acting arc-transitively with $G_v^{\vec{\Gamma}^+(v)}$ a cyclic group of order p such that, for fixed r, $|G_v|$ grows exponentially with the number of vertices, in sharp contrast with case (ii) of the theorem.

It is quite probable that the bound in case (ii) of the theorem is not tight. The best lower bound known when $p=2$ is due to Gardiner and Praeger [2]. All the graphs considered in their paper have a natural orientation as digraphs. Using this orientation, one of their construction easily yields an infinite family of G_i-arc-transitive digraphs $\vec{\Gamma}_i$ with out-valence 2, where $|\mathrm{V}(\vec{\Gamma}_i)| = i3^i$, and $|(G_i)_v| = 2^i$.

Using standard arguments, one can easily show that in the context of Theorem 1.1 the vertex-stabiliser G_v is a p-group of order p^s where s is the largest integer such that G acts transitively on the set of s-arcs of $\vec{\Gamma}$ (see Lemma 5.4). Moreover, if g is an element of G mapping v into a neighbouring vertex u, then G_v and G_u are p-subgroups of G, conjugate via g, with the property that

$G_v \cap G_u$ has index p in G_v. Finite groups G containing two such p-subgroups were studied in detail by Glauberman. His results (in particular [4, Section 3]) immediately imply the following theorem.

Theorem 1.2. *Let $\vec{\Gamma}$ and G be as in Theorem 1.1 and let s be the largest integer such that G acts transitively on the set of s-arcs of $\vec{\Gamma}$. Then the vertex-stabiliser G_v is a group of order p^s. Moreover, there exist constants α, κ and $\varepsilon(\iota,\delta) \in \{0,1,\dots,p-1\}$ satisfying $\frac{2}{3}s \leqslant \alpha \leqslant \frac{1}{2}(\alpha+s) \leqslant \kappa \leqslant s$ such that*

$$G_v = \langle x_0, x_1, \dots, x_{s-1} \mid x_0^p = x_1^p = \dots = x_{s-1}^p = 1,\ [x_i, x_j] = \varphi(i,j) \text{ for } 0 \leqslant i < j \leqslant s-1 \rangle$$

where

$$\varphi(i,j) = \begin{cases} 1, & \text{if } 1 \leqslant j-i \leqslant \alpha-1, \\ x_{i+s-\alpha}^{\varepsilon(0,j-i)} \cdots x_{j-s+\alpha}^{\varepsilon(j-i-2s+2\alpha,j-i)}, & \text{if } \alpha \leqslant j-i \leqslant \kappa-1, \\ x_{i+\kappa-\alpha}^{\varepsilon(0,j-i)} \cdots x_{j-\kappa+\alpha}^{\varepsilon(j-i-2\kappa+2\alpha,j-i)} & \text{if } \kappa \leqslant j-i \leqslant s-1. \end{cases}$$

In particular, the nilpotency class of G_v is at most 3. Furthermore, if $p=2$, then $\kappa = s$ and the nilpotency class of G_v is at most 2. The group G is generated by the vertex-stabiliser G_v and an element g, such that $x_{i-1}^g = x_i$ for all $i \in \{1,\dots,s-1\}$.

We should mention that in the case when $\kappa \neq s$ not all the groups with the above presentation can occur as vertex-stabilisers G_v. Further restrictions on the constants $\varepsilon(\iota,\delta)$ can be found in [4, Proposition 3.6].

Note also that the above theorem for the case $p=2$ appeared also as Theorem 1.1 in [7], where a slight misprint occurred at the end of line (R3); instead of the term $\tau_{(h-d+i)+2d-2h+j-1}^{\varepsilon(j-i,2d-2h+j-1)}$, one should have $\tau_{(h-d+i)+2d-2h+j-i}^{\varepsilon(j-i,2d-2h+j-i)}$.

Let us point out that, in many ways, our paper follows Tutte's ingenious approach [13] to cubic arc-transitive graphs, which was later used and improved by several authors (see for example [1,3,4, 12,16]). The essence of this approach is perhaps best captured in a purely group theoretical lemma of Glauberman [3, Lemma 1]. One of the side results of our paper is a modification (proved in Section 6) of this lemma, where one of the original conditions is substituted with one which seems to be easier to check in practice. Since this modification might prove useful elsewhere, we state it as a separate theorem.

Theorem 1.3. *(Compare with [3, Lemma 1].) Let x and g be elements of a group G. Put $x_i = x^{g^i} = g^{-i}xg^i$ for $i \in \mathbb{Z}$ and define $H_i = \langle x_1, \dots, x_i \rangle$ for each $i \geqslant 1$. Let $H_0 = 1$. Suppose that x has prime order p, let t be the largest integer such that $|H_t| = p^t$ and let n be the smallest integer such that $H_{t+n} = H_{t+n+1}$. Suppose that H_t contains no nonidentity normal subgroup of $\langle H_t, g \rangle$. Then $t \leqslant 3n$ and $t \neq 3n-1$. Moreover, if $n=2$ and $p=2$, then $t \leqslant 4$.*

We have tried to prove partial results of the paper in as general a form as possible. The proofs of the main theorems are thus broken up into several lemmas, where the assumptions on the digraphs appearing in these lemmas vary from very weak to the full assumptions of Theorem 1.1. To facilitate smooth reading, we have also included an extended summary (Section 2), where a casual reader can see the main steps and ideas of the proofs without going into much details. This allows us to use a slightly drier style in the rest of the paper, where exact proofs of the results can be found.

We now conclude this section with a few definitions needed in the rest of the paper. Let $\vec{\Gamma}$ be a digraph. If (u,v) is an arc of $\vec{\Gamma}$, then we say that v is an *out-neighbour* of u and that u is an *in-neighbour* of v. We also say that u is the *tail* and v the *head* of (u,v), respectively. The symbols $\vec{\Gamma}^+(v)$ and $\vec{\Gamma}^-(v)$ will denote the set of out-neighbours of v and the set of in-neighbours of v, respectively. A vertex without out-neighbours is called a *sink* and a vertex without in-neighbours a *source* of $\vec{\Gamma}$. The digraph $\vec{\Gamma}$ is said to be of out-valence k if $|\vec{\Gamma}^+(v)| = k$ for every $v \in \mathrm{V}(\vec{\Gamma})$.

A *successor* of an s-arc $(v_0, v_1, \dots, v_s)$ of $\vec{\Gamma}$ is an s-arc $(v_1, \dots, v_s, v_{s+1})$ such that $(v_0, \dots, v_s, v_{s+1})$ is an $(s+1)$-arc of $\vec{\Gamma}$.

If R is an equivalence relation on $\mathrm{V}(\vec{\Gamma})$, then we can construct a *quotient digraph* $\vec{\Gamma}/R$, the vertices of which are the equivalence classes of R, with two equivalence classes U and V of R forming an arc (U, V) in $\vec{\Gamma}/R$ whenever there exist $(u, v) \in \mathrm{A}(\vec{\Gamma})$ with $u \in U$ and $v \in V$.

2. Summary

2.1. Concerning alterexponent

The main tool which enabled us to extend the existing results about the size of the vertex-stabilisers in arc-transitive digraphs is the notion of the alterexponent of a digraph. In this subsection, we will summarise some relevant definitions and results pertaining to this topic, and refer the reader to [9] for details and proofs.

Let $\vec{\Gamma}$ be a digraph. A *walk* of length n in $\vec{\Gamma}$ is a sequence $W = (v_0, a_1, v_1, a_2, \ldots, a_n, v_n)$ of vertices v_i and arcs a_i of $\vec{\Gamma}$ such that, for any $i \in \{1, \ldots n\}$, either $a_i = (v_{i-1}, v_i)$ or $a_i = (v_i, v_{i-1})$. In the first case, we say that the arc a_i is *positively oriented* in W. We say that it is *negatively oriented* in the second case. In many cases (for example if $\vec{\Gamma}$ is asymmetric), no ambiguity arises if we omit the reference to the arcs and simply list the vertices of a walk.

The *sum* $s(W)$ of a walk W is the difference between the number of positively oriented arcs in the walk and the number of negatively oriented arcs in the walk. The kth *partial sum* $s_k(W)$ of the walk W is the sum of the initial walk $(v_0, a_1, v_1, a_2, v_2, \ldots, v_{k-1}, a_k, v_k)$ of length k. By convention, we let $s_0(W) = 0$.

The *tolerance* of a walk W is the set $\{s_k(W)\colon\ k \in \{0, 1, \ldots, n\}\}$. Observe that the tolerance of a walk is always an interval of integers containing 0. We say that two vertices u and v of $\vec{\Gamma}$ are *alter-equivalent with tolerance I* if there is a walk from u to v with sum 0 and tolerance $J \subseteq I$ and denote this by $u\mathcal{A}_I v$.

It transpires that the relation $\mathcal{A}_I$ is an equivalence relation for any interval I containing 0, and is invariant under any automorphism of $\vec{\Gamma}$. The equivalence class containing the vertex v will be denoted by $\mathcal{A}_I(v)$. For an integer $i \in \mathbb{Z} \cup \{\pm\infty\}$ we will abbreviate and write $\mathcal{A}_i$ instead of $\mathcal{A}_{[0,i]}$ (if $i \geqslant 0$) or instead of $\mathcal{A}_{[i,0]}$ (if $i \leqslant 0$).

Since we assume that $\vec{\Gamma}$ is a finite digraph, there exists an integer $e \geqslant 0$ such that $\mathcal{A}_e = \mathcal{A}_{e+1}$ (and then $\mathcal{A}_e = \mathcal{A}_\infty$). The smallest such integer is called the *alterexponent* of $\vec{\Gamma}$ and denoted by $\exp(\vec{\Gamma})$. Note that $\exp(\vec{\Gamma}) = 0$ is equivalent to the fact that $|\vec{\Gamma}^-(v)| \leqslant 1$ for all vertices $v \in \mathrm{V}(\vec{\Gamma})$.

Several properties of the alterexponent can be proved (see [9] and Corollary 4.2). For example, if $\vec{\Gamma}$ is connected and G-vertex-transitive, then $\exp(\vec{\Gamma})$ is the smallest positive integer e such that the setwise stabiliser $G_{\mathcal{A}_e(v)}$ is normal in G. The group $G_{\mathcal{A}_e(v)}$ is the group generated by all vertex-stabilisers in G, $G/G_{\mathcal{A}_e(v)}$ is a cyclic group, and the quotient digraph $\vec{\Gamma}/\mathcal{A}_e$ is isomorphic to a directed cycle $\vec{C}_n$ for some $n \geqslant 1$.

2.2. Concerning alternets and partial line digraphs

We now introduce two, in some sense inverse, operations on digraphs. They generalise the notions of partial line graph and anti-partial line graph operators, introduced in [6] and used in [5,7] to study tetravalent $\frac{1}{2}$-arc-transitive graphs. They are closely related to the notion of alternets, introduced in [18] as a generalisation of alternating cycles [8].

Let $\vec{\Gamma}$ be a digraph. We say that two arcs a and b of $\vec{\Gamma}$ are *related* if they have a common tail or a common head. Let R denote the transitive closure of this relation. The *alternet* of $\vec{\Gamma}$ (with respect to a) is the subdigraph of $\vec{\Gamma}$ induced by the R-equivalence class $R(a)$ of the arc a (i.e. the digraph with vertex-set consisting of all heads and tails of arcs in $R(a)$ and whose arc-set is $R(a)$).

If the alternet with respect to a contains a successor of a, then this alternet is called *degenerate*. It is not hard to see that, if $\vec{\Gamma}$ is connected, asymmetric, G-arc-transitive and has a degenerate alternet, then $\exp(\vec{\Gamma}) = 1$ and $\vec{\Gamma}$ has only one alternet (such digraphs were called alternating-connected in [15]). If, in addition, $G^{\vec{\Gamma}^+(v)}$ is regular, then the arc-stabiliser G_{uv} is trivial and hence is uninteresting from our point of view.

If the alternet of the arc (u, v) is non-degenerate, then it is a connected bipartite digraph where the first bipartition set is $\mathcal{A}_1(u)$ and consists only of sources while the second bipartition set is $\mathcal{A}_{-1}(v)$ and contains only sinks. An important case occurs when this alternet is in fact a complete bipartite digraph in which case we will simply say that the alternet is *complete bipartite*.

We say that $\vec{\Gamma}$ is *loosely attached* if $\vec{\Gamma}$ has no degenerate alternets and the intersection of the set of sinks of one alternet intersects the set of sources of another alternet in at most one vertex (in other words, $\mathcal{A}_1(v) \cap \mathcal{A}_{-1}(v) = v$ for all vertices v).

We define the *digraph of alternets* $\mathrm{Al}(\vec{\Gamma})$ of $\vec{\Gamma}$ as the graph the vertices of which are the alternets of $\vec{\Gamma}$ and with two alternets A and B forming an arc (A, B) of $\mathrm{Al}(\vec{\Gamma})$ whenever the intersection of the set of sinks of A with the set of sources of B is non-empty. We note that, if $\vec{\Gamma}$ has no sinks and no degenerate alternets, then $\mathrm{Al}(\vec{\Gamma})$ could be equivalently defined as the quotient digraph $\mathrm{Q}(\vec{\Gamma}) = \vec{\Gamma}/\mathcal{A}_1$ (see Lemma 3.1).

Finally, let the *partial line digraph* of $\vec{\Gamma}$, denoted by $\mathrm{Pl}(\vec{\Gamma})$, be the digraph the vertices of which are the arcs of $\vec{\Gamma}$ with two arcs $a_1 = (x, y)$ and $a_2 = (w, z)$ forming an arc (a_1, a_2) of $\mathrm{Pl}(\vec{\Gamma})$ whenever $y = w$ and (x, y, z) is a 2-arc of $\vec{\Gamma}$. It is not difficult to see that, under very mild assumptions (see Lemma 3.2), $\mathrm{Q}(\mathrm{Pl}(\vec{\Gamma})) \cong \vec{\Gamma}$, and, if $\vec{\Gamma}$ is loosely attached and the alternets of $\vec{\Gamma}$ are complete bipartite, then also $\mathrm{Pl}(\mathrm{Q}(\vec{\Gamma})) \cong \vec{\Gamma}$ (see Lemma 3.3).

The importance of the Pl and Q operators for the study of arc-transitive digraphs of prime out-valence is revealed in the following fact (see Corollary 5.2). If $\vec{\Gamma}$ is an asymmetric G-arc-transitive digraph of prime out-valence p, then $\vec{\Gamma} \cong \mathrm{Pl}^r(\vec{\Gamma}')$ for some $r \geqslant 0$, where either $\vec{\Gamma}' = \mathrm{Q}^r(\vec{\Gamma})$ is a G-arc-transitive digraph of out-valence p with alternets not complete bipartite, or $\vec{\Gamma}' \cong \vec{W}(n, p)$ for some $n \geqslant 3$.

2.3. Concerning certain combinatorial and group theoretical parameters

Throughout this section, we will assume that p is a prime and that $\vec{\Gamma}$ is a connected asymmetric G-arc-transitive digraph of valence p, such that $G_v^{\vec{\Gamma}^+(v)} \cong \mathbb{Z}_p$, for some $v \in \mathrm{V}(\vec{\Gamma})$.

Let s be the largest integer such that G acts transitively on the set of all s-arcs of $\vec{\Gamma}$. Let g be any element of G such that (v^g, v) is an arc of $\vec{\Gamma}$ and, for $i \in \mathbb{Z}$, let $v_i = v^{g^{-i}}$. Clearly, for every $i \geqslant 0$, $(v_0, v_1, \ldots, v_i)$ is an i-arc of $\vec{\Gamma}$. Moreover, the assumption that G_v acts regularly on $\vec{\Gamma}^+(v)$ implies that the s-arc-stabiliser $G_{(v_0,\ldots,v_s)}$ is trivial and $G_{(v_0,\ldots,v_{s-1})}$ is a cyclic group generated by some element x of order p. For $i \in \mathbb{Z}$, let $x_i = x^{g^i}$ and, for $i \geqslant 0$, let

$$H_i = \langle x_0, \ldots, x_{i-1} \rangle.$$

For $i \leqslant s$, one can follow the approach of Tutte [13] or Sims [12] and show that the group H_i has order p^i and coincides with the $(s-i)$-arc-stabiliser $G_{(v_0,\ldots,v_{s-i})}$.

In this paper, we also consider the group H_i for $i \geqslant s$. In fact, we show that, for $i \geqslant s$, the group H_i coincides with the setwise stabiliser $G_{\mathcal{A}_{i-s}(v_{s-i})}$ of the equivalence class $\mathcal{A}_{i-s}(v_{s-i})$. To summarize, we show that (see Lemma 5.4)

$$H_i = \begin{cases} G_{(v_0,\ldots,v_{s-i})} & \text{for } i \leqslant s, \\ G_{\mathcal{A}_{i-s}(v_{s-i})} & \text{for } i \geqslant s. \end{cases}$$

This result enables us to characterise the alterexponent of the digraph $\vec{\Gamma}$ in a purely group theoretical language. In particular, it follows from Lemma 4.2 and Lemma 5.4 that the following statements are equivalent.

(i) $e = \exp(\vec{\Gamma})$;
(ii) e is the smallest integer such that $H_{s+e} = H_{s+e+1}$;
(iii) e is the smallest integer such that H_{s+e} is normal in G;
(iv) e is the smallest integer such that $H_{s+e} = \langle G_u \colon u \in \mathrm{V}(\vec{\Gamma}) \rangle$.

The third important parameter of $\vec{\Gamma}$ which we would like to mention here is the smallest integer r such that either $\mathrm{Q}^r(\vec{\Gamma}) \cong \vec{W}(n,p)$ for some $n \geqslant 3$ or the alternets of $\mathrm{Q}^r(\vec{\Gamma})$ are not complete bipartite. By Corollary 5.2, such an integer r always exists. We will call this integer r *the height* of $\vec{\Gamma}$. On the other hand, let t be the largest integer such that $|H_t| = p^t$. Then either $t = s + r$, or $\vec{\Gamma} \cong \mathrm{Pl}^r(\vec{W}(n,p))$ for some $n \geqslant 3$, in which case $t = s + r + 1 = s + e$.

This reduction shows that the case when alternets are not complete bipartite is crucial. In this case, it can be shown that $s \leqslant 3e$ and $s \neq 3e - 1$. Moreover, if $p = 2$ and $e = 2$, then $s \leqslant 4$ and, if $e = 1$, then $s = 1$ (see Theorem 7.1).

3. Digraphs of alternets and partial line digraphs

In this section, most of the definitions from Sections 2.1 and 2.2 will be needed. In particular, recall that, for a digraph $\vec{\Gamma}$, we define $\mathrm{Q}(\vec{\Gamma}) = \vec{\Gamma}/\mathcal{A}_1$. Since $\mathcal{A}_1$ is an $\mathrm{Aut}(\vec{\Gamma})$-invariant partition of $\mathrm{V}(\vec{\Gamma})$, there is a natural action of $\mathrm{Aut}(\vec{\Gamma})$ on $\mathrm{V}(\mathrm{Q}(\vec{\Gamma}))$ which preserves the arcs of $\mathrm{Q}(\vec{\Gamma})$. This action is clearly faithful whenever $\vec{\Gamma}$ is loosely attached, which is the first claim of the following lemma.

Lemma 3.1. *Let $\vec{\Gamma}$ be a digraph. Then the following holds.*

(i) *If $\vec{\Gamma}$ is loosely attached and $G \leqslant \mathrm{Aut}(\vec{\Gamma})$, then also $G \leqslant \mathrm{Aut}(\mathrm{Q}(\vec{\Gamma}))$.*
(ii) *If $\vec{\Gamma}$ has no sinks and no degenerate alternets, then the mapping which sends the alternet of an arc (u,v) of $\vec{\Gamma}$ to $\mathcal{A}_1(u)$ is an isomorphism between the digraphs $\mathrm{Al}(\vec{\Gamma})$ and $\mathrm{Q}(\vec{\Gamma})$.*
(iii) *For all $n \geqslant 1$, we have that $\vec{\Gamma}/\mathcal{A}_n \cong \mathrm{Q}(\vec{\Gamma}/\mathcal{A}_{n-1})$, where an isomorphism is given by $\mathcal{A}_n(v) \mapsto \mathcal{A}_1(\mathcal{A}_{n-1}(v))$. In particular, $\vec{\Gamma}/\mathcal{A}_n \cong \mathrm{Q}^n(\vec{\Gamma})$.*
(iv) *$\exp(\mathrm{Q}^n(\vec{\Gamma})) = \exp(\vec{\Gamma}) - n$ for all $n \leqslant \exp(\vec{\Gamma})$. In particular, $\exp(\mathrm{Q}^n(\vec{\Gamma})) = 0$ for all $n \geqslant \exp(\vec{\Gamma})$.*
(v) *If $\vec{\Gamma}$ is connected and vertex-transitive, then $\vec{\Gamma}/\mathcal{A}_{\exp(\vec{\Gamma})}$ is a directed cycle of length $m \geqslant 1$.*

Proof. The proof of part (i) and (ii) are straightforward and left to the reader. Let us now prove (iii). The mapping $\theta : \mathrm{V}(\vec{\Gamma}/\mathcal{A}_n) \to \mathrm{V}(\mathrm{Q}(\vec{\Gamma}/\mathcal{A}_{n-1}))$ defined by $\theta(\mathcal{A}_n(v)) = \mathcal{A}_1(\mathcal{A}_{n-1}(v))$ is clearly surjective. To show that it is well defined and injective, we must show that, for any two vertices $u, v \in \mathrm{V}(\vec{\Gamma})$, we have $\mathcal{A}_n(u) = \mathcal{A}_n(v)$ if and only if $\mathcal{A}_1(\mathcal{A}_{n-1}(u)) = \mathcal{A}_1(\mathcal{A}_{n-1}(v))$. In other words, we must show that $v \in \mathcal{A}_n(u)$ if and only if $\mathcal{A}_{n-1}(v) \in \mathcal{A}_1(\mathcal{A}_{n-1}(u))$.

Let $v \in \mathcal{A}_n(u)$ and let $W = (w_0, \ldots, w_\ell)$ be a walk from $u = w_0$ to $v = w_\ell$ of sum 0 and tolerance $[0,n]$. Let $0 = x_0 < x_1 < \cdots < x_k = \ell$ be the indices for which $w_{x_i} \in \mathcal{A}_n(u)$. We will show that $\mathcal{A}_{n-1}(w_{x_i}) \in \mathcal{A}_1(\mathcal{A}_{n-1}(u))$ by induction on $i \in \{0, \ldots, k\}$. This is clearly true for $i = 0$. Suppose it is true for integers smaller than i. Note that, since $w_{x_i} \in \mathcal{A}_n(u)$, then $w_{x_i - 1} \in \mathcal{A}_{n-1}(w_{x_{i-1}+1})$, hence $\mathcal{A}_{n-1}(w_{x_i}) \in \mathcal{A}_1(\mathcal{A}_{n-1}(w_{x_{i-1}})) = \mathcal{A}_1(\mathcal{A}_{n-1}(u))$, by the induction hypothesis. We have thus shown that $v \in \mathcal{A}_n(u)$ implies $\mathcal{A}_{n-1}(v) \in \mathcal{A}_1(\mathcal{A}_{n-1}(u))$.

Conversely, let v be a vertex of $\vec{\Gamma}$ such that $\mathcal{A}_{n-1}(v) \in \mathcal{A}_1(\mathcal{A}_{n-1}(u))$ and let $W = (\mathcal{A}_{n-1}(w_0), \ldots, \mathcal{A}_{n-1}(w_\ell))$ be a walk from $\mathcal{A}_{n-1}(w_0) = \mathcal{A}_{n-1}(u)$ to $\mathcal{A}_{n-1}(w_\ell) = \mathcal{A}_{n-1}(v)$ of sum 0 and tolerance $[0,1]$ in $\vec{\Gamma}/\mathcal{A}_{n-1}$ (note that ℓ must be even). Clearly, for every $w \in \mathcal{A}_{n-1}(w_1)$, there is a walk in $\vec{\Gamma}$ from u to w of sum 1 and tolerance $[0,n]$. This implies that for every $w \in \mathcal{A}_{n-1}(w_2)$, there is a walk in $\vec{\Gamma}$ from u to w of sum 0 and tolerance $[0,n]$. One can thus show by induction that there is a walk from u to w of sum 0 and tolerance $[0,n]$ for every $w \in \mathcal{A}_{n-1}(w_j)$, j even. In particular, $v \in \mathcal{A}_n(u)$. This proves that θ is a well defined bijection.

Next, we need to show that θ maps the arcs of $\vec{\Gamma}/\mathcal{A}_n$ bijectively onto arcs of $\mathrm{Q}(\vec{\Gamma}/\mathcal{A}_{n-1})$. By the definition of quotient graphs it follows that $(\mathcal{A}_n(u), \mathcal{A}_n(v))$ is an arc of $\vec{\Gamma}/\mathcal{A}_n$ if and only if there exists an arc (u', v') in $\vec{\Gamma}$ such that $u' \in \mathcal{A}_n(u)$ and $v' \in \mathcal{A}_n(v)$. By what we have proved above, the latter is equivalent to $\mathcal{A}_{n-1}(u') \in \mathcal{A}_1(\mathcal{A}_{n-1}(u))$ and $\mathcal{A}_{n-1}(v') \in \mathcal{A}_1(\mathcal{A}_{n-1}(v))$. Hence, $\mathcal{A}_n(u) \sim \mathcal{A}_n(v)$ if and only if there exists u'', v'' such that $\mathcal{A}_{n-1}(u'') \sim \mathcal{A}_{n-1}(v'')$, $\mathcal{A}_{n-1}(u'') \in \mathcal{A}_1(\mathcal{A}_{n-1}(u))$ and $\mathcal{A}_{n-1}(v'') \in \mathcal{A}_1(\mathcal{A}_{n-1}(v))$, if and only if $\mathcal{A}_1(\mathcal{A}_{n-1}(u)) \sim \mathcal{A}_1(\mathcal{A}_{n-1}(v))$, if and only if $\theta(\mathcal{A}_n(u)) \sim \theta(\mathcal{A}_n(v))$. This shows that θ is an isomorphism. The last claim of part (iii) now follows by induction on n.

To show part (iv), it suffices to show that $e = \exp(\vec{\Gamma}) \geqslant 1$ implies $e' = \exp(\mathrm{Q}(\vec{\Gamma})) = e - 1$. By definition and above, e is the smallest integer such that $\mathrm{Q}^e(\vec{\Gamma}) = \mathrm{Q}^n(\vec{\Gamma})$ for all $n \geqslant e$, while e' is the smallest integer such that $\mathrm{Q}^{e'}(\mathrm{Q}(\vec{\Gamma})) = \mathrm{Q}^n(\mathrm{Q}(\vec{\Gamma}))$ for all $n \geqslant e'$. Hence $e' = e - 1$.

It follows from (iii) and (iv) that $\exp(\vec{\Gamma}/\mathcal{A}_e) = 0$. If $\vec{\Gamma}$ is connected and vertex-transitive, then so is $\vec{\Gamma}/\mathcal{A}_e$ and hence $\vec{\Gamma}/\mathcal{A}_e$ must be a directed cycle. □

Recall that the partial line digraph $\mathrm{Pl}(\vec{\Gamma})$ of $\vec{\Gamma}$ is the digraph the vertices of which are the arcs of $\vec{\Gamma}$ with two arcs $a_1 = (x, y)$ and $a_2 = (w, z)$ forming an arc (a_1, a_2) of $\mathrm{Pl}(\vec{\Gamma})$ whenever $y = w$ and (x, y, z) is a 2-arc of $\vec{\Gamma}$. More generally, for $n \geqslant 0$, we denote by $\mathrm{Pl}_n(\vec{\Gamma})$ the digraph the vertices of which are the n-arcs of $\vec{\Gamma}$ with two n-arcs a_1 and a_2 forming an arc (a_1, a_2) of $\mathrm{Pl}(\vec{\Gamma})$ whenever a_2 is a successor of a_1 in $\vec{\Gamma}$. Then $\mathrm{Pl}_0(\vec{\Gamma}) = \vec{\Gamma}$ and $\mathrm{Pl}_1(\vec{\Gamma}) = \mathrm{Pl}(\vec{\Gamma})$.

Lemma 3.2. *Let $\vec{\Gamma}$ be an asymmetric digraph without sinks and sources. Then the following holds.*

(i) *For all $n \geqslant 1$, we have that $\mathrm{Pl}_n(\vec{\Gamma}) \cong \mathrm{Pl}(\mathrm{Pl}_{n-1}(\vec{\Gamma}))$, where an isomorphism is given by*

$$(v_0, \dots, v_n) \mapsto \big((v_0, \dots, v_{n-1}), (v_1, \dots, v_n)\big) \quad \text{for any } n\text{-arc } (v_0, \dots, v_n) \text{ of } \vec{\Gamma}.$$

In particular, $\mathrm{Pl}_n(\vec{\Gamma}) \cong \mathrm{Pl}^n(\vec{\Gamma})$ for all $n \geqslant 0$.

(ii) $\mathrm{Q}(\mathrm{Pl}(\vec{\Gamma})) \cong \vec{\Gamma}$*, where an isomorphism is given by $\mathcal{A}_1(u, v) \mapsto v$ for any $(u, v) \in \mathrm{A}(\vec{\Gamma})$.*

(iii) $\mathrm{Aut}(\vec{\Gamma}) \leqslant \mathrm{Aut}(\mathrm{Pl}(\vec{\Gamma}))$*. Moreover, if $\vec{\Gamma}$ is (G, s)-arc-transitive for some $G \leqslant \mathrm{Aut}(\vec{\Gamma})$ and $s \geqslant 1$, then $\mathrm{Pl}(\vec{\Gamma})$ is $(G, s - 1)$-arc-transitive.*

Proof. The proofs of parts (i) and (iii) are straightforward and left to the reader. Let us now prove part (ii). To show that the mapping from the statement of (ii) is well defined and bijective, observe that, in $\mathrm{Pl}(\vec{\Gamma})$, the equivalence class $\mathcal{A}_1(u, v)$ of a vertex (u, v) of $\mathrm{Pl}(\vec{\Gamma})$ consists of all the arcs of $\vec{\Gamma}$ ending in v. Note that the arcs of $\mathrm{Q}(\mathrm{Pl}(\vec{\Gamma}))$ are precisely the pairs of the form $(\mathcal{A}_1(w, u), \mathcal{A}_1(u, v))$ where (w, u, v) is an arbitrary 2-arc of $\vec{\Gamma}$. Hence arcs of $\mathrm{Q}(\mathrm{Pl}(\vec{\Gamma}))$ map onto arcs of $\vec{\Gamma}$, completing the proof. □

We have seen that Q is a left-inverse of Pl. We now show that, under certain circumstances, it is also a right-inverse. Note that vertices of $\mathrm{Pl}(\mathrm{Q}(\vec{\Gamma}))$ are arcs of $\mathrm{Q}(\vec{\Gamma})$, hence of the form $(\mathcal{A}_1(u), \mathcal{A}_1(v))$ for some $(u, v) \in \mathrm{A}(\vec{\Gamma})$.

Lemma 3.3. *Let $\vec{\Gamma}$ be an asymmetric digraph such that its alternets are complete bipartite. Then $\mathrm{Q}(\vec{\Gamma})$ is also asymmetric. Moreover, if $\vec{\Gamma}$ has no sources and is loosely attached, then $\mathrm{Pl}(\mathrm{Q}(\vec{\Gamma})) \cong \vec{\Gamma}$, where an isomorphism is given by $(\mathcal{A}_1(u), \mathcal{A}_1(v)) \mapsto v$ for $(u, v) \in \mathrm{A}(\vec{\Gamma})$.*

Proof. We first show that $\mathrm{Q}(\vec{\Gamma})$ is also asymmetric. Suppose, by contradiction, that both $(\mathcal{A}_1(u), \mathcal{A}_1(v))$ and $(\mathcal{A}_1(v), \mathcal{A}_1(u))$ are arcs of $\mathrm{Q}(\vec{\Gamma})$. Then there exists (w, x) and (y, z) arcs of $\vec{\Gamma}$ such that $w, z \in \mathcal{A}_1(u)$ and $x, y \in \mathcal{A}_1(v)$. Since alternets are complete bipartite, this implies that both (x, z) and (z, x) are arcs of $\vec{\Gamma}$, which is a contradiction.

We now assume that $\vec{\Gamma}$ has no sources and is loosely attached and show that the mapping $\theta : \mathrm{V}(\mathrm{Pl}(\mathrm{Q}(\vec{\Gamma}))) \to \mathrm{V}(\vec{\Gamma})$ given in the statement is a well defined bijection. Let (u, v) and (x, y) be arcs of $\vec{\Gamma}$ such that $\mathcal{A}_1(u) = \mathcal{A}_1(x)$ and $\mathcal{A}_1(v) = \mathcal{A}_1(y)$. Together with $\mathcal{A}_1(u) = \mathcal{A}_1(x)$, the fact that (u, v) and (x, y) are arcs of $\vec{\Gamma}$ imply that $\mathcal{A}_{-1}(v) = \mathcal{A}_{-1}(y)$. Since $\vec{\Gamma}$ is loosely attached, it follows that $y = v$. This shows that θ is well defined. Since $\vec{\Gamma}$ has no sources, θ is surjective. If $u, w \in \vec{\Gamma}^-(v)$, then $\mathcal{A}_1(u) = \mathcal{A}_1(w)$ and hence θ is injective.

It remains to show that θ is an isomorphism. Let (u, v) and (w, y) be two arcs of $\vec{\Gamma}$. Since $\mathrm{Q}(\vec{\Gamma})$ is asymmetric, $((\mathcal{A}_1(u), \mathcal{A}_1(v)), (\mathcal{A}_1(w), \mathcal{A}_1(y)))$ is an arc of $\mathrm{Pl}(\mathrm{Q}(\vec{\Gamma}))$ if and only if $\mathcal{A}_1(v) = \mathcal{A}_1(w)$. Since the alternets of $\vec{\Gamma}$ are complete bipartite, the latter is equivalent to (v, y) being an arc of $\vec{\Gamma}$. □

4. Alterexponent

Recall the definition of the tolerance of a walk, alterexponent and the relations $\mathcal{A}_I$ from Section 2.1.

The following lemma and its corollary may be viewed as a generalisation of the folklore result stating that, in a connected, bipartite G-arc-transitive graph, the group generated by the vertex stabilisers of two adjacent vertices coincides with the index-2 subgroup of G preserving each bipartition set.

Lemma 4.1. *Let $\vec{\Gamma}$ be an asymmetric G-arc-transitive digraph and let $(v_0, \dots, v_s)$ be an s-arc of $\vec{\Gamma}$. For $j \in \{1, \dots, s\}$, let R_{j-1} be a subgroup of $G_{v_{j-1}}$ acting transitively on $\vec{\Gamma}^+(v_{j-1})$ and let L_j be a subgroup of G_{v_j} acting transitively on $\vec{\Gamma}^-(v_j)$. For $m \in \{0, \dots, j\}$, let $T_m = \langle R_0, L_1, \dots, R_{m-1}, L_m \rangle$. Then $v_t^{T_i} = \mathcal{A}_{[-t,i-t]}(v_t)$ for any $i \in \{1, \dots, s\}$ and $t \in \{0, \dots, i\}$. In particular, $v_0^{T_i} = \mathcal{A}_i(v_0)$.*

Proof. Let $i \in \{1, \dots, s\}$. We will first show that $v_t^{T_i} \subseteq \mathcal{A}_{[-t,i-t]}(v_t)$ for any $t \in \{0, \dots, i\}$. It suffices to show that the equivalence class $\mathcal{A}_{[-t,i-t]}(v_t)$ is preserved by T_i.

Let $g \in G_{v_j}$ for some $j \in \{0, \dots, i\}$, let $w \in \mathcal{A}_{[-t,i-t]}(v_t)$, and consider the walk $(v_t, v_{t\pm1}, \dots, v_{j\mp1}, v_j, v_{j\mp1}^g, \dots, v_t^g)$. This is clearly a walk of sum 0 and tolerance contained in the interval $[-t, i-t]$, showing that $v_t \,\mathcal{A}_{[-t,i-t]}\, v_t^g$. Since the relation $\mathcal{A}_{[-t,i-t]}$ is preserved by every automorphism of $\vec{\Gamma}$, we also see that $v_t^g \mathcal{A}_{[-t,i-t]} w^g$, and thus $v_t \,\mathcal{A}_{[-t,i-t]}\, w^g$. This shows that $\mathcal{A}_{[-t,i-t]}(v_t)$ is preserved setwise by every G_{v_j}, $j \in \{0, \dots, t\}$. But since T_i is generated by subgroups of these vertex-stabilisers, this implies that T_i preserves $\mathcal{A}_{[-t,i-t]}(v_t)$, as required.

Let us now show that $\mathcal{A}_{[-t,i-t]}(v_t) \subseteq v_t^{T_i}$ for any $t \in \{0, \dots, i\}$. Suppose this is not the case. Note that $\mathcal{A}_{[-t,i-t]}(v_t)$ is precisely the set of those vertices w for which there exists a walk from v_0 to w of tolerance $[0, i]$ and sum t (see [9, Lemma 3.6]). Hence, by our assumption, the set of walks W of tolerance $[0, i]$, starting in v_0 and ending in a vertex not in $v_{s(W)}^{T_i}$ is non-empty. Choose a walk $W = (w_0, w_1, \dots, w_n)$ of shortest length in this set and let $t = s(W)$. If $n = 0$, then $w_n = v_0$, $t = 0$, and hence $w_n \in v_t^{T_i}$, contradicting our choice of W. If $n \geqslant 1$, then consider the walk $W' = (w_0, \dots, w_{n-1})$, the sum of which is clearly $t \pm 1$ for some choice of the sign. By minimality of the length n, it follows that $w_{n-1} \in v_{t\pm1}^{T_i}$. Note that both $t = s(W)$ and $t \pm 1 = s(W')$ are contained in $[0, i]$ and hence $\vec{\Gamma}^{\mp}(v_{t\pm1}) \subseteq v_t^{T_i}$. It follows that $\vec{\Gamma}^{\mp}(w_{n-1}) \subseteq v_t^{T_i}$, and thus $w_n \in v_t^{T_i}$, contradicting our choice of W. □

Corollary 4.2. *Let $\vec{\Gamma}$ be a strongly-connected G-arc-transitive digraph. For any $i \in \mathbb{N}$, let $(v_0, \dots, v_i)$ be an i-arc of $\vec{\Gamma}$. Then $T_i = \langle G_{v_0}, \dots, G_{v_i} \rangle$ is the setwise stabiliser of $\mathcal{A}_i(v_0)$. In particular, $e = \exp(\vec{\Gamma})$ is the smallest integer such that T_e is normal in G. Moreover, T_e coincides with the group generated by all vertex-stabilisers G_v, $v \in \mathrm{V}(\vec{\Gamma})$.*

Proof. Let $e = \exp(\vec{\Gamma})$. Note that, since $\vec{\Gamma}$ is strongly-connected and G-arc-transitive, it is also G-vertex-transitive. Using Lemma 4.1 with $R_i = L_i = G_{v_i}$ yields that $v_0^{T_i} = \mathcal{A}_i(v_0)$. Since T_i contains a vertex-stabiliser, it follows that T_i is precisely the setwise stabiliser of $\mathcal{A}_i(v_0)$.

It follows that $T_i \leqslant T_e$ for any $i \in \mathbb{N}$. Since $\vec{\Gamma}$ is strongly-connected, the path $(v_0, \dots, v_i)$ can be extended to include every vertex of $\vec{\Gamma}$ and hence T_e is precisely the subgroup of G generated by all vertex-stabilisers G_v, $v \in \mathrm{V}(\vec{\Gamma})$. In particular, T_e is normal in G.

Suppose now that $0 \leqslant i \leqslant e$ is any integer such that T_i is normal in G (for example $i = e$). Since T_i contains G_{v_0}, the normality of T_i and vertex-transitivity of G imply that $T_i = T_e$, and so the normality of T_i and Lemma 4.1 imply that $\mathcal{A}_i(v) = \mathcal{A}_e(v)$ for all $v \in \mathrm{V}(\vec{\Gamma})$. But then $i = e$ by the definition of the alterexponent. □

We remark that the group generated by all the vertex-stabilisers was studied in some detail also in [11].

5. Prime out-valence

From now on, we will restrict ourselves to asymmetric arc-transitive digraphs of prime out-valence. Continuing with the theme of Lemma 3.3, we show that the case when alternets are complete bipartite is particularly well behaved.

Lemma 5.1. *Let $\vec{\Gamma}$ be a connected asymmetric G-arc-transitive digraph of prime out-valence p such that the alternets of $\vec{\Gamma}$ are complete bipartite. Then either*

(i) *$\vec{\Gamma} = \vec{W}(n, p)$ is a wreath digraph with $n \geqslant 3$, or*
(ii) *$\mathrm{Q}(\vec{\Gamma})$ is also a connected asymmetric digraph of prime out-valence p on which G acts faithfully and arc-transitively and $\mathrm{Pl}(\mathrm{Q}(\vec{\Gamma})) \cong \vec{\Gamma}$.*

Proof. Clearly, $\mathrm{Q}(\vec{\Gamma})$ is a connected digraph. By Lemma 3.3, it is asymmetric. Since $\vec{\Gamma}$ is G-arc-transitive, alternets of $\vec{\Gamma}$ are non-degenerate and must intersect in either 1 or p vertices. In the latter case, $\vec{\Gamma}$ is a wreath digraph $\vec{W}(n, p)$ with $n \geqslant 3$. In the former case, $\vec{\Gamma}$ is loosely connected and it follows that $\mathrm{Q}(\vec{\Gamma})$ has out-valence p. By Lemma 3.1, G acts faithfully and arc-transitively. Finally, $\mathrm{Pl}(\mathrm{Q}(\vec{\Gamma})) \cong \vec{\Gamma}$ follows from Lemma 3.3. □

In other words, when the alternets of $\vec{\Gamma}$ are complete bipartite, we either have a wreath digraph or we can get a reduced graph $\mathrm{Q}(\vec{\Gamma})$, from which $\vec{\Gamma}$ can be reconstructed. Note that, under the hypothesis of Lemma 5.1, we have that $|\mathrm{V}(\mathrm{Q}(\vec{\Gamma}))| = |\mathrm{V}(\vec{\Gamma})|/|A_1(v)| \leqslant |\mathrm{V}(\vec{\Gamma})|/p$. Hence, repeating this procedure must terminate. This is the content of our next result.

Corollary 5.2. *Let $\vec{\Gamma}$ be a connected asymmetric G-arc-transitive digraph of prime out-valence p. Then there exists some $r \geqslant 0$ such that $\mathrm{Q}^r(\vec{\Gamma})$ is also a connected asymmetric digraph of prime out-valence p on which G acts faithfully and arc-transitively and such that either the alternets of $\mathrm{Q}^r(\vec{\Gamma})$ are not complete bipartite, or $\mathrm{Q}^r(\vec{\Gamma}) = \vec{W}(n, p)$ is a wreath digraph with $n \geqslant 3$. Moreover, $\mathrm{Pl}^r(\mathrm{Q}^r(\vec{\Gamma})) \cong \vec{\Gamma}$.*

If $\mathrm{Q}^r(\vec{\Gamma})$ is a wreath digraph $\vec{W}(n, p)$ with $n \geqslant 3$, then $\vec{\Gamma}$ is isomorphic to $\mathrm{Pl}^r(\vec{W}(n, p))$. Hence, it is natural to now turn our attention to the other case, that is when alternets are not complete bipartite.

Lemma 5.3. *Let $\vec{\Gamma}$ be a connected asymmetric G-arc-transitive digraph of prime out-valence p. Let v be a vertex of $\vec{\Gamma}$, let $e = \exp(\vec{\Gamma})$ and let $(v_0, \dots, v_e)$ be an e-arc of $\vec{\Gamma}$. If alternets of $\vec{\Gamma}$ are not complete bipartite, then the pointwise stabiliser of $\mathcal{A}_e(v)$ is trivial and G_v contains no nonidentity normal subgroup of $\langle G_{v_0}, \dots, G_{v_e}\rangle$.*

Proof. We show that the pointwise stabiliser H of $\mathcal{A}_e(v)$ is trivial. In view of Lemma 3.1(v), we know that $\vec{\Gamma}/\mathcal{A}_e$ is isomorphic to a directed cycle $\vec{C}_m$ on m vertices for some $m \geqslant 1$. Let $w \in \vec{\Gamma}^+(v)$ and let $x \in \mathcal{A}_e(w)$. Clearly H fixes all in-neighbours of x.

Let X be the set of vertices y such that $\vec{\Gamma}^-(x) = \vec{\Gamma}^-(y)$. Clearly, $|X| \leqslant p$. Moreover, since $\vec{\Gamma}$ is arc-transitive, $|X|$ must divide p. Since alternets are not complete bipartite, we must have $|X| = 1$. It follows that H must fix x. Hence H fixes $\mathcal{A}_e(w)$ pointwise. Repeating the argument yields that H fixes every vertex pointwise, hence H is trivial.

We now show that G_v contains no nonidentity normal subgroup of $T_e = \langle G_{v_0}, \dots, G_{v_e}\rangle$. By Corollary 4.2, T_e is the setwise stabiliser of $\mathcal{A}_e(v)$. Let $K \leqslant G_v$ be a normal subgroup of T_e. Since K is normal in T_e, it fixes every vertex in the orbit of v under T_e, namely $\mathcal{A}_e(v)$. By the previous claim, this shows that $K = 1$. □

In the next lemma, we prove some basic results about the stabiliser of an i-arc in asymmetric arc-transitive digraphs of prime out-valence p. For some of them, we require that the vertex-stabiliser acts regularly (in particular, cyclically) on the p out-neighbours of the vertex.

Lemma 5.4. *Let $\vec{\Gamma}$ be a connected asymmetric G-arc-transitive digraph of prime out-valence p. Let $e = \exp(\vec{\Gamma})$, let s be the largest integer such that $\vec{\Gamma}$ is (G, s)-arc-transitive and let $\alpha = (v = v_0, \dots, v_s)$ be an s-arc of $\vec{\Gamma}$. Let $g \in G$ such that $\alpha^{g^{-1}} = (v_1, \dots, v_s, (v_s)^{g^{-1}})$ is a successor of α. For $0 \leqslant i \leqslant s$, let $G_i = G_{(v_0,\dots,v_{s-i})}$ be the pointwise stabiliser of $(v_0, \dots, v_{s-i})$. Then there exists an element $x \in G_1$ of order p. Let $H_0 = 1$ and let $H_i = \langle x, x^g, \dots, x^{g^{i-1}}\rangle$ for $i \geqslant 1$.*

(i) *$|G_i| = p^i|G_0|$, for $0 \leqslant i \leqslant s$. In particular, $|G_v| = p^s|G_0|$.*
(ii) *If q is a prime dividing $|G_0|$, then $q < p$.*
(iii) *$G_i = \langle H_i, G_0\rangle$ for $0 \leqslant i \leqslant s$.*

If we also assume that $G_v^{\vec{\Gamma}^+(v)}$ is a cyclic group of order p, then

(iv) *$G_{s-1} = G_{(v_0,v_1)}$ fixes the alternet of (v_0, v_1) pointwise.*
(v) *$G_0 = 1$. In particular, $|G_v| = p^s$ and $G_i = H_i$ for $0 \leqslant i \leqslant s$.*
(vi) *For $n \geqslant 0$, $H_{s+n} = \langle G_v, \dots, G_{v^{g^n}}\rangle$. In particular, H_{s+n} is the setwise stabiliser of $\mathcal{A}_n(v_{-n})$.*
(vii) *$|\mathcal{A}_i(v)| \geqslant p^i$ for $0 \leqslant i \leqslant e$.*

Proof. Let us first prove (i). For $1 \leqslant i \leqslant s$, G_i acts transitively on the p successors of $(v_0, \dots, v_{s-i})$. In particular, $|G_i| = p|G_{i-1}|$. Induction completes the proof. Note that it follows that there exists an element $x \in G_1$ of order p.

Let us now turn to the proof of (ii). Suppose that there exist y be an element of G_0 of prime order $q \geqslant p$. If y acts transitively on $\vec{\Gamma}^+(v_s)$, then G is $(s+1)$-arc-transitive, contradicting the assumption. Since $q \geqslant p$, it follows that y fixes $\vec{\Gamma}^+(v_s)$ pointwise. Repeating the argument and using connectivity shows that y must fix all vertices.

We show (iii) by induction on i. This is clear for $i = 0$. Note that, by (ii), x fixes v_{s-1} but not v_s. Hence, for $0 \leqslant i \leqslant s$, $x^{g^{i-1}}$ fixes v_{s-i} but not v_{s-i+1}. In particular, for $0 \leqslant i \leqslant s$, $x^{g^{i-1}}$ is in G_i but not in G_{i-1}. By the induction hypothesis, we have that $\langle H_i, G_0\rangle = \langle H_{i-1}, x^{g^{i-1}}, G_0\rangle$ is contained in G_i but is different from G_{i-1}. Since the latter has index p in the former, this shows that $G_i = \langle H_i, G_0\rangle$.

To prove (iv), observe that, in this case, G_{uv} fixes any arc a in the alternet of uv.

Let us now prove (v). Since $G_v^{\vec{\Gamma}^+(v)}$ is a regular permutation group, and since $\vec{\Gamma}$ is a strongly connected graph, it may be easily deduced that G acts regularly on the set of s-arcs of $\vec{\Gamma}$. In particular, $G_0 = 1$.

Note that (vi) is clear for $n = 0$. By induction, we have that $H_{s+n+1} = \langle H_{s+n}, H_{s+n}^g\rangle = \langle G_v, \dots, G_{v^{g^n}}, G_{v^{g^{n+1}}}\rangle$. The second statement follows from Corollary 4.2.

The proof of (vii) is by induction on i. The claim is vacuous for $i = 0$. Suppose that $0 < i \leqslant e$ and $|\mathcal{A}_{i-1}(v)| \geqslant p^{i-1}$. Now, let $(v_0, \dots, v_i)$ be an i-arc starting at $v = v_0$. By (v), G_{v_i} is a p-group. By Corollary 4.2, it fixes $\mathcal{A}_i(v)$ setwise but, since $i \leqslant e$, it does not fix $\mathcal{A}_{i-1}(v)$ setwise. Hence, the orbit of $\mathcal{A}_{i-1}(v)$ under G_{v_i} has size at least p. It follows that $|\mathcal{A}_i(v)| \geqslant p|\mathcal{A}_{i-1}(v)|$. The induction hypothesis completes the induction step. □

6. On a lemma of Glauberman

As announced in the introduction, one of the side results (as well as a step toward the proof of Theorem 1.1) is a modification of the following lemma due to Glauberman.

Lemma 6.1. *(See [3, Lemma 1].) Let x and g be elements of a group G. For $i \in \mathbb{Z}$, let $x_i = x^{g^i} = g^{-i}xg^i$ and, for $i \geqslant 1$, let $H_i = \langle x_1, \dots, x_i\rangle$. Let $H_0 = 1$. Suppose that x has prime order p and that there exist positive integers n and t such that*

(i) *$\langle H_t, g\rangle = G$,*
(ii) *$|H_i : H_{i-1}| = p$ for $1 \leqslant i \leqslant t$,*

(iii) *H_t contains no nonidentity normal subgroup of G, and*
(iv) *H_t contains no nonidentity subgroup of $Z(H_{t+n})$.*

Then $t \leqslant 3n$ and $t \neq 3n-1$. Moreover, if $n=2$, $p=2$, and $t=6$, then H_t contains a nonidentity normal subgroup of H_8.

We will now show that, if condition (iv) is replaced by (iv′): $|H_{t+1} : H_t| \neq p$ and (iv″): n is the smallest integer such that $H_{t+n} = H_{t+n+1}$, then a conclusion even stronger than condition (iv) holds. More precisely, we prove the following.

Lemma 6.2. *Let x and g be elements of a group G. For $i \in \mathbb{Z}$, let $x_i = x^{g^i} = g^{-i}xg^i$ and, for $i \geqslant 1$, let $H_i = \langle x_1, \ldots, x_i\rangle$. Let $H_0 = 1$. Suppose that x has prime order p, let t be the largest integer such that $|H_t| = p^t$ and let n be the smallest integer such that $H_{t+n} = H_{t+n+1}$. Suppose that H_t contains no nonidentity normal subgroup of $\langle H_t, g\rangle$. Then H_t contains no nonidentity normal subgroup of H_{t+n}. Moreover, $|H_{t+1} : H_t| > p$.*

Proof. We may assume without loss of generality that $G = \langle H_t, g\rangle = \langle x, g\rangle$. Since the order of $H_1 = \langle x\rangle$ is p, it follows that $t \geqslant 1$. Further, since by hypothesis H_t is not normal in G, we see that $n \geqslant 1$.

Clearly, $H_{t-1} \leqslant H_t \cap (H_t)^{g^{-1}} \leqslant H_t$. Since H_t is not normal in $\langle H_t, g\rangle$ and H_{t-1} is maximal in H_t, the equality $H_{t-1} = H_t \cap (H_t)^{g^{-1}}$ holds.

Now consider the coset digraph $\vec{\Gamma} = \vec{\Gamma}(G, H_t, g^{-1})$, that is, the digraph $\vec{\Gamma}$ with $\mathrm{V}(\vec{\Gamma}) = G/H_t = \{H_t y\colon\ y \in G\}$ and $\mathrm{A}(\vec{\Gamma}) = \{(H_t y, H_t g^{-1} h y)\colon\ h \in H_t,\ y \in G\}$. The group G then acts by right multiplication on $\vec{\Gamma}$ vertex-transitively and arc-transitively, and since H_t contains no nonidentity normal subgroup of G, also faithfully. Since G is generated by H_t and g, $\vec{\Gamma}$ is connected.

Let u denote the vertex $H_t \in G/H_t$, and let $v = H_t g^{-1} \in G/H_t$. Then $G_u = H_t$, $G_v = H_t^{g^{-1}}$, and $G_{uv} = H_t \cap (H_t)^{g^{-1}} = H_{t-1}$. In particular, $|G_u : G_{uv}| = p$, implying that $\vec{\Gamma}$ has out-valence p and $G_u^{\vec{\Gamma}^+(u)}$ is a cyclic group of order p.

If $\vec{\Gamma}$ is a graph, then the fact that $G_u^{\vec{\Gamma}^+(u)}$ acts regularly on $\vec{\Gamma}^+(u) = \vec{\Gamma}^-(u)$, together with the connectivity of $\vec{\Gamma}$, implies that $G_{uv} = 1$. But then $t = 1$, and the claim of the lemma follows immediately. We may thus assume that $\vec{\Gamma}$ is asymmetric.

Let us now determine the alterexponent $e = \exp(\vec{\Gamma})$. For $i \geqslant 0$ let $v_i = u^{g^{-i}}$. Then $v_0 = u$, $v_1 = v$, and $(v_0, v_1, \ldots, v_s)$ is an s-arc of $\vec{\Gamma}$ for any $s \geqslant 0$. By Corollary 4.2, the equivalence class $\mathcal{A}_s(v_0)$ coincides with the orbit $v_0^{T_s}$ under the group $T_s = \langle G_{v_0}, G_{v_1}, \ldots, G_{v_s}\rangle$, and thus $|\mathcal{A}_s(v_0)| = |T_s : G_{v_0}|$. Therefore e is the minimal integer s such that $T_s = T_{s+1}$. On the other hand,

$$T_s = \langle G_{v_0}, G_{v_1}, \ldots, G_{v_s}\rangle = \bigl\langle H_t, H_t^{g^{-1}}, \ldots, H_t^{g^{-s}}\bigr\rangle = \langle x_{1-s}, x_{2-s}, \ldots, x_t\rangle = H_{t+s}^{g^{-s}},$$

and therefore $e = n$. Since $|\mathcal{A}_1(v_0)| = |T_1 : G_{v_0}| = |H_{t+1} : H_t|$, which, by definition of t, is not p, it follows that the alternets of $\vec{\Gamma}$ are not complete bipartite. Furthermore, since $|\mathcal{A}_1(v_0)| \geqslant p$, this implies that $|H_{t+1} : H_t| > p$.

To conclude the proof, we may now use Lemma 5.3, which states that the vertex-stabiliser $G_v = H_t$ contains no nontrivial subgroups which are normal in $\langle G_{v_0}, \ldots, G_{v_e}\rangle = H_{t+n}^{g^{-s}} = H_{t+n}$. $\square$

We now have all the tools to prove our main results. Theorem 1.3 follows directly from the previous two lemmas. It remains only to prove Theorem 1.1.

7. Proof of Theorem 1.1

We first need the following result.

Theorem 7.1. *Let $\vec{\Gamma}$ be a connected asymmetric G-arc-transitive digraph of prime out-valence p and let v be a vertex of $\vec{\Gamma}$. If $G_v^{\vec{\Gamma}^+(v)}$ is a cyclic group of order p and, in addition, alternets of $\vec{\Gamma}$ are not complete bipartite,* *then* $|G_v| = p^s$ *for some positive integer* s *with* $s \leqslant 3\exp(\vec{\Gamma}) = 3e$ *and* $s \neq 3e-1$. *Moreover, if* $p = 2$ *and* $e = 2$, *then* $s \leqslant 4$ *and, if* $e = 1$, *then* $s = 1$.

Proof. Let s be the largest integer such that $\vec{\Gamma}$ is s-arc-transitive and let $\alpha = (v = v_0, \ldots, v_s)$ be an s-arc of $\vec{\Gamma}$. Let $g \in G$ such that $\alpha^{g^{-1}} = (v_1, \ldots, v_s, (v_s)^{g^{-1}})$ is a successor of α. By Lemma 5.4, the pointwise stabiliser of $(v_0, \ldots, v_{s-1})$ has order p. Let x be one of its generator. Let $H_0 = 1$ and let $H_i = \langle x, x^g, \ldots, x^{g^{i-1}} \rangle$ for $i \geqslant 1$. Again, by Lemma 5.4, $G_v = H_s$ has order p^s. But, by Corollary 4.2, $H_{s+1} = \langle H_s, H_s^g \rangle = \langle G_v, G_{v^g} \rangle$ is the setwise stabiliser of $\mathcal{A}_1(v)$. In particular, $|H_{s+1} : H_s| = |\mathcal{A}_1(v)|$ and hence $|H_{s+1} : H_s| > p$ since alternets are not complete bipartite. It follows that s is the largest integer such that $|H_s| = p^s$.

By Corollary 4.2, $H_{s+n} = \langle G_v, G_{v^g}, \ldots, G_{v^{g^n}} \rangle$ is the setwise stabiliser of $\mathcal{A}_n(v^{g^n})$. In particular, e is the smallest integer such that $H_{s+n} = H_{s+n+1}$. Clearly $G = \langle g, H_s \rangle$ and H_s contains no nonidentity normal subgroup of G. Use Theorem 1.3 with $t = s$ and $n = e$ to conclude, leaving only the case where $e = 1$.

If $e = 1$, then Lemma 5.3 implies that the pointwise stabiliser of $\mathcal{A}_1(v)$ is trivial. It follows from Lemma 5.4 that G_{vw} fixes $\mathcal{A}_1(v)$ pointwise and hence $G_{vw} = 1$ and $s = 1$. $\square$

Let $\vec{\Gamma}$ be a finite connected asymmetric G-arc-transitive digraph of prime out-valence p, let v be a vertex of $\vec{\Gamma}$ and suppose that $G_v^{\vec{\Gamma}^+(v)}$ is a cyclic group of order p. Then Theorem 1.1 claims that either $\vec{\Gamma} \cong \mathrm{Pl}^r(\vec{W}(n,p))$ for some $n \geqslant 3$ and $r \geqslant 0$, or $|G_v| \leqslant |\mathrm{V}(\vec{\Gamma})|^3$. We now prove this.

Let $e = \exp(\vec{\Gamma})$. The fact that $|\mathrm{V}(\vec{\Gamma})| \geqslant |\mathcal{A}_e(v)| \geqslant p^e$ follows from Lemma 5.4. By Corollary 5.2, there exists some $r \geqslant 0$ such that $\vec{\Gamma} \cong \mathrm{Pl}^r(\vec{\Gamma}')$, where $\vec{\Gamma}' = \mathrm{Q}^r(\vec{\Gamma})$ is also a finite connected asymmetric digraph of prime out-valence p on which G acts faithfully and arc-transitively and such that either the alternets of $\vec{\Gamma}'$ are not complete bipartite, or $\vec{\Gamma}'$ is a wreath digraph. Note that, by Lemma 3.1, $e' = \exp(\vec{\Gamma}') \leqslant e$. Also, $|\mathrm{V}(\vec{\Gamma}')| \leqslant |\mathrm{V}(\vec{\Gamma})|$ from which it follows that $|G_v| \leqslant |G_{v'}|$ for any vertex v' of $\vec{\Gamma}'$. If the alternets of $\vec{\Gamma}'$ are not complete bipartite, then, by Theorem 7.1, $|G_{v'}|$ divides $p^{3e'}$. It follows that $|G_v| \leqslant |G_{v'}| \leqslant p^{3e'} \leqslant p^{3e} \leqslant |\mathrm{V}(\vec{\Gamma})|^3$. This concludes the proof.

Acknowledgment

The authors acknowledge the financial support of the Slovenian Research Agency ARRS, grant No. J1-0540.

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