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GRADED DEDEKIND RINGS

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1. Introduction

Generalized Rees rings over Dedekind rings were introduced in [3] as a particular case of graded Dedekind rings. It was discovered soon that these rings simplified the study of certain Dedekind rings (cf. e.g. [4]). Extending the concept of a generalized Reesring to higher dimensions and to the non-commutative case provided new examples of rings with interesting properties (see [2], [6]).

We are interested in the following application. Let Λ be an H.N.P. order with center R (in the sense of [5]). Then there is a finite number of maximal ideals $M_1, \dots, M_k$ of R such that Λ_{M_i} is not an Azumaya algebra. Denote $\mathrm{rad}(\Lambda M_i)$ by J_i and let $J = J_1 \cdots J_k$. It is well known that J is an invertible Λ ideal. Now it is easily proved that $\Delta = \sum_{i \in \mathbb{Z}} J^i X^i$ is a $\mathbb{Z}$-graded unramified maximal order over a graded Dedekind ring S. Under some mild extra conditions Δ is even an Azumaya algebra.

The close relation between Λ and Δ makes it possible to relate invariants of Λ to invariants of the (easier) ring Δ. We will give some applications of this principle in a subsequent paper. However since S is not necessarily a generalized Rees ring it is necessary in the first place to further the study of arbitrary $\mathbb{Z}$-graded Dedekind rings. The theory of graded Dedekind rings is not completely parallel to the ungraded case. As a matter of fact, the approximation property does not hold and this has a drastic consequence, namely: the graded class group of a graded semilocal Dedekind ring need not be zero. Another problem arises because there is no direct generalization of Steinitz's theorem available. The aim of this paper is to study the graded module category and the graded class group of a graded Dedekind ring and to derive some results which may be used as substitutes for those parts of the ungraded theory that cannot be recovered in a trivial way. The author thanks F. Van Oystaeyen for some comments.

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2. Notations and conventions

(a) Let R be an integral domain. We denote by $\mathrm{Frac}(R)$ the set of fractional ideals of R. The set of invertible fractional ideals of R is denoted by $\mathrm{Inv}(R)$ and the set of maximal ideals of R is denoted by $\Omega(R)$.

(b) Let R be a Dedekind ring, $\alpha \in \mathbb{Q}$ and $I \in \mathrm{Frac}(R)$. We may write I as $M_1^{n_1} \cdots M_k^{n_k}$ for some $M_1, \ldots, M_k \in \Omega(R)$. Then I^α is defined as $M_1^{\lceil n_1\alpha \rceil} \cdots M_k^{\lceil n_k\alpha \rceil}$ where $\lceil \beta \rceil$ with $\beta \in \mathbb{Q}$ stands for the smallest integer $\geq \beta$. A more general definition of these 'rational powers' is provided in the appendix.

(c) In this paper 'graded' always means '$\mathbb{Z}$-graded'. Constructions of a graded nature, paralleling similar constructions in the ungraded theory are denoted by the ungraded symbol but with a sub- or superscript g, e.g. $Q_g(R)$ means the graded quotientring etc.

(d) $\simeq$ means 'isomorphism'. If this symbol is used in connection with graded objects it always means graded isomorphism.

(e) Throughout we use terminology and notations of [1].

3. Graded discrete valuation rings

3.1. Definition. Let K be a graded field. A graded subring R of K is a graded discrete valuation ring of K if and only if for every homogeneous x in K either x or x^{-1} is in R.

3.2. Definition. Let K be a graded field. A graded valuation is a map $v : h(K) \to T$ where Γ is an ordered group satisfying the following properties.

(a) $\forall a, b \in h(K) : v(ab) = v(a) + v(b)$.

(b) $\forall a, b \in h(K) : \deg(a) = \deg(b) \Rightarrow v(a+b) > \mathrm{Min}(v(a), v(b))$.

If $\Gamma \cong \mathbb{Z}$ in the foregoing, then v is called a graded discrete valuation. Properties of graded valuations and graded valuation rings may be found in [1]. Let us only mention the bijective correspondence between graded valuations and graded valuation rings. A graded valuation ring corresponding to a graded discrete valuation is called a graded discrete valuation ring.

3.3. Theorem. *Let $K = K_0[X, X^{-1}]$ be a graded field with $\deg X = 1$. A graded discrete valuation ring is of one of the following types.*

(a) $K_0[X]$,

(b) $K_0[X^{-1}]$,

(c) $\sum_{n \in \mathbb{Z}} M^{-n\alpha} X^n$ *where $\alpha \in \mathbb{Q}$ and M is the maximal ideal of a discrete valuation ring R_0 of K_0.*

Proof. Let $v : h(K) \to \mathbb{Z}$ be the graded valuation corresponding to R. Since $v(K_0) \subset \mathbb{Z}$

two possibilities may occur: $\upsilon(K_0)=0$ or $\upsilon(K_0)=n\mathbb{Z}$ for some $n\in\mathbb{N}\setminus\{0\}$.

First, if $\upsilon(K_0)=0$, then $R\supset K_0$ and since $X\in R$ or $X^{-1}\in R$, it follows that $R\supset K_0[X]$ or $R\supset K_0[X^{-1}]$. Now $K_0[X]$ and $K_0[X^{-1}]$ are both graded discrete valuation rings [1] hence maximal graded subrings of K. So $R=K_0[X]$ or $R=K_0[X^{-1}]$.

Secondly, if $\upsilon(K_0)=n\mathbb{Z}$, then R_0 is a discrete valuation ring with corresponding valuation $\upsilon\mid K_0$.

It is possible to normalize υ such that $\upsilon(K_0)=\mathbb{Z}$. In the sequel we suppose all valuations to be normalized in this way. Let us denote $\upsilon(X)$ by α. By definition $h(R)=\{x\in h(K)\mid \upsilon(x)\geq 0\}$ and so

$$\begin{aligned} R_i &= \{x\in K_i \mid \upsilon(x)\geq 0\} \\ &= \{yX^i \mid y\in K_0, \upsilon(yX^i)\geq 0\} \\ &= \{yX^i \mid y\in M^{-i\alpha}\} = M^{-i\alpha}X^i. \qquad \square \end{aligned}$$

In the sequel all graded discrete valuation rings are supposed to be of the latter type.

If we replace X by $\pi^k X$ where π is the uniformizing element of R_0 we may always assume that $0\leq\alpha<1$. The type of R is defined to be the number $\alpha \operatorname{Mod} 1$ and it will be denoted by $t(R)$. Note that $t(R)$ and R_0 completely determine R.

3.4. Theorem. *Let R be a graded valuation ring with $t(R)=p/e$, $0\leq p<e$ and $(p,e)=1$. (If $t(R)=0$ we put $p=0$ and $e=1$.) Let m denote the maximal ideal of R_0 and M the graded maximal ideal of R. Then:*

(a) *$M^e=mR$. Hence it makes sense to call e the ramification index of R.*

(b) *The homogeneous units of R have degrees ke with $k\in\mathbb{Z}$.*

(c) *The uniformizing elements of R have degrees p' where $pp'\equiv 1 \operatorname{Mod} e$.*

(d) *$R/M\cong R_0/m[X^e,X^{-e}]$ with* $\deg X=1$.

Proof. From Theorem 3.3 we infer that $R\cong\sum_{n\in\mathbb{Z}} m^{n\alpha}X^n$. We first establish (b). Clearly $m^{-n\alpha}X^n$ contains a unit iff

$$m^{-n\alpha}m^{n\alpha}=R_0 \Leftrightarrow m^{\lceil -n\alpha\rceil+\lceil n\alpha\rceil}=R_0 \Leftrightarrow \lceil -n\alpha\rceil+\lceil n\alpha\rceil=0 \Leftrightarrow n\alpha\in\mathbb{Z}\Leftrightarrow e\mid n.$$

Now it is clear that the minimal value attained by the valuation of an element of $m^{-n\alpha}X^n$ is $\lceil -n\alpha\rceil+n\alpha$. So for $m^{\lceil -n\alpha\rceil}x^n$ to contain a uniformizing element, $\lceil -n\alpha\rceil+n\alpha$ should be as small as possible. The lowest value that $\lceil -n\alpha\rceil+n\alpha$ can possibly take is clearly $1/e$. For this to happen it is necessary that $e\mid 1-np$ or equivalently $np\equiv 1 \operatorname{Mod} e$. Conversely, if $np\equiv 1 \bmod e$, then a direct calculation shows that $\lceil -n\alpha\rceil+n\alpha=1/e$. This also proves (a). It remains to prove (d). We know that $R/M=R_0/m[X^f,X^{-f}]$ (R/M is a graded field). To find f we must find the smallest i such that $R_i\neq M_i$. Since $M_i=\{x\in R_i\mid \upsilon(x)\geq 1/e\}$ it is clear that $M_i\neq R_i \Leftrightarrow \exists x\in R_i:\upsilon(x)=0 \Leftrightarrow R$ contains a homogeneous unit $\Leftrightarrow i=ke$. This proves the statement. $\square$

3.5. Remarks. (a) From the foregoing it is clear that the valuation of a homogeneous element is not independent of its degree. In fact it is easy to show that for $a \in h(K)$

$$v(a) \equiv \alpha \deg a \operatorname{Mod} 1.$$

(b) From [7] it follows that all units of R are homogeneous. So the adjective 'homogeneous' in 3.4(b) is superflous.

4. Graded Dedekind rings

A lot of equivalent characterizations of graded Dedekind rings are to be found in [1]. We use the following definition.

4.6. Definition. A graded domain is a graded Dedekind ring if and only if
(a) R is (graded) integrally closed.
(b) R is (graded) Noetherian.
(c) gr-$K\dim R = 1$.

4.7. Proposition. *Let $(R_i)_{i\in I}$ be a family of graded discrete valuation rings in some fixed graded field. Suppose: $\bigcap R_{i,0}$ is a Dedekind ring, $R_{i,0} \neq R_{j,0}$ for $i \neq j$ and $t(R_i) = 0$ for almost all i. Then $R = \bigcap_{i\in I} R_i$ is a graded Dedekind ring.*

Proof. R is graded integrally closed since each R_i is graded integrally closed. Let e be the least common multiple of the ramification indices of the R_i's. (This definition makes sense since $t(R_i) = 0$ for almost all i.) From $R_{i,0} \neq R_{j,0}$ for $i \neq j$ it follows that $R^{(e)}$ is a generalized Rees ring. So $R^{(e)}$ is graded Noetherian and has graded Krull dimension one because R_0 has the corresponding ungraded properties $R(e) = IR$ for some invertible R-ideal. This implies that R is generated by $R_0, R_1, \ldots, R_{e-1}$ over $R(e)$. So R is finitely generated and gr-integral over $R(e)$. This shows that R is Noetherian and had graded Krull dimension 1. □

In the sequel we only consider graded Dedekind rings which arise in this way. From [1] it follows that the only two which are not of this type are $k[X]$ and $k[X^{-1}]$.

4.8. Remark. (a) The condition $R_{i,u} \neq R_{j,0}$ is necessary. Take for example $R_1 = \sum_{n\in\mathbb{Z}} M^n X^n$ and $R_2 = \sum_{n\in\mathbb{Z}} M^{-n} X^n$ where M is the maximal ideal of a discrete valuation ring. Then $R_1 \cap R_2$ has graded Krull dimension two.

(b) From 4.7 it follows that $R = \sum_{n\in\mathbb{Z}} I^{n/e} X^n$. Thus R is a so-called lepidopterous Rees ring (cf. [2]). (It is possible to do this in a slightly more general way, cf. the Appendix.)

(c) From 4.7 it follows that there is a bijective correspondence between maximal ideals of R_0 and graded maximal ideals of R. If $M \in \Omega_g(R)$, then the graded localization of R at M is denoted by R_M. If $m = M \cap R_0$, then $R_M = R_{R_0 \setminus m}$. The corresponding ramification index and graded valuation will be denoted by e_M and v_M respectively.

4.9. Definition. Let R be a graded Dedekind ring with graded field of quotients $K_0[X, X^{-1}]$ ($\deg X = 1$). We say that R satisfies the graded approximation property if the following condition holds.

Let S be a finite subset of $\Omega_g(R)$ and $(n_P)_{P \in S}$ a set of integers. Then there exists an $x \in h(K)$ such that $e_p v_p(x) = n_p$ for all $p \in S$ and $v_p(x) \geq 0$ for $p \notin S$.

4.10. Theorem. *Let R be a graded Dedekind ring. Then R satisfies the graded approximation property if and only if $(e_p, e_q) = 1$ for $p, q \in \Omega_g(R)$, $p \neq q$.*

Proof. Assume that R satisfies the graded approximation property. If $p, q \in \Omega_g(R)$, $p \neq q$, there exists a $\tau \in h(R)$ such that $v_p(\tau) = 0$ and $v_q(\tau) = 1/e_q$. 3.5(a) implies that $\deg \tau \equiv 0 \bmod e_p$ and $\deg \tau \equiv 1 \bmod e_q$; this is only possible when $(e_p, e_q) = 1$.

Conversely, just like in the ungraded case, it suffices to show that $h(P_1) \not\subset h(P_1^2) \cup \cdots \cup h(P_n)$ for $P_1, \ldots, P_n \in \Omega_g(R)$. Let p' be the degree of a uniformizing element of R_{P_1}. Take a k such that $k \equiv p' \bmod e_{P_1}$ and $k \equiv 0 \bmod e_{P_i}$, $i = 1, \ldots, n$. In this case $(P_i)_k = (P_{i,0})^{a_i} R_k$ for $i = 1, \ldots, n$ with $a_1 > 0$ for $i = 2, \ldots, n$ and $(P_1^2)_k = (P_{1,0})^{a_1'} R_k$ with $a_1' > a_1$. Then it follows from the ungraded approximation theorem that

$$(P_1)_k \not\subset (P_1^2)_k \cup \cdots \cup (P_n)_k. \quad \square$$

We will now investigate the graded module category of a graded Dedekind ring.

4.11. Proposition. *If R is a graded Dedekind ring and M a finitely generated graded R-module, then $M \cong N \otimes T$ where N is a graded torsion free R-module and T is a graded torsion R-module.*

Proof. Exactly as in the ungraded case. □

4.12. Proposition. *Suppose that R is a graded Dedekind ring and T is a graded finitely generated torsion R-module. Then $T \cong \bigoplus_{i=1}^n R/P_i^{n_i}$ with $P_i \in \Omega_g(R)$.*

Proof. Let $\mathfrak{a}$ be the annihilator of T. We may write $\mathfrak{a}$ as $\mathfrak{b}\mathfrak{c}$ with $\mathfrak{b} + \mathfrak{c} = R$. Choose elements of degree zero $s \in \mathfrak{b}$, $t \in \mathfrak{c}$ such that $s + t = 1$. Now it is that $sT = tT = T$. Let $x \in sT \cap tT$. Then $x = (s+t)x = sx + tx = 0$. So by induction $T \cong T_1 \oplus \cdots \oplus T_n$ were $\operatorname{anh}(T_i) = P_i^{k_i}$ for some $P_i \in \Omega_g(R)$. The R-modules annihilated by some power of a $P \in \Omega_g(R)$ are in one-one correspondence with the torsion modules over the ring R_q which is a graded P.I.D. The proof may now be carried further exactly as in the ungraded case [8]. □

4.13. Proposition. *If R is a graded Dedekind and M is a graded R-lattice (i.e. a graded torsion free finitely generated R-module), then $M \cong \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n$ where $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n \in \mathrm{Frac}_g(R)$.*

Proof. Exactly as in the ungraded case. □

4.14. Proposition. *Let R be a graded discrete valuation ring. For $(n_i)_{i=1,\ldots,k}$, $(m_i)_{i=1,\ldots,k} \in \mathbb{Z}$ we have $R(n_1) \oplus \cdots \oplus R(n_k) \cong R(m_1) \oplus \cdots \oplus R(m_k)$ if and only if there is a permutation $\varphi\{1,\ldots,k\} \to \{1,\ldots,k\}$ such that $n_{\varphi(i)} \equiv m_i \bmod e$ where e is the ramification index of R.*

Proof. Write M and N for $R(n) \oplus \cdots \oplus R(n_t)$ and $R(m_1) \oplus \cdots \oplus R(m_k)$ respectively. Denote the graded maximal ideal of R by P. 3.4 implies that

$$M/PM \cong k[X^e, X^{-e}](n_1) \oplus \cdots \oplus k[X^e, X^{-e}](n_k)$$

and

$$N/PN \cong k[X^e, X^{-e}](m_1) \oplus \cdots \oplus k[X^e, X^{-e}](m_k).$$

Comparing $\dim_k(M/PM)_i$ to $\dim_k(N/PN)_i$ yields the desired result. The converse is trivial. □

4.15. Definition. Let R be a graded Dedekind ring, M and N two graded R-lattices. We say that M and N are in the same genus (notation: $M \sim N$) if $M_p \cong N_p$ for all $P \in \Omega_g(R)$.

4.16. Definition. Let R be a graded Dedekind ring and M a graded R lattice of rank n. Then $\det M$ denotes the graded module of rank 1: $\Lambda^n M$. This means that if $M \cong \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n$, then $\det M \cong \mathfrak{a}_1 \cdots \mathfrak{a}_n$.

4.17. Lemma. *Let R be a graded Dedekind ring and consider $(\mathfrak{a})_i$, $(\mathfrak{b}_i) \in \mathrm{Frac}_q(R)$ $(i = 1, \ldots, n)$ such that $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n \sim \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_n$, $t \in R_0$. Then the following are equivalent*

(a) $\mathfrak{a}_1 \cdots \mathfrak{a}_n \mathfrak{b}_n^{-1} \cdots \mathfrak{b}_n^{-1} = tR$,

(b) $(\mathfrak{a}_1)_0 \cdots (\mathfrak{a}_n)_0 (\mathfrak{b}_1)_0^{-1} \cdots (\mathfrak{b}_n)_0^{-1} = tR_0$.

Proof. It suffices to check this locally. So we may assume that R is a graded discrete valuation ring. Let $r(R) = p/e$ with $(p,e) = 1$, $0 \le p < e$, and let p' denote the degree of a uniformizing element Π of R. Thus $\mathfrak{a}_i = \Pi^{r_i} R$ and $\mathfrak{b}_i = \Pi^{s_i} R$. Since $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n \sim \mathfrak{b}_n \oplus \cdots \oplus \mathfrak{b}_n$ we may suppose that $r_i - s_i = k_i e$ (by 4.14). Thus $\mathfrak{a}_1 \cdots \mathfrak{a}_n \mathfrak{b}_1^{-1} \cdots \mathfrak{b}_n^{-1} = \Pi^{e \Sigma k_i} R$. Theorem 3.4 implies that $\Pi^e = \pi u$ where π is a uniformizing element of R_0 and u is a homogeneous unit. Hence $\Pi^{e \Sigma k_i} R = \pi^{\Sigma k_i} R$. Furthermore

$$(\mathfrak{a}_i)_0 = (\Pi^{r_i} R_i)_0 = \Pi^{r_i} R - n_i p' \quad \text{and} \quad (\mathfrak{b}_i)_0 = (\Pi^{s_i} R_i)_0 = \Pi^{s_i} R - s_i p'.$$

So

$$
\begin{aligned}
(\mathfrak{a}_i)_0(\mathfrak{b}_i)_0^{-1} &= \Pi^{k_i e}(R - r_i p)(R - s_i p')^{-1} \\
&= \Pi^{k_i e}\pi^{\lceil (p/e) r p' \rceil} X^{-r_i p'} \pi^{-\lceil (p/e) s_i p' \rceil} X^{s_i p'} R_0 \\
&= \pi^{(pp'+1)k_i} u^{k_i} X^{-k_i e p'} R_0 \\
&= \pi^{(pp'+1)k_i} (\pi^{-pp'} X^{p'e})^{k_i} X^{-k_i e p'} R_0 = \pi^{k_i} R_0.
\end{aligned}
$$

Thus $\mathfrak{a}_{1,0} \cdots \mathfrak{a}_{n,0} \mathfrak{b}_{1,0}^{-1} \cdots \mathfrak{b}_{n,0}^{-1} = \pi^{\Sigma k_i} R_0$. □

4.18. Corollaries. (a) *Let M, N be two graded R-lattices with $M \sim N$. If $f \in \mathrm{Hom}_g(M, N)$ then f is an isomorphism if and only if f_0 is an isomorphism* ($f_0 = f \mid M_0$).

(b) $\det M \cong \det N$ *if and only if* $M_0 \cong N_0$.

(c) $\mathfrak{a}, \mathfrak{b} \in \mathrm{Frac}_g(R)$. *Then* $\mathfrak{a} \sim \mathfrak{b}$ *implies that* $\mathfrak{a} = I\mathfrak{b}$ *for some* $I \in \mathrm{Frac}(R_0)$.

Proof. (a) We may suppose that $M \cong \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n$ and $N \cong \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_n$ with $\mathfrak{a}_i, \mathfrak{b}_i \in \mathrm{Frac}_g(R)$. Then t may be represented as an element of degree zero of $(\mathfrak{a}_i^{-1}\mathfrak{b}_j)_{j=1,\ldots,n}^{i=1,\ldots,n}$. The statement that f_0 is an isomorphism is equivalent to $R_0 \det f = \mathfrak{a}_{1,0}^{-1} \cdots \mathfrak{a}_{n,0}^{-1} \mathfrak{b}_{1,0} \cdots \mathfrak{b}_{n,0}$. By 4.17 this is equivalent to $R \det f = \mathfrak{a}_1^{-1} \cdots \mathfrak{a}_n^{-1} \mathfrak{b}_1 \cdots \mathfrak{b}_n$. So f is an isomorphism.

(b) With notations as in (a) we have that $\det M \cong \mathfrak{a}_1 \cdots \mathfrak{a}_n$ and $\det N \cong \mathfrak{b}_1 \cdots \mathfrak{b}_n$. So (b) is a direct consequence of 4.17 and Steinitz's theorem.

(c) $\mathfrak{a} \sim \mathfrak{b}$ implies that $\mathfrak{a}\mathfrak{b}^{-1} \sim R$ and since $(\mathfrak{a}\mathfrak{b}^{-1})_0 R \hookrightarrow \mathfrak{a}\mathfrak{b}^{-1}$, (a) implies that $\mathfrak{a}\mathfrak{b}^{-1} = (\mathfrak{a}\mathfrak{b}^{-1})_0 R$. □

4.19. Lemma. *Let R be a graded Dedekind ring and consider graded R-lattices M, N such that $M \sim N$ and $M \cong \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n$, $\mathfrak{a}_i \in \mathrm{Frac}_g(R)$. Then there exists $(\mathfrak{b}_i)_i \in \mathrm{Frac}_g(R)$ such that $N \cong \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_n$ and $b_i \sim a_i$ for $i = 1, \ldots, n$.*

Proof. In view of Proposition 4.14 we may suppose that N is of the form $\gamma_1 \oplus \cdots \oplus \gamma_n$ with $(\gamma_i)_i \in \mathrm{Frac}_g(R)$. We use the approximation property of R_0. Choose $t_1, \ldots, t_n \in R_0$ with the following properties:

(a) $t_1 = 1$.

(b) Let $p \in \Omega_g(R)$ be such that the following condition holds: $(\mathfrak{a}_1)_p \neq R_p$ or $(\exists i)$: $(\gamma_i)_p \neq R_p$. Then there exists an i such that $(\gamma_i)_p \cong (\mathfrak{a}_1)_p$ (by 4.14). We require:

$$
\begin{aligned}
v_p(t_i) + v_p(\gamma_i) \leq \mathrm{Min}(&v_p(t_1) + v_p(\gamma_1), \ldots, v_p(t_{i-1}) + v_p(\gamma_{i-1}), \\
&v_p(t_{i+1}) + v_p(\gamma_{i+1}), \ldots, v_p(t_n) + v_p(\gamma_n)).
\end{aligned}
$$

Note that there is only a finite number of p's satisfying the condition.

(c) For primes different from the above we demand that $0 \leq v_p(t_2), \ldots, 0 \leq v_p(t_n)$. Let $\mathfrak{b}_1$ denote $t_1\gamma_1 + \cdots + t_n\gamma_n$. Then it is easy to see that $\mathfrak{b}_1 \sim \mathfrak{a}_1$. Put $N' = \ker(N \xrightarrow{t_1, \ldots, t_n} b_1)$. Thus $N \cong N' \oplus \mathfrak{b}_1$ with $N' \sim \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_n$. The proof proceeds by induction. □

4.20. Lemma. *Suppose that $I_{11}, I_{12}, I_{21}, I_{22} \in \mathrm{Frac}(R)$ where R is a Dedekind ring. If $I_{11}I_{22}+I_{12}I_{21}=R$, then there exists $(a_{ij})_{j=1,2}^{i=1,2}$ with $a_{11}a_{22}-a_{12}a_{21}=1$.*

Proof. Define $A=\{p\in\Omega(R)\mid \exists I_{ij}: v_p(I_{ij})\neq 0\}$. Choose a'_{11}, a'_{22} such that the following conditions hold.

If $p\in A$, then $v_p(a'_{11})=v_p(I_{11})$ and $v_p(a'_{22})=v_p(I_{22})$. If $p\in A^c$, then $v_p(a'_{11})\geq 0$ and $v_p(a'_{22})\geq 0$. Choose a'_{21} and a'_{12} such that: If $p\in A$, then $v_p(a'_{21})=v_p(I_{21})$ and $v_p(a'_{12})=v_p(I_{12})$. If $p\in A^c$ we distinguish two cases:

(a) If $v_p(a'_{11})\neq 0$ or $v_p(a'_{22})\neq 0$, then $v_p(a'_{12})=v_p(a'_{21})=0$.

(b) If $v_p(a'_{11})=v_p(a'_{22})=0$, then $v_p(a'_{12})\geq 0$ and $v_p(a'_{21})\geq 0$.

The choice of the a_{ij}'s guarantees that $Ra'_{11}a'_{22}+Ra'_{12}a'_{21}=R$. The statement of the lemma is a trivial consequence of this. □

4.21. Lemma. *Let R be a graded Dedekind ring. Let $I, J\in \mathrm{Frac}(R_0)$ and $\mathfrak{a},\mathfrak{b}\in \mathrm{Frac}_g(R)$. Then $I\mathfrak{a}\oplus J\mathfrak{b}\cong \mathfrak{a}\oplus IJ\mathfrak{b}$.*

Proof. Let M and N denote $I\mathfrak{a}\oplus J\mathfrak{b}$ and $\mathfrak{a}\oplus IJ\mathfrak{b}$ respectively. We can write $\mathrm{Hom}(M,N)$ as

$$\begin{pmatrix} I^{-1}R & J^{-1}\mathfrak{b}^{-1}\mathfrak{a} \\ J\mathfrak{a}^{-1}\mathfrak{b} & IR \end{pmatrix}$$

Since $(I^{-1}R)_0(IR)_0+(J^{-1}\mathfrak{b}^{-1}\mathfrak{a})_0=R_0$, Lemmas 4.20 and 4.17 imply that there exists a graded isomorphism between M and N. □

We are now ready to prove the main theorem of this paper.

4.22. Theorem. *Consider a graded Dedekind ring R and graded R lattices M and N. The following statements are equivalent.*

(a) $M\cong N$.

(b) $M\sim N$ *and* $\det M\cong \det N$.

(c) $M\sim N$ *and* $M_0\cong N_0$.

(d) $M\sim N$ *and* $\det M_0\cong \det N_0$.

Proof. The equivalence of (c) and (d) is Steinitz's theorem since R_0 is a Dedekind ring. The equivalence of (b) and (c) follows from 4.18. So it remains to establish the equivalence of (a) and (b). To this end we may assume that $M\cong \mathfrak{a}_1\oplus\cdots\oplus\mathfrak{a}_n$. From 4.19 and 4.20 we retain that $N\cong I_1\mathfrak{a}_1\oplus\cdots\oplus I_n\mathfrak{a}_n$ with $I_i\in\mathrm{Frac}(R_0)$. From 4.22 we retain that $N\cong (I_1\cdots I_n\mathfrak{a}_1)\oplus\mathfrak{a}_2\oplus\cdots\oplus\mathfrak{a}_n$. Since $\det M\cong\det N$ we obtain $I_1\cdots I_n=R$. □

4.23. Theorem. (Cancellation for graded Dedekind rings). *Let R be a graded Dedekind ring. If M, N, P are graded R-lattices, then $M\oplus P\cong N\oplus P$ implies $M\cong N$.*

Proof. An easy consequence of 4.22. □

5. Class groups of graded Dedekind rings

In [1] several different classgroups of graded Krull domains have been introduced.

(a) $\mathrm{Cl}_g(R)$: the graded isomorphism classes of graded divisorial R-ideals.

(b) $\mathrm{Cl}^g(R)$: the (ungraded) isomorphism classes of graded divisorial R-ideals.

(c) $\mathrm{Cl}(R)$: the isomorphism classes of divisorial R ideals.

In [1] it has been proved that $\mathrm{Cl}^g(R) \cong \mathrm{Cl}(R)$. We calculate $\mathrm{Cl}^g(R)$ and $\mathrm{Cl}_g(R)$ for graded Dedekind rings.

5.24. Definition. Let R be a graded Dedekind ring. The genus group of R is defined as

$$g(R) = \sum_{p \in \Omega_g(R)} \mathbb{Z}/e_p\mathbb{Z}.$$

$g_0(R)$ is defined to be $g(R)/(\ldots, 1, \ldots, 1, \ldots)$.

5.25. Lemma. *Let R be a graded Dedekind ring. The proof of* 4.7 *implies that $R(e) = IR$ where e is the l.c.m. of the $(e_p)_p \in \Omega_g(R)$ and $I \in \mathrm{Frac}(R_0)$. The following diagram with canonical arrows is exact.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \langle \bar{I} \rangle & \longrightarrow & \langle R(1) \rangle & \longrightarrow & \langle \ldots, 1, \ldots, 1, \ldots \rangle & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Cl}(R_0) & \longrightarrow & \mathrm{Cl}_g(R) & \longrightarrow & g(R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Cl}(R_0)/\langle \bar{I} \rangle & \longrightarrow & \mathrm{Cl}^g(R) & \longrightarrow & g_0(R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. Exactness of the columns and the first row follows by definition. Exactness of the middle row is a consequence of 4.18. Exactness of the 3rd row follows from the snake lemma.

5.26. Corollary. (a) *Suppose that R satisfies the graded approximation property. Then $\mathrm{Cl}^g(R) \cong \mathrm{Cl}(R_0)/\langle I \rangle$. This generalizes a result of* [1].

(b) *Suppose that R is graded semilocal. In this case R is a graded principal ideal domain if and only if R satisfies the graded approximation property.*

Proof. (a) If R satisfies the graded approximation property, then 4.10 implies that $g(R)_0=0$. Hence $\mathrm{Cl}^g(R)\cong \mathrm{Cl}(R_0)/\langle I\rangle$ by 5.24.

(b) If R is semilocal, then R_0 is also semilocal. (b) is then an consequence of 4.10 and 5.25.

6. Appendix: lepidopterous Rees rings

We want to give a structure theorem for graded integrally closed rings containing a generalized Rees ring. The foregoing indicates that such rings may be viewed as generalizations of graded Dedekind rings. Let R_0 be an integrally closed ring and consider $I\in \mathrm{Inv}(R_0)$.

(a) For $p\in \mathbb{Z}$ we define $I^{1/p}$ to be the sum of all ideals J that satisfy $J^p\subset I$. We claim that $(I^{1/p})^p\subset I$. Actually it suffices to check that $J_1^p\subset I$, $J_2^p\subset I$ entails $(J_1+J_2)^p\subset I$. Since $(J_1+J_2)^p=\sum_{i=0}^p J_1^iJ_2^{p-i}$ the problem reduces to the validity of $J_1^iJ_2^{p-i}\subset I$ for $i=1,\dots,p$. Now $(J_1^iJ_2^{p-i}I^{-1})^p\subset (J_1^p)^i(J_2^p)^{p-i}I^{-p}\subset R_0$ and since R_0 is integrally closed this implies that $J_1^iJ_2^{p-i}\subset I$.

(b) Let $\alpha=P/q\in Q$, $q\geq 0$. We define I^α as $(I^p)^{1/q}$. To prove that this is well defined it suffices to show that $(I^p)^{1/q}=(i^{np})^{1/q}=(I^{np})^{1/np}$ for all $n\in \mathbb{N}$. Now $((I^{pn})^{1/qn})^{qn}\subset I^{pn}$ and exactly as in (a) this implies that $((I^{pn})^{1/qn})^q\subset I^p$ and hence $(I^{pn})^{1/qn}\subset I^{p/q}$. The other inclusion follows from $((I^p)^{1/q})^{qr}\subset I^{pn}$. In case R_0 is a Dedekind ring one easily deduces that this definition coincides with the one given in the foregoing.

6.27. Proposion. *Let R_0 be an integrally closed ring. The following properties hold.*

(a) $\forall\alpha,\beta\in Q$: $I^\alpha I^\beta\subset I^{\alpha+\beta}$.

(b) $\forall\alpha\in Q$: $I^\alpha J^\alpha\in (IJ)^\alpha$.

(c) $\forall\alpha,\beta\in Q$: $\alpha<\beta<\gamma: I^\alpha\cap I^\gamma\subset I^\beta$.

(This is a kind of continuity property).

Proof. The proofs are based on elementary techniques. As an example let us prove (d).

(a) Let $p,q,n\in\mathbb{Z}: p\leq n\leq q$. We prove that $I^p\cap I^q\subset I^n$. n may be written as $(ap+bq)/(a+b)$ for some $a,b\in\mathbb{Z}$. Since R is integrally closed it is again sufficient to prove that $(I^p\cap I^q)^{a+b}\subset I^{ap+bq}$. This inclusion follows from $I^{(p-q)b}\cap I^{(q-p)a}\subset R_0$ which is easily proved: take $x\in I^{(p-q)b}\cap I^{(q-p)a}$, then $x^a\in I^{(p-q)ab}$ and $x^b\in I^{(q-b)ab}$. Hence $x^{a+b}\in R$ and so $x\in R$.

(b) Let $\alpha,\beta,\gamma\in Q$, $\alpha\leq\beta\leq\gamma$. We may write $\alpha=p/t$, $\beta=r/t$, $\gamma=q/t$ for some $t\in\mathbb{N}$, $p,n,q\in\mathbb{Z}$. Then $(I^\alpha\cap I^\gamma)^t\subset I^p\cap I^q\subset I^n\subset (I^\beta)^t$. □

The foregoing remarks now lead to the structure theorem.

6.28. Theorem. *Let R be a graded integrally closed ring such that $R^{(e)}$ is a generalized Rees ring for some positive integer e. Then $R \cong \sum_{n\in\mathbb{Z}} I^{n/e}X^n$ with $I \in \mathrm{Inv}(R_0)$ and $e \in \mathbb{N}$. Conversely, every graded ring of this type (with R_0 integrally closed) is integrally closed.*

Proof. Put $Q_g(R) = k[X, X^{-1}]$. We may write R as $\sum_i I_iX^i$ where I_i is a R_0 module in k. The fact that R contains a generalized Rees ring implies that $Q(R_0) = k$ and the I_i's are fractional ideals in k. Since R is integrally closed we have that $x \in h(R)$ if and only if $x^e \in h(R^{(e)})$. So $I_i = \{x \in R_i \mid x^e \in I_{ei}\}$. Now $I_{ei} = I^i$ for a certain $I \in \mathrm{Inv}(R_0)$. So $I_i = \{x \in R_0 \mid x^e \in I^i\} = I^{i/e}$. Hence $R \cong \sum_n I^{n/e}X^n$. The opposite direction is almost obvious for if R' is the integral closure of R, then $R' \cong \sum I^{n/e}X^n$ and hence $R' = R$. So R is integrally closed.

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