Amenable Representations and Reiter’s Property for Kac Algebras

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In this paper, we study amenable unitary corepresentations of Kac algebras. We also study some sorts of “noncommutative” Reiter’s properties. As an application, we find some new equivalent conditions for the amenability of Kac algebras.

Key Words: Kac algebras; amenability; Reiter’s property; amenable representations.

1. INTRODUCTION AND NOTATION

Further to our study of amenability of “quantum groups” (in [11, 12, 14]), we will investigate in this paper amenable unitary corepresentations. However, due to a technical difficulty (namely, the existence of the contragredient corepresentation of a unitary corepresentation; see Definition 3.3), we will only deal with Kac algebras (see Remark 5.2(c) for a discussion of the general case of locally compact quantum groups).

We will begin this paper with the study of an amenability-like property (we call it property (A)) for unitary corepresentations. This will then be used to study (Bekka) amenability of unitary corepresentations. We will also investigate certain Kac-algebra versions of Reiter’s property for representations and relate them to the (Bekka) amenability. As an application of all these studies, we obtained some interesting equivalent conditions for the amenability of Kac algebras (Theorem 5.1).

In this paper, we may assume materials from the following literatures without recalling them explicitly: [1–4, 11–13]. We draw the readers’ attention to the following convention that we use here.
1.1. In this paper, the inner product of a Hilbert space is assumed to be conjugate-linear in the first variable and linear in the second one. For any \( x, y, z \) in a Hilbert space \( K \) and any \( t \in \mathcal{L}(K) \), we denote by \( \omega_{x,y} \) and \( \theta_{x,y}(z) = x\langle y, z \rangle \) respectively. Moreover, throughout this paper, we will use the “leg notation”, \( U_{ij} \), in a similar fashion as in [1, p. 428].

2. AMENABILITY

Notation 2.1. Throughout this paper, \((\mathcal{S}, \mathcal{A}, \kappa, \varphi)\) is a Kac algebra as defined in [4, 2.2.5] (for simplicity, we will also use \( \mathcal{S} \) to denote this Kac algebra) and \( H \) is the Hilbert space given by \( \varphi \) while \((\mathcal{S}, \mathcal{A}, \kappa, \varphi)\) is the dual Kac algebra of \( \mathcal{S} \) (see [3, 3.7.4]). Moreover, \( V \in \mathcal{S} \otimes \mathcal{S} \subseteq \mathcal{S}(H \otimes H) \) is the canonical regular multiplicative unitary associate with \( \mathcal{S} \) such that \( \delta(x) = V(x \otimes 1) V^* \). \( \mathcal{S} \) and \( \mathcal{S}^\# \) are respectively the norm closures of \{\( (\text{id} \otimes \kappa)(V) : v \in \mathcal{S}(H) \}\} and \{\( (\text{id} \otimes \varphi)(V) : \omega \in \mathcal{S}(H) \}\} which are Hopf \( \mathcal{C}^* \)-algebras with coproducts given by \( \delta(x) = V(x \otimes 1) V^* \) and \( \delta(y) = V^*(1 \otimes y) V \) (see [1, 3.8]). In this case, \( \mathcal{S}^\# \) is called a reduced dual Hopf \( \mathcal{C}^* \)-algebra of \( \mathcal{S} \). On the other hand, we denote by \( \mathcal{S}^p \) the universal \( \mathcal{C}^* \)-algebra corresponding to the unitary corepresentations of \( \mathcal{S} \) as defined in [1, A.5] and \( V \in \mathcal{M}(\mathcal{S}^p \otimes \mathcal{S}) \) is the unitary as in [1, A.6(c)]. More precisely, for any unitary corepresentation \( U \) of \( \mathcal{S} \) (i.e. \( U \in \mathcal{M}(\mathcal{S}^p \otimes \mathcal{S}) \) is a unitary such that \( (\text{id} \otimes \delta)(U) = U_{12} U_{13} \)), there exists a unique representation \( \pi_U \) of \( \mathcal{S}^p \) such that \( (\pi_U \otimes \text{id})(V') = U \) (and vice versa). In particular, we denote by \( \pi_1 \) (respectively, \( \pi_\varphi \)) the representation corresponding to the trivial corepresentation \( 1 \in \mathcal{M}(\mathcal{C} \otimes \mathcal{S}) \) (respectively, the unitary corepresentation \( V \) of \( \mathcal{S} \) on \( H \); see Remark 2.2(a) below and [1, A.2(a)]).

Remark 2.2. (a) Note that \( V \) is a biregular irreducible multiplicative unitary (see e.g. [1, 6.11(d)]) and so the results in [1] apply.

(b) \( \mathcal{S} \) and \( \mathcal{S}^\# \) are Kac \( \mathcal{C}^* \)-algebras (see [5]) and \( V \in \mathcal{M}(\mathcal{S}^\# \otimes \mathcal{S}) \) ([1, 3.6(c)]). On the other hand, \( \mathcal{S}^\#_p \) is a Hopf \( \mathcal{C}^* \)-algebra with a coproduct \( \delta_p \) such that both \( \delta_p(\mathcal{S}^\#_p)(1 \otimes \mathcal{S}^\#_p) \) and \( \delta_p(\mathcal{S}^\#_p)(\mathcal{S}^\#_p \otimes 1) \) are total subsets of \( \mathcal{S}^\#_p \otimes \mathcal{S}^\#_p \) (see [1, A.6(e)]). Moreover, \{\( (\text{id} \otimes \varphi)(V') : \omega \in \mathcal{S}(H) \}\} is dense in \( \mathcal{S}^\#_p \) (see [1, A.5 and A.6]).

(c) There is a canonical one to one correspondence between unitary corepresentations of \( \mathcal{S} \) and those of \( \mathcal{S} \) (see e.g. [1, A.6(d)]). In particular, for any unitary corepresentation \( U \in \mathcal{S}(K) \otimes \mathcal{S} \) of \( \mathcal{S} \) on a Hilbert space \( K \) (i.e. \( U \) is a unitary such that \( (\text{id} \otimes \delta)(U) = U_{12} U_{13} \)), \( U \) will automatically lie inside \( \mathcal{M}(\mathcal{S}(K) \otimes \mathcal{S}) \). In fact, in this case, \( U \in \mathcal{M}(\pi_U(\mathcal{S}^\#_p) \otimes \mathcal{S}) \) (see [1, A.3(d)]).
If $\mathcal{S} = L^\alpha(G)$ (for a locally compact group $G$), then $\hat{\mathcal{S}}_p = C^*(G)$ and there is an one to one correspondence between unitary representations of $G$ and non-degenerate $*$-representations of $L^1(G) = L^\alpha(G)_*$ (or equivalently, those of $C^*(G)$). This, in turn, is in one to one correspondence with unitary corepresentations of $L^\alpha(G)$. This is the reason behind the study of unitary corepresentations in the literatures.

Let us first recall the following definition of amenability of $\mathcal{S}$.

**Theorem and Definition 2.3 (Enock–Schwartz).** With the notations as above, the following conditions are equivalent.

(I) $\hat{\mathcal{S}}_g$ is unital (which is equivalent to $\hat{\mathcal{S}} = \hat{\mathcal{S}}_p$ as $\hat{\mathcal{S}}_g$ is an ideal of $\hat{\mathcal{S}}_p$).

(II) $\pi_1$ is weakly contained in $\pi_\nu$.

(III) There exists a net of unit vectors $\{\xi_i\} \subseteq H$ such that $\|V(\xi_i \otimes \eta) - \xi_i \otimes \eta\|$ converges to 0 for any $\eta \in H$.

(IV) $\hat{\mathcal{S}}_e$ has a bounded left (or right) approximate identity.

(V) There exist a right invariant mean $m$ on $\mathcal{S}$ and a functional $f \in \hat{\mathcal{S}}^*$ such that $m((f \otimes \text{id})(V)) \neq 0$.

$\mathcal{S}$ is said to be amenable if one of the above conditions holds.

The equivalences of (I)-(IV) were proved in [3] and their equivalence with (V) can be found in [11, 3.6(a)].

In [11, 3.1], we defined a more general notion of amenable Hopf $C^*$-algebras which can be shown to be equivalent to the above (see [11, 3.5 and 3.6]). In particular, the above theorem is also true in the case of locally compact quantum groups (as defined in [9]).

Furthermore, there are similar equivalent statements as (I)-(IV) concerning unitary corepresentations. These equivalent properties will be needed (explicitly or implicitly) throughout the whole paper and so we will give it a name.

**Proposition and Definition 2.4.** Suppose that $U \in \mathcal{S}(K) \overline{\otimes} \mathcal{S}$ is a unitary corepresentation of $\mathcal{S}$ on a Hilbert space $K$. Let $T = \pi_\nu(\hat{\mathcal{S}}_p)$. Then the following conditions are equivalent.

(i) There exists $f \in T^*$ such that $(f \otimes \text{id})(U) = 1$ (in fact, we can take $f \in T^*_+$ with $f(1) = 1$).

(ii) $\pi_1$ is weakly contained in $\pi_\nu$.  

There exists a net \( \{ \xi_i \} \subseteq K \) with \( \| \xi_i \| = 1 \) such that \( \| U(\xi_i \otimes \eta) - \xi_i \otimes \eta \| \) converges to zero for any \( \eta \in H \).

(iv) There exists a bounded net \( \{ \omega_i \} \subseteq \mathcal{L}(K) \) (in fact, we can assume that \( \| \omega_i \| = 1 \) and \( \omega_i \geq 0 \)) such that \( (\omega_i \otimes \text{id})(U) \) converges strongly (and equivalently, weak-*-converges) to 1.

If \( U \) satisfies one of these equivalent conditions, then we say that \( U \) has property (A).

The arguments for (i) implies (ii) and (ii) implies (iii) are more or less the same as that for Theorem 2.3. The fact that (iii) implies (iv) is almost obvious (by taking \( \omega_i = w_i \)). Finally, (iv) implies (i) because \( \{ w_i \} \), considered as a bounded nets in \( T \), will have a weak-*-limit point \( f \in T \) and condition (i) follows from the facts that \( U \in M(T \otimes S) \) (Remark 2.2(c)) and \( S \) separates points of \( S \). Note that if \( \omega_i \) are all positive with \( \omega_i(1) = 1 \), then \( f \) is positive and \( f(1) = 1 \).

Example 2.5. Suppose that \( G \) is a locally compact group and \( \mathcal{S} \) is the reduced group von Neumann algebra of \( G \) (under the left regular representation). Consider for any \( r \in G \), the one dimensional representation \( \pi_r \) of \( C_0(G) = S^\wedge \) defined by \( \pi_r(f) = f(r) \). Then \( \pi_r \) will not have property (A) unless \( r = e \). In particular, if \( G \) is a locally compact Abelian group, all the one dimensional representations in \( G^\wedge \) \( \{ e \} \) will not have property (A).

We would like to add one more equivalent condition to property (A). Let us first recall the following simple lemma from [15] (see also [12, Sect. 3]).

Lemma 2.6. Suppose that \( (\mathcal{A}, A_g) \) is a Hopf von Neumann algebra. There is an one to one correspondence between (not necessarily unitary) corepresentations of \( (\mathcal{A}, A_g) \) on a Hilbert space \( K \) (i.e. \( X \in \mathcal{L}(K) \otimes \mathcal{A} \) such that \( (\text{id} \otimes A_g)(X) = X_{12}X_{13} \)) and left operator \( A_g \)-module structures on the column Hilbert space \( K_c \) in the sense of [18, p. 1453] (we will call such structure a right coaction of \( A \) on \( K_c \); see [12, Sect. 3]).

This correspondence is actually given by the following completely isometric isomorphisms: \( \mathcal{L}(K) \otimes \mathcal{A} \cong (\mathcal{L}(K) \otimes \mathcal{A}_g)^* \cong \text{CB}(\mathcal{A}_g, \mathcal{L}(K)) \cong \text{CB}(\mathcal{A}_g, \text{CB}(K_c, K_c)) \cong \text{CB}(\mathcal{A}_g \otimes K_c, K_c) \). In particular, if \( U \) is a unitary corepresentation of \( \mathcal{S} \) on \( K \), then the corresponding right coaction \( \gamma_U \) (as a completely bounded map from \( K_c \) to \( \text{CB}(\mathcal{S}_g; K_c) \)) is given by \( \gamma_U(\xi)(v) = (\text{id} \otimes v)(U) \xi = \pi_U((\text{id} \otimes v)(V')) \xi \). In this case, we denote \( v: \xi = \gamma_U(\xi) \).

Definition 2.7. (a) Let \( U \) and \( \gamma_U \) be as above. Suppose that for any \( \varepsilon > 0 \) and any finite subset \( \{ v_1, \ldots, v_n \} \) of \( \mathcal{S}_g \) (the set of all normal positive
functionals on $\mathcal{F}$ with norm 1), there exists a unit vector $\eta \in K$ such that 
$\|v_i \cdot \eta - \eta\| < \varepsilon$ for $i = 1, \ldots, n$. Then $\gamma_{\nu}$ is said to have an approximate fixed vector in $K$.

(b) Suppose that $(\hat{\mathcal{F}}, H, \hat{J}, \hat{P})$ is the canonical standard form for $\mathcal{F}$ (given by $\phi$; see e.g. [3, 2.1.1]). $\mathcal{F}$ is said to satisfy the Reiter's property $(P_2)$ if $\gamma_{\nu}$ has an “approximate fixed vector in $\hat{P}$” (i.e. we can choose $\eta \in \hat{P} \subseteq H$ in part (a) above).

Remark 2.8. (a) It is easily seen that $\gamma_{\nu}$ has an approximate fixed vector in $K$ if and only if there exists a net $\{g_i\}$ in $K$ with $\|g_i\| = 1$ such that for any $v \in \mathcal{F}^*_s$, $\|v \cdot g_i - 1\| \to 0$ as $i \to \infty$. Similar thing is true for the Reiter’s property $(P_2)$ as well.

(b) Note that in the case when $\mathcal{F} = L^e(G)$ for a locally compact group $G$, $\gamma_{\nu}(1_v)$ is the convolution product of $v \in \mathcal{F}^*_s = P(G) \subseteq L^1(G)$ and $\xi \in H = L^2(G)$. In this case, the Reiter’s property $(P_2)$ above is not exactly the same as that defined in [7, p. 46]. In particular, we are considering the action of $r \in L^1(G)$ instead of $r \in G$ (which does not exist for Kac algebras) on $H = L^2(G)$. However, it can be shown that property $(P_2)$ in [7, p. 46] can also be expressed in term of a net of unit vectors “uniformly approximately fixed” by any compact subset of $G$. By using an integration argument (as well as the fact that the set of continuous functions with compact support is dense in $L^1(G)$), it is not hard to see that property $(P_2)$ in [7] will give a net in $\hat{P} = L^2(G)_s$ satisfying part (a) above (and thus implies the property $(P_2)$ here). Moreover, they are in fact equivalent because both are equivalent to the amenability of $G$ (see Proposition 2.9 below as well as [7, p. 46 and 3.2.1]).

(c) In the light of part (a), we can call $\mathcal{F}$ to have Reiter’s property $(P_1)$ if it has a left approximate invariant mean but this terminology is clearly redundant.

PROPOSITION 2.9. (a) A unitary corepresentation $U$ of $\mathcal{F}$ on a Hilbert space $K$ has property (A) if and only if $\gamma_{\nu}$ has an approximate fixed vector in $K$.

(b) $\mathcal{F}$ is amenable if and only if it satisfies the Reiter’s property $(P_2)$.

Proof. (a) Suppose that $\{\eta_j\}$ is a net of unit vectors in $K$ satisfying condition (iii) of Proposition 2.4. Then it is clear that $(\omega_{\nu, \eta_j} \otimes \text{id})(U)$ will weak-*converge to 1. Hence for any $v \in \mathcal{F}^*_s$, we have $1 \geq \|\omega_{\nu, \eta_j} \otimes \text{id}(U) \eta_j\| \geq |\langle \eta_j, (\text{id} \otimes v)(U) \eta_j \rangle|$ which converges to 1. This implies that $\|\omega_{\nu, \eta_j} \otimes \text{id}(U) \eta_j\|$ converges to 1 and

$$\|v \cdot \eta - \eta\|^2 = \|\omega_{\nu, \eta_j} \otimes \text{id}(U) \eta_j - \eta_j\|^2 = \|\text{id} \otimes v(U) \eta_j\|^2 + 1 - \langle (\text{id} \otimes v)(U) \eta_j, \eta_j \rangle - \langle \eta_j, (\text{id} \otimes v)(U) \eta_j \rangle.$$
converges to 0. Conversely, let \( \mathcal{S} \) be the collection of all finite subsets of \( \mathcal{S}^*_+ \) and let \( I \) be the index set \( \mathcal{S} \times \mathbb{R}^+ \) with the ordering given by \((F, r) \leq (F', r')\) if \( F \subseteq F' \) and \( r' \leq r \). For any \( i = (F, r) \in I \), let \( \eta_i \in K \) such that \( \|\eta_i\| = 1 \) and \( \|v \cdot \eta_i - \eta_i\| \leq r \) for all \( v \in F \). Let \( \omega_i = \omega_{\eta_i} \). Clearly, \( \|\omega_i\| = 1 \). We claim that \((\omega_i \otimes \text{id})(U)\) will weak*-converge to 1. In fact, for any \( \varepsilon > 0 \) and any \( v \in \mathcal{S}^*_+ \setminus \{0\} \), we can take \( F_0 = \{v / v(1)\} \). Therefore, if \( i = (F, r) \geq (F_0, \varepsilon / v(1)) \), we have \( \|(\text{id} \otimes v / v(1))(U) \eta_i - \eta_i\| \leq r \) and so \( |\varepsilon / v(1)| \leq v(1) \) \( r \leq \varepsilon \). This proved part (a).

(b) The sufficiency follows directly from part (a). To show the necessity, we first recall that \( V \in \mathcal{G} \otimes \mathcal{S} \). Therefore, in the argument of part (a), we can find for each \( i \in I \), a unique \( \zeta_i \in \hat{P} \) such that \( \omega_{\eta_i, \zeta_i} = \omega_{\eta_i} \) (see [8, 2.10]). Now the same argument as in the first half of part (a) will give the required net in Remark 2.8(a).

Moreover, we have another equivalent formulation for amenability in part (c) below.

**Proposition 2.10.** (a) If \( U \) has property (A), then \( \|(\text{id} \otimes g)(U)\| = \|g\| \) for any \( g \in S^*_+ \) (see Remark 2.2(c)).

(b) Suppose that there exists a left invariant mean on \( \mathcal{S} \). Then \( U \) has property (A) if and only if there exists \( \lambda > 0 \) such that \( \|(\text{id} \otimes v)(U)\| > \lambda \|v\| \) for all \( v \in \mathcal{S}^*_+ \).

(c) \( \mathcal{S} \) is amenable if and only if the canonical map from \( S^*_+ \) to \( M(\hat{S}) \) (and equivalently, the map from \( \mathcal{S}^*_+ \) to \( \hat{S} \)) is norm preserving.

**Proof.** (a) It is obvious that \( \|(\text{id} \otimes g)(U)\| \leq \|g\| \). Now let \( f \in \pi_{\zeta_i}(\hat{S}^*_+) \) be such that \( \|f\| = f(1) = 1 \) and \( (f \otimes \text{id})(U) = 1 \) (Proposition 2.4(i)). Then \( \|g\| = (f \otimes g)(U) \leq \|(\text{id} \otimes g)(U)\| \).

(b) The necessity follows from part (a). To show the sufficiency, let \( \{v_i\}_{i \in I} \) be a left approximate invariant mean on \( \mathcal{S} \) (see [11, 1.14]). By the hypothesis, for each \( i \in I \), there exists \( \zeta_i \in K \) with \( \|\zeta_i\| = 1 \) such that \( r_i := \|(\text{id} \otimes v_i)(U) \zeta_i\| > \lambda \). Let \( \eta_i \) be the unit vector \( \text{id} \otimes v_i)(U) \zeta_i / r_i \) in \( K \). For any \( v \in \mathcal{S}^*_+ \), we have

\[
\|v \cdot \eta_i - \eta_i\| = (1 / r_i) \|(\text{id} \otimes v)(U)((\text{id} \otimes v_i)(U) \zeta_i) - (\text{id} \otimes v_i)(U) \zeta_i\| \\
= (1 / r_i) \|(\text{id} \otimes (v \cdot v_i - v_i))(U) \zeta_i\| \\
\leq (1 / \lambda) \|v \cdot v_i - v_i\|,
\]

which converges to zero. Thus \( v_i \) has an approximate fixed vector in \( K \) and Proposition 2.9(a) completes the proof of this part.
Again, we need only to show the sufficiency. By the hypothesis, \( v(1) = [(\text{id} \otimes v)(V)] \) for \( v \in \mathcal{S}_+^* \subseteq \mathcal{S}_+^* \). This implies that for any hermitian functional \( \mu \in \mathcal{S}_* \),
\[
|\mu(1)| = |v_0(1) - v_1(1)| = |(\|((\text{id} \otimes v_0)(V)) - ((\text{id} \otimes v_1)(V))\|) - \|((\text{id} \otimes \mu)(V))\|
\]
where \( v_0, v_1 \in \mathcal{S}_+^* \) and \( \mu = v_0 - v_1 \). Now take any \( \omega \in \mathcal{S}_* \). We have
\[
\|((\text{id} \otimes \mu^*)(V))\| = \|((\text{id} \otimes \omega)(V^*))\| = |\hat{k}((\text{id} \otimes \omega)(V))| = \|((\text{id} \otimes \omega)(V))\| \quad (1)
\]
(where \( \hat{k} \) is the coinvolution of \( \mathcal{S}_* \)). Let \( \mu_0 \) and \( \mu_1 \) be respectively the real and the imaginary parts of \( \omega \). Then,
\[
|\mu(1)| = \|((\text{id} \otimes (\omega + \omega^*)(V)/2)(V)| \leq \|((\text{id} \otimes \omega)(V))\|
\]
and the same is true for \( \mu_i \). Hence,
\[
|\pi((\text{id} \otimes \omega)(V^*))| = |\omega(1)| \leq |\mu_0(1)| + |\mu_1(1)| \leq 2 \|((\text{id} \otimes \omega)(V))\| = 2 \|\pi_\omega((\text{id} \otimes \omega)(V^*))\|
\]
(where \( V^* \) is the unitary in Notation 2.1). Thus, we showed that \( \pi_\omega \) is weakly contained in \( \pi_\omega \).

The idea of the proof of part (c) comes from \([17, 8.3.7(ii)](\) (the statement concerning \( \mathcal{S}_+^* \) in this part has already been proved in \([8a, 7.6]\). The author thanks P. Desmedt for this information). Note that we need to consider \( U = V \) in part (c) since for a general unitary corepresentation \( U \), there may not be a bounded linear map \( \chi \) from \( \pi_\omega(\mathcal{S}_*) \) to \( \mathcal{S}(K) \) such that \( \chi((\text{id} \otimes \omega)(U)) = (\text{id} \otimes \omega)(U^*) \) (yet we will see in the next section a situation in which there is an injective \( * \)-anti-homomorphism \( \chi \) satisfying this equation). We note also that in the argument of part (c), it suffices to show the existence of a \( \lambda > 0 \) such that \( \|((\text{id} \otimes \omega^*)(U))\| \leq \lambda \|((\text{id} \otimes \omega)(U))\| \).

3. AMENABLE UNITARY COREPRESENTATIONS

In this section, we will study amenability of unitary corepresentations that is defined in analogy to that of locally compact groups. First of all, let us recall the following definitions and theorems of Bekka (see \([2, 1.1, 3.3, 3.5, \text{and} 5.1]\)).

**Proposition and Definition 3.1 (Bekka).** Let \( G \) be a locally compact group and \( \pi \) a unitary representation of \( G \) on a Hilbert space \( K \). \( \pi \) is said to be amenable if one of the following equivalent conditions holds.

(i) There exists \( m \in \mathcal{S}(K)_+^* \) with \( m(1) = 1 \) such that \( m(\pi(r) T \pi(r^{-1})) = m(T) \) for any \( r \in G \) and \( T \in \mathcal{S}(K) \) (such a \( m \) is called a \( G \)-invariant mean).
There exists $M \in \mathcal{L}(K)^+$ with $M(1) = 1$ such that $M[\int_{G} f(r)\pi(r) T \pi(r^{-1}) \, dr] = M(T)$ for any $T \in \mathcal{L}(K)$ and any $f \in L^1(G)$, with $\int_{G} f(r) \, dr = 1$ (such a $M$ is called a topological invariant mean).

(iii) $\pi_1$ (the trivial representation of $G$ on $C$) is weakly contained in $\pi \otimes \bar{\pi}$ (where $\bar{\pi}$ is the contragredient representation of $\pi$). In the case of Kac algebras, we can define similar properties as (ii) and (iii) above but we do not know if they are equivalent—see the discussion after Proposition 4.7 (note that it is still not known whether the existence of a left or a right invariant mean on $\mathcal{S}$ will imply $\mathcal{S}$ to be amenable).

Before we define such properties, we need the following notation.

**Notation 3.2.** From now on, $U \in \mathcal{L}(K) \otimes \mathcal{S}$ is a unitary corepresentation of $\mathcal{S}$ on a Hilbert space $K$ and $\bar{U}$ is the unitary corepresentation of $\mathcal{S}$ on the conjugate Hilbert space $\bar{K}$ given by $\bar{U} = (\tau \otimes \kappa)(U)$ where $\tau$ is the canonical anti-isomorphism from $\mathcal{L}(K)$ to $\mathcal{L}(\bar{K})$. Moreover, if $W$ is another unitary corepresentation of $\mathcal{S}$ on a Hilbert space $L$, then we denote by $U \otimes W$ the unitary corepresentation $U_{13}W_{23}$ on $K \otimes L$.

**Definition 3.3.** A unitary corepresentation $U$ of $\mathcal{S}$ is said to be

(i) weakly Bekka amenable if there exists $M \in \mathcal{L}(K)^+$ with $M(1) = 1$ such that $M[(\text{id} \otimes \alpha_U(x))] = M(x)$ for any $\alpha \in \mathcal{S}_+^+$ and any $x \in \mathcal{L}(K)$ (where $\alpha_U(x) = U(x \otimes 1)U^*$ is a coaction of $\mathcal{S}$ on $\mathcal{L}(K)$). Those $M$ satisfying the above condition are called $\alpha_U$-invariant means.

(ii) Bekka amenable if $\pi_1$ is weakly contained in $\pi_{U \otimes \bar{U}}$ (in other words, $U \otimes \bar{U}$ has property (A)).

The following proposition justified the use of the term “weak Bekka amenability” (cf. [2, 3.5, 5.1 and 2.2]).

**Proposition 3.4.** Let $U$ be a unitary corepresentation of $\mathcal{S}$ on a Hilbert space $K$.

(a) If $U$ is Bekka amenable, then it is weakly Bekka amenable.

(b) If there is a right invariant mean on $\mathcal{S}$, then $U$ is weakly Bekka amenable.

**Proof.** (a) Note that any unitary corepresentation $W$ of $\mathcal{S}$ on a Hilbert space $L$ having property (A) is weakly Bekka amenable. More precisely, suppose that $\{\zeta_i\}$ is a net of unit vectors in $L$ such that $\|W(\zeta_i \otimes \eta) - \zeta_i \otimes \eta\|$ converges to 1 (for any $\eta \in H$). Then the net $\{\omega_{\zeta_i \otimes \bar{\zeta}}\}$ has a subnet weak-*-convergent to some $m \in \mathcal{S}(L)_+$. Now the same argument as in [11, 3.6] shows that $m$ is an $\alpha_p$-invariant mean. Thus, the
The hypothesis gives us an \( \alpha_\tau \circ \tau \)-invariant mean \( m \). It is obvious that the functional \( M \) on \( \mathcal{L}(K) \) defined by \( M(x) = m(x \otimes 1) \) is then an \( \alpha_\tau \)-invariant mean.

(b) Take any \( \omega \in \mathcal{L}(K)^*_+ \) with \( \omega(1) = 1 \) and consider the map \( \Phi_\omega \) from \( \mathcal{L}(K) \) to \( \mathcal{S} \) defined by \( \Phi_\omega(x) = (\omega \otimes \text{id}) \alpha_\tau(x) \). It is easy to see that \( \Phi_\omega \) is a completely positive map such that \( \Delta \circ \Phi_\omega = (\Phi_\omega \otimes \text{id}) \circ \alpha_\tau \) and \( \Phi_\omega(1) = 1 \). Thus, if \( m \) is a right invariant mean on \( \mathcal{S} \), then \( M = m \circ \Phi_\omega \) is an \( \alpha_\tau \)-invariant mean.

The idea of the proof of part (b) comes from [2, 2.2]. We remark here that in general the existence of an \( \alpha_\tau \)-invariant mean is strictly weaker than property (A) (c.f. Theorem 2.3(V)) since Bekka amenability is in general, strictly weaker than property (A) (see Remark 3.11(a)).

In the case of locally compact groups, the left regular representation is amenable if and only if the group is amenable (cf. [2, 2.2]). The same is true for Kac algebras even though the argument is very different. To show this, we will need the following lemma.

**Lemma 3.5.** Suppose that \( \pi_\tau \) is the extension of \( \pi_\tau \) to \( \hat{S} \times \times \) and \( e \) is the support projection of \( \pi_\tau \). Then \( \hat{\delta}^{**}(e) \geq 1 \).

In fact, if \( (\hat{S}_\mu, V', \mathcal{S}) \) is as in Notation 2.1, then [13, 3.2(c)] tells us that it is a Fourier duality (in the sense of [13, 3.1]). If \( \mu \) is the canonical representation of \( \mathcal{S} \) on \( \mathcal{L}(H) \), then \( (\mu, \pi_\tau) \) is a \( V' \)-covariant representation in the sense of [13, 4.1] and the above lemma follows directly from [13, 5.5].

**Proposition 3.6.** \( V \) is Bekka amenable if and only if \( \mathcal{S} \) is amenable.

**Proof.** Suppose that \( \mathcal{S} \) is amenable. Then \( V \) will have property (A). In particular, there exists \( \omega_\mu \in \mathcal{L}(H)^*_+ \) satisfying 2.4(iv). It is obvious that \( \omega_\mu \circ \tau^{-1} \in \mathcal{L}(\hat{H})^*_+ \) will satisfy the corresponding condition for \( \hat{V} \) and so \( \hat{V} \) has property (A). Now let \( \{ \xi_i \} \subseteq H \) and \( \{ \zeta_j \} \subseteq \hat{H} \) be two nets of unit vectors that satisfy the corresponding properties of 2.4(iii). Then \( \{ \xi_i \otimes \zeta_j \} \) is a net of unit vectors in \( H \otimes \hat{H} \) and

\[
\|V_{13}\hat{V}_{23}(\xi_i \otimes \zeta_j \otimes \eta) - \xi_i \otimes \zeta_j \otimes \eta\| \\
\leq \|V_{13}\hat{V}_{23}(\xi_i \otimes \zeta_j \otimes \eta) - V_{13}(\xi_i \otimes \zeta_j \otimes \eta)\| + \|V_{13}(\xi_i \otimes \zeta_j \otimes \eta) - \xi_i \otimes \zeta_j \otimes \eta\| \\
\leq \|\hat{V}(\zeta_j \otimes \eta) - \zeta_j \otimes \eta\| + \|V(\xi_i \otimes \eta) - \xi_i \otimes \eta\|
\]

which converges to 0. Hence \( V \circ \hat{V} \) has property (A) and \( V \) is Bekka amenable. To show the converse, let us suppose as above that \( \pi_\tau \) is the extension of \( \pi_\tau \) to \( \hat{S}_\mu \times \times \) and \( e \) is the support projection of \( \pi_\tau \). Then
ker\(\pi_v = (1-e)\mathbf{S}^{**}_p \cap \mathbf{S}_p\) and so for any \(x \in \ker \pi_v\), we have \(x = (1-e)x\) (for simplicity, \(\mathbf{S}_p\) and \(M(\mathbf{S}_p \otimes \mathbf{S}_p)\) are identified with the corresponding subspaces of \(\mathbf{S}^{**}_p\) and \(\mathbf{S}^{**}_p \oplus \mathbf{S}^{**}_p\) respectively). Thus,

\[
\begin{align*}
(\pi_v \otimes \text{id}) \delta_p(x) &= (\tilde{\pi}_v \otimes \text{id}) \delta^{**}_p(x) \\
&= (\tilde{\pi}_v \otimes \text{id}) \delta^{**}_p((1-e)x) \\
&= (\tilde{\pi}_v \otimes \text{id})((e \otimes 1) \delta^{**}_p((1-e)x)) = 0
\end{align*}
\]

(note that \((e \otimes 1) \delta^{**}_p((1-e)x) = 0\) by Lemma 3.5). Therefore, \(\ker \pi_v \subseteq \ker(\pi_v \otimes \pi_v) \circ \delta_p = \ker \pi_{v \circ v} \subseteq \ker \pi_v\) and this proved the proposition (by Theorem 2.3(II)).

**Remark 3.7.** The argument of the above proposition also shows that if two unitary corepresentations \(U\) and \(W\) have property (A), then so are \(\tilde{U}\) and \(U \circ W\). Moreover, if \(V \circ U\) has property (A) for a unitary corepresentation \(U\), then \(\mathcal{S}\) is amenable.

**Lemma 3.8.** Let \(U\) be a unitary corepresentation of \(\mathcal{S}\) on \(K\).

(a) For any \(\xi, \xi' \in K\) and \(\eta, \eta' \in H\), we have \(\langle \xi \otimes \eta, \tilde{U}(\xi' \otimes \eta') \rangle = \langle \xi' \otimes \eta, U^*(\xi \otimes \eta') \rangle\) (where \(\xi\) and \(\xi'\) are the elements in \(\tilde{K}\) corresponding to \(\xi\) and \(\xi'\) respectively).

(b) For any \(v \in \mathcal{S}_v\), \((\text{id} \otimes v)((U \circ \tilde{U})^*) = \sigma \circ (\tau \otimes \tau^{-1})((\text{id} \otimes v)(U \circ \tilde{U}))\) (where \(\sigma\) is the flip of the two variables).

(c) Let \((\tilde{\mathcal{S}}, H, \tilde{J}, \tilde{P})\) be the canonical standard form for \(\mathcal{S}\) and \(J_K\) be the conjugate-linear isometry on \(K \otimes \tilde{K}\) given by \(J_K(\xi \otimes \tilde{\xi}) = \xi \otimes \tilde{\xi}\). Then \((J_K \otimes \tilde{J}) U_{13} \tilde{U}_{23}(J_{-} \otimes \tilde{J}) = (U_{13} \tilde{U}_{23})^*\).

**Proof.**

(a) This part follows from the following sequence of equalities.

\[
\begin{align*}
\langle \xi \otimes \eta, \tilde{U}(\xi' \otimes \eta') \rangle &= \langle \tilde{\xi}, \tau(\text{id} \otimes \omega_{v',v} \circ \kappa)(U) \tilde{\xi}' \rangle \\
&= \langle \xi', (\text{id} \otimes \omega_{v',v} \circ \kappa)(U) \xi \rangle = \langle \eta, \kappa(\omega_{\xi',\xi} \otimes \text{id})(U) \eta' \rangle \\
&= \langle \eta, (\omega_{\xi',\xi} \otimes \text{id})(U^*) \eta' \rangle = \langle \xi' \otimes \eta, U^*(\xi \otimes \eta') \rangle.
\end{align*}
\]

(b) For any \(\eta, \eta' \in H\) and any \(\xi, \xi', \zeta, \zeta' \in K\), we have, by part (a),

\[
\begin{align*}
\langle \xi \otimes \tilde{\xi}, (\text{id} \otimes \omega_{v',v})(\tilde{U}_{12} U_{13}^*)(\xi' \otimes \tilde{\xi}') \rangle
&= \langle \xi \otimes \zeta' \otimes \eta, U_{23} U_{13}^* (\xi' \otimes \zeta \otimes \eta') \rangle
= \langle \zeta' \otimes \xi \otimes \eta, U_{23} U_{13} (\xi \otimes \zeta' \otimes \eta') \rangle
= \langle \xi' \otimes \tilde{\xi} \otimes \eta, U_{13} U_{23} (\xi \otimes \tilde{\xi} \otimes \eta') \rangle
\end{align*}
\]
\[= \langle \zeta' \otimes \xi', (id \otimes \omega_{\kappa, \eta'})(U_{13} \bar{U}_{23})(\zeta \otimes \xi) \rangle \]
\[= \langle \zeta \otimes \xi, \sigma \circ (\tau \otimes \tau^{-1})(id \otimes \omega_{\kappa, \eta})(U_{13} \bar{U}_{23})(\zeta' \otimes \xi') \rangle. \]

Now part (b) follows from linearity and continuity.

(c) We recall from [4, 3.6.6] that \( \hat{J} \kappa^* \hat{J} = \kappa(x) \) for any \( x \in U \). For any \( \eta, \eta' \in H \) and any \( \zeta, \zeta', \xi' \in K \), by part (a) and the fact that \( U_{13} \bar{U}_{23} \) is a unitary corepresentation of \( S \), we have

\[ \langle \zeta \otimes \xi \otimes \eta, (J_k \otimes \hat{J}) U_{13} \bar{U}_{23}(J_k \otimes \hat{J})(\zeta' \otimes \xi' \otimes \eta') \rangle \]
\[= \langle \zeta \otimes \xi \otimes \hat{J} \eta, U_{13} \bar{U}_{23}(\zeta' \otimes \xi' \otimes \hat{J} \eta') \rangle \]
\[= \langle \xi \otimes \xi' \otimes \hat{J} \eta, U^*_{13} U^*_{23}(\zeta \otimes \xi \otimes \hat{J} \eta') \rangle \]
\[= \langle \eta, \hat{J} \kappa((\omega_{\zeta, \eta} \otimes \xi \otimes \xi \otimes id)(U_{13} \bar{U}_{23})) \hat{J} \eta' \rangle \]
\[= \langle \xi \otimes \xi' \otimes \eta, (U_{13} \bar{U}_{23})^* (\zeta' \otimes \xi' \otimes \eta') \rangle. \]

As in the case of locally compact groups, any unitary corepresentation on a finite dimensional Hilbert space is automatically (Bekka) amenable (see [2, 1.3]). Before we give this result, we want to recall the following interesting lemma from [9, 9.5] (note that our convention of Hilbert space is different from that in [9] and hence we have a virtually different statement here). This lemma can be avoided in the proof of Proposition 3.10 since we are dealing with finite dimensional Hilbert space but it is more convenient to use it (and we will need it in the next remark anyway).

**Lemma 3.9 (Kustermann–Vaes).** Let \( K \) and \( L \) be two Hilbert spaces and \( \{e_i\}_{i \in A} \) be an orthonormal basis for \( K \). For any \( \zeta, \eta \in K \) and \( X, Y \in \mathcal{L}(K \otimes L) \), the net \( \{\sum_{i \in A}(\omega_{\zeta, \eta} \otimes id)(X)(\omega_{\zeta, \eta} \otimes id)(Y)\}_{\mathcal{F}(A)} \) (where \( \mathcal{F}(A) \) is the collection of all finite subsets of \( A \)) converges strongly to \( (\omega_{\zeta, \eta} \otimes id)(XY) \).

**Proposition 3.10.** Any unitary corepresentation \( U \) of \( S \) on a finite dimensional Hilbert space \( K \) is Bekka amenable.

**Proof.** Let \( \{e_1, ..., e_n\} \) be an orthonormal basis for \( K \) and let \( \zeta = \sum_{i=1}^n e_i \otimes \bar{e}_i \in K \otimes \bar{K} \). Now for any \( \eta, \eta' \in H \), we have by Lemmas 3.8(a) and 3.9,
\[
\langle \eta, (\omega_{\zeta, \zeta} \otimes \text{id})(U_{13} \hat{U}_{23}) \eta' \rangle = \sum_{i,j=1}^{n} \langle e_i \otimes e_i \otimes \eta, U_{13} \hat{U}_{23} (e_j \otimes e_j \otimes \eta') \rangle \\
= \sum_{i,j=1}^{n} \langle e_i \otimes e_j \otimes \eta, U_{13} U_{23}^* (e_j \otimes e_i \otimes \eta') \rangle \\
= \sum_{i,j=1}^{n} \langle \eta, (\omega_{e_i, e_j} \otimes \text{id})(U)(\omega_{e_j, e_i} \otimes \text{id})(U^* \eta') \rangle \\
= \sum_{i=1}^{n} \langle \eta, (\omega_{e_i, e_i} \otimes \text{id})(1) \eta' \rangle = n \langle \eta, \eta' \rangle.
\]

Thus, if \( \zeta_0 = (1/\sqrt{n}) \zeta \), then \( (\omega_{\zeta_0, \zeta_0} \otimes \text{id})(U_{13} \hat{U}_{23}) = 1 \) and the proposition follows from Proposition 2.4.

**Remark 3.11.** (a) This proposition, together with Example 2.5 and Remark 3.7, shows that property (A) is strictly stronger than Bekka amenability.

(b) Suppose that \( \{e_i\}_{i \in J} \) is an orthonormal basis for any Hilbert space \( K \) and \( U \) is any unitary corepresentation of \( S \) on \( K \). Let \( \zeta \in K \) be such that \( \| \zeta \| = 1 \) and \( \zeta = \sum_{e \in J} e_i \otimes \bar{e}_i \) (where \( J \) is a finite subset of \( A \)). Consider \( \zeta = \xi \otimes \tilde{\xi} \in K \otimes \bar{K} \) and \( \omega_j = \omega_{\zeta, \zeta} \). Then a similar argument as in the above proposition shows that \( \langle \eta, (\omega_j \otimes \text{id})(U_{13} \hat{U}_{23}) \eta' \rangle \) will converge to \( \langle \eta, \eta' \rangle \) for any \( \eta, \eta' \in H \). Hence, \( (\omega_j \otimes \text{id})(U_{13} \hat{U}_{23}) \) weak-*-converges to \( 1 \) (note that any \( \nu \in \mathcal{S}_+ \) is of the form \( \omega_{\zeta, \zeta} \) for some \( \zeta \) in the self dual cone of the canonical standard form for \( \mathcal{S} \); see \([3, 2.1.1]\)). Therefore, the boundedness of the net in Proposition 2.4(iv) is essential.

4. REITER’S PROPERTY FOR UNITARY COREPRESENTATIONS

In the case of a locally compact group \( G \), it is well known that for any representation \( \pi \) of \( G \) on \( K \), \( \pi \otimes \bar{\pi} \) is unitary equivalent to the corresponding representation \( \hat{\pi} \) of \( G \) on the Hilbert space \( HS(K) \) of Hilbert–Schmidt operators on \( K \) (more precisely, \( \hat{\pi}(T) = \pi(r) T \pi(r^{-1}) \)). In the case of a Kac algebra \( \mathcal{S} \), it is not obvious that \( \alpha_U \) (which corresponds to the integral form of \( \hat{\pi} \)) will induce a sort of corepresentation of \( \mathcal{S} \) on \( HS(K) \) (which require that \( \alpha_U(HS(K)) \mathcal{S}_+ \subseteq HS(K) \); here we identify \( \text{CB}(\mathcal{S}_+; \mathcal{L}(K)) \) with \( \mathcal{L}(K) \otimes \mathcal{S} \)). Nevertheless, there is a right coaction \( \gamma_U^\mathcal{S} = \gamma_U \otimes \phi \) of \( \mathcal{S} \) on \( (K \otimes \hat{K}) \oplus \mathcal{S} \). It is natural to ask if there is any relation between \( \gamma_U^\mathcal{S} \) and \( \alpha_U \).
LEMMA 4.1. Let $\Psi$ be the canonical isometric isomorphism from $K \otimes \hat{K}$ to $\text{HS}(K)$. For any $\Xi \in K \otimes \hat{K}$ and $\nu \in \mathcal{S}_*$, we have

$$\Psi(\gamma^U(\Xi))(\nu) = \alpha_\nu(\Psi(\Xi))(\nu).$$

(2)

Proof. Let $\zeta, \zeta', \eta, \eta' \in K$ and $x, y \in H$. We will first prove the above equality for $\Xi$ and $\nu$ being finite sums of $\zeta \otimes \eta$ and $\omega_{x, y}$ respectively. Observe that by Lemma 3.8(a),

$$\langle \zeta', \gamma^U(\zeta \otimes \eta)(\omega_{x, y}) \rangle = \langle \zeta', (\text{id} \otimes \text{id} \otimes \omega_{x, y})(U_{13} \hat{U}_{23})(\zeta \otimes \eta) \rangle$$

$$= \langle U^*(\zeta' \otimes x), U^*(\eta' \otimes y) \rangle$$

$$= \langle \zeta' \otimes x, \alpha_\nu(\theta_{\zeta' y})(\eta' \otimes y) \rangle$$

$$= \omega_{x, y}[(\text{id} \otimes \omega_{x, y}) \alpha_\nu(\Psi(\zeta \otimes \eta))]$$

$$= \text{Tr}[\theta^*_x \alpha_\nu(\Psi(\zeta \otimes \eta))(\omega_{x, y})].$$

Therefore, by linearity, we know that

$$\langle \Xi', \gamma^U(\Xi)(\nu) \rangle = \text{Tr}[\Psi(\Xi')^* \alpha_\nu(\Psi(\Xi))(\nu)]$$

(3)

for any $\Xi', \Xi \in K \otimes \hat{K}$ and $\nu = \sum_{i=1}^n \omega_{x_i, y_i}$ (where $x_1, ..., x_n, y_1, ..., y_n \in H$). By letting $\Xi'$ converge to any element in $K \otimes \hat{K}$, we see that equation (3) holds for any $\Xi' \in K \otimes \hat{K}$ and any $\Xi$ and $\nu$ as above (note that $\Psi$ is an isometry). This implies that

$$\text{Tr}[\Psi(\Xi')^* \Psi(\gamma^U(\Xi))(\nu)] = \text{Tr}[\Psi(\Xi')^* \alpha_\nu(\Psi(\Xi))(\nu)]$$

for any $\Xi' \in K \otimes \hat{K}$ and so equation (2) holds for $\Xi \in K \otimes \hat{K}$ and $\nu$ of the form $\sum_{i=1}^n \omega_{x_i, y_i}$. Now, for any $\Xi \in K \otimes \hat{K}$ and $\nu \in \mathcal{S}_*$, there exists a net $\{\Xi_i\}$ in $K \otimes \hat{K}$ and a net $\{\nu_j\}$ of the above form which converge to $\Xi$ and $\nu$ respectively. Then the continuity of $\gamma^U$ (which can be considered as a completely bounded map from $(K \otimes \hat{K}) \otimes \mathcal{S}_*$ to $(K \otimes \hat{K})_\nu$) as well as that of $\Psi$ ensure that $\Psi(\gamma^U(\Xi_i))(\nu_j)$ converges to $\Psi(\gamma^U(\Xi))(\nu)$ in the Hilbert–Schmidt norm and hence converges in the operator norm. On the other hand, the continuity of $\Psi^*$ means that $\Psi(\Xi_i)$ converges to $\Psi(\Xi)$ in the Hilbert–Schmidt norm and thus in the operator norm. Therefore, the continuity of $\alpha_\nu$ (considered as a complete bounded map from $\mathcal{L}(H) \otimes \mathcal{S}_*$ to $\mathcal{L}(H)$) shows that $\alpha_\nu(\Psi(\Xi_i))(\nu_j)$ will converge to $\alpha_\nu(\Psi(\Xi))(\nu)$ in the operator norm. Hence equation (2) holds for any $\Xi \in K \otimes \hat{K}$ and $\nu \in \mathcal{S}_*$. 

Definition 4.2. Let $U$ be a unitary corepresentation of $\mathcal{S}$ on $K$. We say that $U$ satisfies

(a) Property ($\hat{P}_2$) (cf. [2, 4.1]) if for any $\varepsilon > 0$ and any finite subset $\{v_1, ..., v_n\}$ of $\mathcal{S}_+^+$, there exists a self-adjoint Hilbert–Schmidt operator $R$ on $K$ with $\|R\|_\mathcal{S} = 1$ such that $\|v_k \cdot R - R\|_\mathcal{S} < \varepsilon$ for all $k = 1, ..., n$ (where $v \cdot R := \alpha_u(R)(v) \in HS(K)$ by Lemma 4.1);

(b) Property ($d_2$) (cf. [2, p. 398]) if for any $v \in \mathcal{S}_+^+$, $\|(id \otimes v) \circ \alpha_u\|_\mathcal{S} = \|v\|$ (where $\|\cdot\|_\mathcal{S}$ is the norm in $\mathcal{S}(HS(K))$).

Remark 4.3. (a) In the light of Lemma 4.1, property ($\hat{P}_2$) is apparently stronger than $c_u$ having an approximate fixed vector in $K \bar{\otimes} K$ but we will see in the proof of Proposition 4.5 that they are actually the same. Moreover, ($\hat{P}_2$) can also be reformulated as follows: there exists a net $\{R_i\}$ in $HS(K)$ such that $\|R_i\|_\mathcal{S} = 1$ and $\|v \cdot R_i - R(1) \|_\mathcal{S}$ converges to 0 for any $v \in \mathcal{S}_+^+$.

(b) Note that in the case of $\mathcal{S} = L_\infty(G)$, property ($\hat{P}_2$) above is different from ($P_2$) in [2] in two places. The first one is that we are considering the action of $\mathcal{S}_+^+ = P(G)$ instead of $G$ on $HS(K)$. The second one is that the Hilbert Schmidt operators $R$ above are only required to be self adjoint instead of positive. A similar comment as Remark 2.8(b) applies for the first difference. The second difference is a technical difficulty in the case of Kac algebras. Let us give a brief discussion here.

(i) If $r \in G$, $T \in HS(K)$ and $T_r = \pi(r) T \pi(r^{-1})$, then the Powers–Stømer inequality (see e.g. [2, 4.2]) implies that

$$\|\pi(r)(T^*T)^{1/2} \pi(r^{-1}) - (T^*T)^{1/2}\|_\mathcal{S} \leq \|T_r^* T_r - T^*T\|_1 \leq 2 \|\pi(r) T \pi(r^{-1}) - T\|_\mathcal{S} \|T\|_\mathcal{S}.$$ 

Thus, we can obtain positive elements satisfying [2, 4.1] from arbitrary (possibly non-positive) elements satisfying the same condition. However, in the general case, $(id \otimes v) \circ \alpha_u$ need not respect products nor square roots.

(ii) One may try to get round this problem of positivity by using a sort of standard form argument as in the proof of Proposition 2.9(b). However, suppose that $(\pi_{\nu \otimes G}(\mathcal{S}_\nu)^+, K \bar{\otimes} K, J, P)$ is in the standard form such that $P$ is generated by $\xi \otimes \overline{\xi}$ (this is the only choice if we want $P$ to represent positive elements in $HS(K)$). Then the conjugate linear map $J$ should be the map $J_x$ in Lemma 3.8(c), i.e. $J(\xi \otimes \overline{\xi}) = \xi \otimes \overline{\xi}$ (as $P$ is total in $176$ CHI-KEUNG NG

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Now the condition for standard form and Lemma 3.8(c) imply that \( \pi_{U \circ \hat{o}}(\hat{S}_p)' = J \pi_{U \circ \hat{o}}(\hat{S}_p)^* J = \pi_{U \circ \hat{o}}(\hat{S}_p)^* \), i.e. \( \pi_{U \circ \hat{o}}(\hat{S}_p) \) is commutative.

(iii) Nevertheless, \((P_2)_a\) in \([2]\) is stronger than \((\hat{P}_2)\) by a similar consideration as Remark 2.8(b) and both of them are actually equivalent to the Bekka amenability of the representation \( \pi \) (see Proposition 4.5 below as well as \([2, 4.3]\)).

(c) Property \((d_2)\) in the above definition is an inferior version of property \((D_2)\) in \([2]\) because we consider \( v \) in \( \mathcal{S}^*_\alpha \) instead of \( S^*_\alpha \) or \( M(\mathcal{S}) = S^* \) if and only if \( \hat{\mathcal{S}} \) is amenable by \([11, 2.1(c), 3.13(a) \text{ and } 2.12]\)). In fact, \((\id \otimes f) \circ \alpha_v \) apparently does not make sense for \( f \) in \( S^*_\alpha \). Nevertheless, using equality (4) in the next proposition, we can define property \((D_2)\) as follows: \( \| \alpha_v(U \circ \hat{o}) \| = \| f \| \) for any \( f \in S^*_\alpha \) (recall that \( U \circ \hat{o} \in M(\mathcal{X}(K \otimes \check{K}) \otimes S) \)). Furthermore, because of Proposition 2.10(a) and Proposition 4.4 below, property \((D_2)\) is equivalent to property \((d_2)\).

**Proposition 4.4.** A unitary corepresentation \( U \) is Bekka amenable if and only if it has property \((D_2)\).

**Proof.** First of all, we note that by Lemma 4.1, we have the following equality.

\[
\| (\id \otimes v) \alpha_v \| = \sup \{ \| \alpha_v(x) \| : x \in HS(K); \| x \| \leq 1 \}
= \sup \{ \| \iota(x) \| : x \in K \otimes \check{K}; \| x \| \leq 1 \}
= \| (\id \otimes v)(U \circ \hat{o}) \|.
\]

Therefore, the necessity follows from Proposition 2.10(a). The sufficiency follows from a similar argument as Proposition 2.10(c) together with Lemma 3.8(b). More precisely, we replace \( \hat{V} \) and \( \hat{k} \) in the proof of Proposition 2.10(c) by \( U \circ \hat{o} \) and \( \sigma \circ (\tau \otimes \tau^{-1}) \) respectively and notice that the corresponding equality of (1) holds because of Lemma 3.8(b) and the fact that \( \sigma \circ (\tau \otimes \tau^{-1}) \) is an isometry.

The following is another characterisation for Bekka amenability.

**Proposition 4.5.** Let \( U \) be a unitary corepresentation of \( \mathcal{S} \) on \( K \). \( U \) is Bekka amenable if and only if it has property \((\hat{P}_2)\).

**Proof.** The sufficiency is a direct consequence of Proposition 2.9(a) and Lemma 4.1. To show the necessity, we first note that by Proposition 2.9(a) and Lemma 4.1, for any \( \varepsilon \in \mathbb{R}_+ \), \( v_1, \ldots, v_n \in \mathcal{S}^*_\alpha \), there exists \( R \in HS(K) \) (not necessarily self adjoint) such that \( \| R \| = 1 \) and \( \| v_i \cdot R - R \|_2 < \varepsilon \) for
Let $S$ and $T$ be respectively the real and the imaginary parts of $R$. Then because $\|v_i^* R - R\|_2 < \epsilon$ (as $\alpha_v$ is a $*$-homomorphism and $v_i$ is positive), we have $\|v_i S - S\|_2 < \epsilon$ and $\|v_i T - T\|_2 < \epsilon$. On the other hand, since $\|S\|_2 + |T|_2^2 = \text{Tr}(S^2 + T^2)$, we know that $\|S\|_2 < 1/\sqrt{2}$ or $|T|_2 < 1/\sqrt{2}$ and there exist $\alpha, \beta \in [0, 1]$ such that $\|S\|_2 - \alpha \|T\|_2 = 1/\sqrt{2}$.

Now by the Cauchy–Schwarz inequality, we have

$$\|\lambda S + \mu T\|_2^2 = \lambda^2 \|S\|_2^2 + \mu^2 \|T\|_2^2 + 2\lambda \mu \text{Tr}(ST) \geq (\lambda \|S\|_2 - \mu \|T\|_2)^2 = 1/2.$$  

If we take $Q = (\lambda S + \mu T)/\|\lambda S + \mu T\|_2 \in HS(K)_a$, then

$$\|v_i Q - Q\|_2 \leq (\lambda \|v_i S - S\|_2 + \mu \|v_i T - T\|_2)/\|\lambda S + \mu T\|_2 < \sqrt{2} \epsilon.$$

This completes the proof.

In the case of locally compact groups, Bekka showed in [2, 6.5] that a unitary representation $\pi$ is amenable if and only if it satisfies $(P_p)_a$ for all $p \in [1, \infty)$ (and equivalently, for some) where $(P_p)_a$ is expressed in terms of $I_p = \{T \in \mathcal{L}(K) : \text{Tr}(T|_p) < \infty\}$ (note that $\pi(r)(I_p) \pi(r^{-1}) \subseteq I_p$). However, it is not known if we still have $\alpha_v(I_p)(\mathcal{S}_v^+ \subseteq \mathcal{S}_v^+)$, and a similar argument as for [12, 3.6] can be employed to show that this defines a left coaction. We denote by $\beta_v$ the corresponding left coaction on $TC(K)$ ($\cong \mathcal{L}(K)_a$). In the case of a locally compact group $G$, the left coaction $\beta_v \in CB(TC(K); CB(L^1(G); TC(K)))$ is non-degenerate and corresponds to a completely bounded non-degenerate anti-representation $\mu \in CB(L^1(G); CB(TC(K)_a; TC(K))$ which is given by $\mu(r)(T) = \pi(r^{-1}) T \pi(r)$. Notice that this $\mu$ is not far from the restriction of $\alpha_v$ on $I_2$ and in particular, $\|\pi(r) T \pi(r^{-1}) - T\|_2 = \|T - \mu(T)\|_2$. Therefore, in this case, we can take another look at the case of $\alpha_v(I_p)_a$ in terms of $\mu$.

**Definition 4.6.** $U$ is said to have property $(\tilde{P})$ if for any $\epsilon > 0$ and any finite subset $\{v_1, ..., v_n\}$ of $\mathcal{S}_v^+$, there exists $T \in TC(K)_a$ with $\|T\|_2 = 1$ such that $\|T \cdot v_k - T\|_2 < \epsilon$ for all $k = 1, ..., n$ (where $T \cdot v = \beta_v(T)(v)$).

It is natural to ask if we can define $(\tilde{A}_v)$ as in Definition 4.2(b). However, for any $v \in \mathcal{S}_v^+$ and $\alpha_v \in \mathcal{L}(K)_a$ with $\alpha_v(1) = 1$, we have

$$\|v\| = \alpha_v(1) \|v(1) = (\alpha_v \otimes v) \alpha_v(1) \leq \sup_{1 \otimes \|v\| \leq 1} |(\omega \otimes v) \alpha_v(x)| \leq \sup_{1 \otimes \|v\| \leq 1} \|\beta_v(\cdot)\|_1 \|v\|.$$
Hence \( \| \beta_v (\cdot) (v) \|_1 = \| v \| \) and such “condition \((d)\)” will be satisfied automatically by any unitary corepresentation \( U \). On the other hand, \((\tilde{P}_1)\) turns out to be interesting.

**Proposition 4.7.** A unitary corepresentation \( U \) is weakly Bekka amenable if and only if it satisfies property \((\tilde{P}_1)\).

**Proof.** Notice first of all the following reformulation of \((\tilde{P}_1)\) in terms of \( L(K) \): there exists a net \( \{ \omega_i \} \) in \( L(K) \) with \( \omega_i (1) = 1 \) such that for any \( \nu \in L^+_s(K) \), \( \| (\omega_i \otimes \nu) \circ \sigma_U - \omega_i \| \) converges to zero (i.e. \( \{ \omega_i \} \) is an “approximate \( \sigma_U \)-invariant mean” on \( L(K) \)). Now using a similar argument as that for \([3, 2.8.4]\) (which is a kind of Namioka’s argument; see the proof of \([7, 2.4.2]\)), the existence of an approximate \( \alpha_U \)-invariant mean is equivalent to the existence of an \( \alpha_U \)-invariant mean on \( L(K) \). These give the required equivalence.

In the case of locally compact groups, there is a natural relation between \((P_1)\) and \((P_2)\) in that we can transform the required elements in \( HS(K) \) for condition \((P_2)\) to the required elements in \( TC(K) \) for condition \((P_1)\) by taking square and using the fact that \((\pi (r) T \pi (r^{-1}))^2 = \pi (r) T^2 \pi (r^{-1}) \) as well as the Powers-Stømer inequality (see the proof of \([2, 4.3]\)). However, there seems to have no such analogy for Kac algebras and as mentioned above, we do not know if the weak Bekka amenability is the same as Bekka amenability.

### 5. AN APPLICATION

Using the results in the above, we can add the following equivalent conditions to Theorem 2.3.

**Theorem 5.1.** The following conditions are all equivalent to the amenability of \( \mathcal{S} \).

- \((VI)\) The canonical map from \( S^* \) to \( M(\hat{S}) \) is norm preserving.
- \((VII)\) There exist a left approximate invariant mean \( \{ v_i \}_{i \in I} \) on \( \mathcal{S} \) and a \( \lambda > 0 \) such that \( \| (id \otimes v_i)(V) \| \geq \lambda \) for all \( i \in I \) (in fact, we can take \( \lambda = 1 \)).
- \((VIII)\) \( \mathcal{S} \) has the Reiter’s property \((P_2)\) (see Definition 2.7(b)).
- \((IX)\) \( V \) satisfies property \((d)\) (or equivalently, \((D)\); see Remark 4.3(c)).
- \((X)\) There exist a left approximate invariant mean \( \{ v_i \}_{i \in I} \) on \( \mathcal{S} \) and a \( \lambda > 0 \) such that \( \| (id \otimes id \otimes v_i)(V_1 V_2) \| \geq \lambda \) for all \( i \in I \) (again, we can take \( \lambda = 1 \)).
V satisfies property $(\hat{P}_2)$ (see Definition 4.2(a)).

There exists a finite dimensional (and equivalently, one dimensional) representation of $\hat{S}$.

Proof. We first recall from Propositions 2.10(c) and 2.9(b) that the amenability of $\mathcal{S}$ is equivalent to condition (VI) and also to (VIII). If $\mathcal{S}$ is amenable, then 2.3(V) and [11, 1.14] (see also [3, 2.4]) ensure the existence of a left approximate invariant mean on $\mathcal{S}$ which satisfies the relation in (VII) because of (VI). Conversely, the argument in Proposition 2.10(b) shows that condition (VII) implies the amenability of $\mathcal{S}$. The equivalence of the amenability of $\mathcal{S}$ with condition (IX) follows from Remark 4.3(c), Propositions 4.4 and 3.6. Suppose that $\mathcal{S}$ is amenable. Then as above, we have a left approximate invariant mean on $\mathcal{S}$ and condition (IX), together with equality (4), implies (X). Again, using a similar argument as for Proposition 2.10(b), we see that condition (X) gives an approximate fixed vector for $\gamma^\vee$. Therefore, the argument of Proposition 4.5 shows that (X) is stronger than (XI). If condition (XI) holds, then Propositions 4.5 and 3.6 show that $\mathcal{S}$ is amenable. Finally, it is clear that the amenability of $\mathcal{S}$ will imply condition (XII) since we have the trivial representation $\pi_1$ of $\hat{S}_p = \hat{S}$.

As noted at the end of Section 2, the “$\mathcal{S}_+$-version” of (VI) was shown to be equivalent to the amenability of $\mathcal{S}$ in [8a, 7.6].

Remark 5.2. (a) In the case of a locally compact quantum group $\mathcal{S}$ (in the sense of [9]), conditions (VII) and (VIII) are still equivalent to the amenability of $\mathcal{S}$. Note that Notation 2.1 and Remark 2.2 are still valid in this general case as $V$ will be a manageable multiplicative unitary in the sense of [21] (see [9]) and we can use the results in [21] and [13]. Therefore, everything in this paper that does not involve the coinvolutions, is true in the general case. We do not know if condition (VI) is still equivalent to the amenability of $\mathcal{S}$ in general (it is certainly weaker than the amenability because of Proposition 2.10(a)). The same comment applies for condition (XII) (yet there is a partial generalisation for this in the sense that only one dimensional representations are considered; see [14]). Moreover, we do not know if similar conditions as (IX)–(XI) can be formulated in the general case.

(b) Note that conditions (VI) and (IX) as well as conditions (VII) and (X) are in pairs (and conditions (VIII) and (XI) are similar). They are
roughly the same statements concerning \(V\) and \(V \odot \check{\rho}\) respectively but there is no obvious way to go directly from ones of them to the others.

(c) A direct attempt to extend the materials in this paper to the case of locally compact quantum groups is to replace the coinvolution \(\kappa\) by the unitary antipode \(R\). It is easy to see that all the results in Section 2 except Proposition 2.10(c) and all the results in Section 3 up to Remark 3.7 hold for this general situation. However, Lemma 3.8(a) seems to break down in this case. Since almost everything after Lemma 3.8(a) depends on it, those results cannot be extended in an obvious way. On the other hand, S. Vaes suggested to us the following way to generalise Definition 3.3: replace \(U \odot \check{U}\) by the canonical unitary implementation of \(\alpha_U\) as given in [19]. Nevertheless, he then showed in [20] that such unitary is again \(U \odot \check{U}\) where \(\check{U}\) is the one given by the direct extension above (i.e. replacing \(\kappa\) with \(R\)). We do not know whether there is a generalisation for the materials from Lemma 3.8 onward.

We end this paper with the following direct consequence of condition (XII) of Theorem 5.1. The first part of which stresses the fact that the \(C^*\)-algebraic structure of the reduced dual Hopf \(C^*\)-algebra of a Kac algebra determines the amenability of that Kac algebra. This fact is well known in the case of discrete groups (using [10, 4.2]). The case for locally compact groups might also be known as we were told that part (b) below is already known although it is not stated explicitly anywhere (the author thanks Prof. A. Lau for this information).

**Corollary 5.3.** (a) Let \(\mathcal{T}\) be an amenable Kac algebra and \(\hat{T}\) be the reduced dual Hopf \(C^*\)-algebra of \(\mathcal{T}\) as defined in Notation 2.1. If \(\hat{T}\) is isomorphic to \(\mathcal{S}\) as \(C^*\)-algebras, then \(\mathcal{S}\) is also amenable.

(b) Let \(G\) be a locally compact group. Then \(G\) is amenable if and only if there exists a finite dimensional representation of \(C^*_r(G)\).

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