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Subsets of $\mathbb{R}$ which support hypergroups with polynomial characters[1]

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Abstract

It is shown that if a hypergroup $(H, *)$, with H an infinite subset of $\mathbb{R}$, has polynomial characters of every degree and if either H is compact or the polynomials are orthogonal with respect to a measure supported on H, then those polynomials are essentially the Jacobi polynomials. Related results are obtained for polynomial product formulas.

Keywords: Jacobi polynomials; Hypergroups; Product formulas; Measure algebras; Orthogonal polynomials

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0. Introduction

The authors are engaged in a program to identify and classify those families of orthogonal polynomials in one or several variables which satisfy a certain type of product formula. The most famous examples are the Chebyshev polynomials and the Legendre polynomials.

The formulas of interest here are called hypergroup product formulas because such a formula gives rise in a natural way to a type of probability preserving measure algebra called a hypergroup. A family of orthogonal polynomials has a hypergroup product formula if and only if the homomorphisms of the hypergroup (which by the Riesz representation theorem are given by bounded functions) are given by the polynomials. (All these notions are discussed in detail below.)

Two examples of families of polynomials which have product formulas not of hypergroup type are also described.

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Whether or not a family of orthogonal polynomials has a hypergroup product formula depends in a strong way on the geometry of the support of the orthogonality measure. Of course, only some measures supported on such a set will yield orthogonal polynomials with the desired product formula. There are but a few (up to affine transformation) sets in each dimension which are at this time known to support such structures. A consequence of the results below is that the only such sets in $\mathbb{R}$ are compact intervals; the authors so far know of five (and their affine images) in $\mathbb{R}^2$ (these are discussed in [5]).

The results presented here are essential for the authors' results in the more complicated two-variable case which is discussed in [5]; the sharpness of those results (and others in higher dimensions) depend in a strong way on the sharpness of the one-variable result.

Now, one of the hypergroup axioms requires that the Banach algebra have an identity in the form of a unit point mass at some point e in H. It follows from a result in the article by the authors and Markett [3, Theorem 5.3] that the condition that H be a real interval can be weakened to require only that H be a subset of $\mathbb{R}$ and that e be an accumulation point of H. That article also contains a more detailed proof than the one given in [4]. (A similar result is also the essential content of [17, Theorem 5.3].)

In this article we now remove all restrictions on the set H except that it be an infinite compact subset of $\mathbb{R}$ (Theorem 1); the hypothesis of compactness can be replaced by the assumption that the polynomials be orthogonal with respect to a measure on H. The plan of the paper is as follows: Section 1 which contains Theorem 1, also includes definitions and the example of the Jacobi polynomials. Section 2 is devoted to product formulas. Two additional examples of product formulas are given: the generalized Chebyshev polynomials and the continuous q-ultraspherical polynomials. These examples illuminate the conditions given in Theorems 2 and 3. Section 3 is devoted to the question of which hypergroups on $\{0, 1, 2, \dots\}$ have duals and contains a result which is an improvement on [17, Theorem 5.3]. Section 4 contains the proofs of Theorems 1–3.

The situation for polynomials of two or more variables is richer. For instance, with one variable, a positive measure with finite moments determines a family of orthogonal polynomials which is unique up to multiplicative constants; with two or more variables, certain subspaces of polynomials are determined uniquely, but each subspace admits infinitely many different orthogonal bases. Nevertheless, the work here has been generalized to polynomials of two and more variables, and the results will be presented elsewhere.

1. Hypergroups

In this article, the term "hypergroup" is synonymous with "convo" as defined by Jewett [11, Section 4] and with the "DJS-hypergroup", a term recently coined by Litvinov and Ross (DJS = Dunkl, Jewett, Spector).

Let H be a locally compact Hausdorff space, let $M(H)$ denote the bounded regular real-valued Borel measures on H with $\|\cdot\|$ denoting the total variation norm; we use $M_{\mathrm{p}}(H)$ for the probability measures on H. $C(H)$ denotes the continuous functions on H, and $C_{\mathrm{c}}(H)$ the members of $C(H)$ with compact support. We write $\mathrm{supp}(\mu)$ for the support of μ and δ_t for the unit point mass supported at t.

If $M(H)$ is a Banach algebra with multiplication $*$ (usually called a *convolution*), then $(H, *)$ is a *hypergroup* if the following axioms hold (these are essentially identical to Jewett's definition of a *convo* in [11]; that article should be consulted for the terminology used in the axioms).

Axiom 1. A convolution of probability measures is a probability measure.

Axiom 2. The mapping $(\mu, \nu) \to \mu * \nu$ is positive-continuous from $M(H) \times M(H)$ into $M(H)$. (If H is compact, this is equivalent to the mapping being weak-$*$ continuous.)

Axiom 3. There is an element $e \in H$ such that $\delta_e * \mu = \mu * \delta_e = \mu$ for every $\mu \in M(H)$.

Axiom 4. There is a homeomorphic mapping $s \to s^\vee$ of H into itself such that $s^{\vee\vee} = s$ and $e \in \operatorname{supp}(\delta_s * \delta_t)$ if and only if $t = s^\vee$.

Axiom 5. For $\mu, \nu \in M(H)$ $(\mu * \nu)^\vee = \nu^\vee * \mu^\vee$, where $\mu^\vee$ is defined by

$$\int_H f(s)\,\mathrm{d}\mu^\vee(s) = \int_H f(s^\vee)\,\mathrm{d}\mu(s).$$

Axiom 6. The mapping $(s,t) \to \operatorname{supp}(\delta_s * \delta_t)$ is continuous from $H \times H$ into the space of compact subsets of H as topologized in [13]; see [6, 11, 14] for more about hypergroups.

A *character* for $(H, *)$ is a bounded continuous nonzero function ϕ on H such that for every $s \in H$, $\phi(s^\vee) = \overline{\phi(s)}$ and

$$\int_H \phi\,\mathrm{d}(\delta_s * \delta_t) = \phi(s)\phi(t) \quad (s, t \in H),$$

the set of characters is denoted by $H\hat{}$.

We say $\mathscr{P} = \{p_n\}_{n \in \mathbb{N}_0}$ ($\mathbb{N}_0 = \{0, 1, 2, \ldots\}$) is an *algebraically complete family of polynomials* if for every $n \in \mathbb{N}_0$, p_n has exact degree n. We say $(H, *)$ is a *continuous polynomial hypergroup* if H is a subset of $\mathbb{R}$ and if $H\hat{}$ contains an algebraically complete family of polynomials. (The term *continuous* is used because these hypergroups are associated with the continuous variable x in $p_n(x)$; we will discuss the role of the *discrete* variable n below.)

We say that $\mathscr{P}$ has a *product formula on the set* $H \subset \mathbb{R}$ if for each s and t in H, there is $\mu_{s,t} \in M(H)$ such that

$$\int_H p_n\,\mathrm{d}\mu_{s,t} = p_n(s)p_n(t) \tag{1}$$

for every $n \in \mathbb{N}_0$. We say (1) is a *positive product formula* if $\mu_{s,t}$ is a nonnegative measure for every $s, t \in H$.

Two algebraically complete families of polynomials $\mathscr{P} = \{p_n\}_{n \in \mathbb{N}_0}$ and $\mathscr{Q} = \{q_n\}_{n \in \mathbb{N}_0}$ are *linearly equivalent* (written $\mathscr{P} \cong \mathscr{Q}$) if there are real numbers a and b such that $q_n(t) = p_n(at + b)$ ($n \in \mathbb{N}_0$). We say two hypergroups $(H, *)$ and $(K, \circ)$ with $H, K \subset \mathbb{R}$ are *linearly equivalent* (written $(H, *) \cong (K, \circ)$)

if there is an affine function $h(r) = ar + b$ with constants $a \neq 0$ and b such $h(H) = K$ and

$$\int_K f(z)\,\mathrm{d}(\delta_x \circ \delta_y)(z) = \int_H f(h(r))\,\mathrm{d}(\delta_s * \delta_t)(r)$$

for every $f \in C(K)$ where $s = h^{-1}(x)$ and $t = h^{-1}(y)$. Clearly, if $\hat{H} = \mathscr{P}$ and $\hat{K} = \mathscr{Q}$, then $(H, *) \cong (K, \circ)$ if and only if $\mathscr{P} \cong \mathscr{Q}$.

Example 1 (*Jacobi polynomials*). An important class of product formulas was established by Gasper [9] for the normalized Jacobi polynomials

$$R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1),$$

which are orthogonal with respect to a measure supported on $[-1, 1]$ (see [9]). These include Chebyshev, Legendre, and ultraspherical or Gegenbauer polynomials as special cases. We write

$$\mathscr{R}^{(\alpha,\beta)} = \{R_n^{(\alpha,\beta)}(x)\}_{n \in \mathbb{N}_0}$$

for this system. The measures $\mu_{s,t} = \mu_{s,t}^{(\alpha,\beta)}$ are all positive if and only if (α, β) belongs to

$$E_J = \{(\alpha, \beta)\colon \alpha \geqslant \beta > -1 \text{ and either } \beta \geqslant -\tfrac{1}{2} \text{ or } \alpha + \beta \geqslant 0\}.$$

We note for later reference that $\mu_{s,1} = \delta_s$ for every $s \in [-1, 1]$, and that $(s, t) \to \operatorname{supp}(\mu_{s,t})$ is a continuous mapping. Actually, Gasper obtains explicit absolutely continuous measures $\mu_{s,t}$ for $-1 < s, t < 1$, but the product formula is readily extended to $-1 \leqslant s, t \leqslant 1$. (See Laine's comment [12, pp. 136–137].) Implicit in Gasper's article is that for each $(\alpha, \beta) \in E_J$ there is a hypergroup, which we denote $J(\alpha, \beta)$ with $H = [-1, 1]$, $e = 1$, $x^\vee = x$, and character set $\mathscr{R}^{(\alpha,\beta)}$.

The following result completely describes the category of continuous polynomial hypergroups.

Theorem 1. *Suppose that $(H, *)$ is a hypergroup where H is an infinite subset of $\mathbb{R}$ and that $\hat{H}$ contains an algebraically complete family of polynomials $\mathscr{P}$. Assume one of the following holds.*

(i) *H is compact.*

(ii) *$\mathscr{P}$ is orthogonal with respect to a positive Borel measure on H. Then $(H, *)$ is linearly equivalent to $J(\alpha, \beta)$ for some $(\alpha, \beta) \in E_J$.*

Remark 1. The requirement that H be infinite is inserted since there are finite hypergroups with polynomial characters (for instance, those associated with Krawtchouk polynomials [7] or the Chebyshev polynomials of degree $\leqslant N$ on the set $\{x_j\colon j = 0, \ldots, N\}$, where $x_j = \cos(j\pi/N)$). We cannot simply require that $\mathscr{P}$ contain infinitely many polynomials, for there is a hypergroup on $H = [0, 1]$ with the set of even polynomials $\{R_n^{(\alpha,\beta)}(2x^2 - 1)\colon n \in \mathbb{N}_0\}$ as characters.

Remark 2. The hypergroup immediately above is equivalent by the nonlinear change of variables $t = 2x^2 - 1$ to $J(\alpha, \beta)$. This suggests the following.

Questions. (1) To what extent can the hypothesis of algebraic completeness be weakened if linear equivalence of hypergroups is replaced by a more general notion of equivalence?

(2) If all the characters of a hypergroup are polynomials (but not necessarily including one of every degree) what can be concluded?

2. Product formulas

Actually, the proof of Theorem 1 does not require all the hypergroup axioms. In this section we relate measure algebras to product formulas and determine which algebraically complete families of orthogonal polynomials have product formulas of a prescribed type. For instance we have:

Theorem A [3, Theorem 5.3]. *Suppose a family $\mathscr{P}$ of orthogonal polynomials has a positive product formula on a set $H \subset \mathbb{R}$ and satisfying the following two conditions.*

(i) *There is $e \in H$ such that for every $t \in H$, $\mu_{e,t}$ is concentrated on a single point $v(t)$.*

(ii) *e is an accumulation point of H, and for every $n > 2$ and $t \in H$*

$$\lim_{s \to e,\, s \in H} \frac{1}{s-e} \int_H [r - v(t)]^n \, d\mu_{s,t}(r) = 0.$$

Then $\mathscr{P} \cong \mathscr{R}^{(\alpha,\beta)}$ for some $(\alpha, \beta) \in E_J$.

(The condition $H \subset \mathbb{R}$ is not explicitly stated in [3], but it is required in the proof.)

Assume that an algebraically complete family of polynomials $\mathscr{P} = \{p_n\}_{n \in \mathbb{N}_0}$ has a product formula on a set H, and assume that there is $A > 0$ such that $\|\mu_{s,t}\| \leqslant A$ for every $s, t \in H$. Define the operation $*$ on $M(H)$ as follows: if ν and $\lambda \in M(H)$, then $\nu * \lambda$ is defined by its action on $f \in C_c(H)$ by

$$\int_H f \, d(\nu * \lambda) = \int_H \int_H \left[\int_H f \, d\mu_{s,t} \right] d\nu(s) \, d\lambda(t); \tag{2}$$

this is equivalent to setting

$$\delta_s * \delta_t = \mu_{s,t}. \tag{3}$$

If we define

$$\nu\hat{}(n) = \int_H \bar{p}_n \, d\nu \quad (n \in \mathbb{N}_0) \tag{4}$$

Eq. (2) entails

$$(\nu * \lambda)\hat{} = \nu\hat{}\lambda\hat{}. \tag{5}$$

Lemma 1. *If H is compact, the operation $*$ is commutative, associative, and continuous on $M(H)$ with respect to the total variation norm.*

Proof. If $f \in \mathscr{P}$

$$\int_H f \, d(\delta_x * \delta_y) = \int_H f \, d(\delta_y * \delta_x) \quad (x, y \in H), \tag{6}$$

so if f is any polynomial, it is in the linear span of $\mathscr{P}$, hence Eq. (6) still holds for f. Finally, Eq. (6) is valid for each $f \in C(H)$ by the Stone–Weierstrass Theorem. Thus $\delta_x * \delta_y = \delta_y * \delta_x$ and commutativity follows. A similar argument establishes associativity. Eq. (2) leads directly to $\|\nu * \lambda\| \leqslant A\|\nu\| \cdot \|\lambda\|$. $\square$

If $(M(H), *)$ is a Banach algebra, we refer to it as the *Banach algebra associated with the product formula* (1). If $(M(H), *)$ is a hypergroup, we say that (1) is a *hypergroup product formula.*

We need another definition: The product formula (1) has an *identity element* e if $\mu_{t,e} = \delta_t$ for every $t \in H$.

Many algebraically complete families of polynomials have product formulas; we cite two additional examples.

Example 2 (*Generalized Chebyshev polynomials*). Laine [12] established product formulas for the generalized Chebyshev polynomials given by

$$T_n^{(\alpha,\beta)}(x) = \begin{cases} R_k^{(\alpha,\beta)}(2x^2-1) & \text{if } n = 2k, \\ xR_k^{(\alpha,\beta+1)}(2x^2-1) & \text{if } n = 2k+1, \end{cases}$$

which are orthogonal on $[-1,1]$ with respect to the measure

$$\mathrm{d}m^{(\alpha,\beta)}(x) = (1-x^2)^\alpha |x|^{2\beta+1}.$$

The product formula has identity element 1 and it is positive provided that $\alpha > \beta \geqslant -\frac{1}{2}$. Denoting the measures in the product formula here also by $\mu_{s,t} = \mu_{s,t}^{(\alpha,\beta)}$ (this is a different measure than in Example 1) we have $\mu_{s,1} = \delta_s$. (Actually, Laine obtains explicit absolutely continuous measures $\mu_{s,t}$ for $-1 < s,t < 1$, $st \neq 0$ but the product formula is readily extended to $-1 \leqslant s,t \leqslant 1$.) We also note that if $\beta > -\frac{1}{2}$ and $-1 < s,t < 1$ then $\operatorname{supp}(\mu_{s,t})$ is a set symmetric about 0 (see [12, Eq. (2.2)]); thus the measure algebra associated with the generalized Chebyshev polynomials does not satisfy hypergroup Axioms 4 and 6, so it is not a hypergroup. However, $L^1([-1,1], m^{(\alpha,\beta)})$ is a hypercomplex system in the sense of Berezanskii and Kalyuzhnyi ([2, Theorem 3.7, p. 216]).

Example 3 (*Continuous q-ultraspherical polynomials*). These polynomials introduced in [1] are denoted $C_n(x;\beta|q)$. We fix β and q in $(0,1)$. The continuous q-ultraspherical polynomials are orthogonal with respect to a measure with support $[-1,1]$. The normalized continuous q-ultraspherical polynomials are defined by

$$R_n(x;\beta|q) = C_n(x;\beta|q)/C_n(e;\beta|q) \quad (n \in \mathbb{N}_0),$$

where

$$e = (\beta^{1/2} + \beta^{-1/2})/2 > 1.$$

These have a product formula on $[-1,1]$ (see [10, Eq. (8.4.1)] and [1, Eq. (3.23)]) with nonnegative measures $\mu_{s,t} = \mu_{s,t}^{\beta|q}$ with the property that

$$\operatorname{supp}(\mu_{s,t}) = [-1,1] \quad (s,t \in [-1,1]). \tag{7}$$

The product formula can be extended to $H = [-1,1] \cup \{\pm e\}$ by defining

$$\mu_{e,t} = \delta_t \quad \text{and} \quad \mu_{-e,t} = \delta_{-t};$$

e is the identity element for this product formula. The measure algebra associated with the q-ultraspherical polynomials is not a hypergroup because for $-1 \leqslant s \leqslant 1$, there is no $s^\vee$ which satisfies Axiom 4.

Theorems 2 and 3 give characterizations of the Jacobi polynomials in terms of their product formulas. Each theorem contains a set of conditions (labeled "B") on a product formula which are weaker than the assumption that the Banach algebra associated with the product formula is a hypergroup. The conditions in Theorem 3 are more general, but more technical than those in Theorem 2. Theorem 3 contains Theorem 2 as a special case. We shall use the notation: $N_\varepsilon(t) = (t - \varepsilon, t + \varepsilon)$.

Theorem 2. *The following are equivalent for an algebraically complete family $\mathscr{P}$ of orthogonal polynomials.*

(A) $\mathscr{P} \cong \mathscr{R}^{(\alpha,\beta)}$ *for some* $(\alpha,\beta) \in E_J$.

(B) $\mathscr{P}$ *has a positive product formula on an infinite compact set* $H \subset \mathbb{R}$ *with identity* e *that satisfies the following two conditions*:

(i) *corresponding to every* $t \in H$ *and* $\varepsilon > 0$ *there is* $\eta > 0$ *such that if* $s \in N_\eta(e) \cap H$ *then* $\operatorname{supp}(\mu_{s,t}) \subset N_\varepsilon(t)$, *and*

(ii) *for every* $t \in H$, $e \in \operatorname{supp}(\mu_{t,t})$.

Remark 1. That condition (i) is required can be shown by considering the case of the generalized Chebyshev polynomials (Example 2) with $\alpha \geqslant \beta > -\frac{1}{2}$. In this case, if $-1 < s,t < 1$, then $\operatorname{supp}(\mu_{s,t})$ is symmetric about 0, while $\operatorname{supp}(\mu_{1,t}) = \operatorname{supp}(\delta_t) = \{t\}$, thus $\operatorname{supp}(\mu_{s,t})$ does not converge to $\{t\}$ in the space of compact subsets of H as $s \to e$.

Remark 2. That condition (ii) is required is shown by the example of the continuous q-ultraspherical polynomials, because if $t \in [-1,1]$, then $e \notin \operatorname{supp}(\mu_{t,t}) = [-1,1]$.

We need an additional definition for Theorem 3: If A and B are subsets of H, then $A \cdot B$ is the closure of $\bigcup \{\operatorname{supp}(\mu_{s,t})\colon s \in A \text{ and } t \in B\}$.

Theorem 3. *The following are equivalent for an algebraically complete family $\mathscr{P}$ of orthogonal polynomials.*

(A) $\mathscr{P} \cong \mathscr{R}^{(\alpha,\beta)}$ *for some* $(\alpha,\beta) \in E_J$.

(B) $\mathscr{P}$ *has a positive product formula on an infinite compact set* $H \subset \mathbb{R}$ *with identity element* e *that satisfies the following two conditions*:

(i) *for every* $n > 2$ *and* $t \in H$

$$\lim_{s \to e,\, s \in H} \frac{1}{s - e} \int_H (r - t)^n \, \mathrm{d}\mu_{s,t}(r) = 0, \text{ and}$$

(ii) *the only nonempty compact subset* K *of* H *which satisfies* $K \cdot H \subset K$ *is* $K = H$.

Remark 3. In the case of the continuous q-ultraspherical polynomials (Example 3), $K = [-1,1]$ has the property that $K \neq H$ but $K \cdot H \subset K$.

3. Discrete polynomial hypergroups and duality

A family $\mathscr{P} = \{p_n\}_{n \in \mathbb{N}_0}$ of orthogonal polynomials may also play the role of characters of a hypergroup $(\mathbb{N}_0, \circ)$, which we denote $\mathbb{N}_0(\mathscr{P})$, by defining a product $\circ$ on $\ell^1(=M(\mathbb{N}_0))$ with the property that if u and v belong to ℓ^1

$$w = u \circ v \leftrightarrow \sum_{n=0}^{\infty} w_n p_n(x) = \left(\sum_{n=0}^{\infty} u_n p_n(x)\right)\left(\sum_{n=0}^{\infty} v_n p_n(x)\right)$$

for all x such that $\{p_n(x)\}$ is bounded. We use the term *discrete polynomial hypergroup* to refer to such structures in contrast to the continuous polynomial hypergroups defined above. Many researchers reserve the term "polynomial hypergroups" to this class of objects. There are discrete polynomial hypergroups which we denote by $\hat{J}(\alpha, \beta)$ associated with the Jacobi polynomials for (α, β) belonging to a strictly larger set than E_J (see [9]). If $(\alpha, \beta) \in E_J$, we can say that $J(\alpha, \beta)$ and $\hat{J}(\alpha, \beta)$ have each other as dual hypergroups. There are also discrete polynomial hypergroups associated with the generalized Chebyshev polynomials (Szwarc [16]), and the q-continuous ultraspherical polynomials for $0 < |\beta|, q < 1$ (see [10, Section 8.5] and the references given there). In fact, the category of discrete polynomial hypergroups is quite large; for instance, Szwarc [15, 16] gives simple conditions on the recurrence relation for a family $\mathscr{P}$ of orthogonal polynomials which ensure that $\mathbb{N}_0(\mathscr{P})$ is a hypergroup. This is in contrast to the category of continuous polynomial hypergroups which is completely described by Theorem 1. Consequently, most discrete polynomial hypergroups will not support a theory of harmonic analysis which relies on any analog for hypergroups of Pontryagin duality. Indeed, Zeuner [17] offers an answer to the question:

Which discrete polynomial hypergroups $\mathbb{N}_0(\mathscr{P})$ have the property that there is a continuous polynomial hypergroup $(H, *)$ with characters $\mathscr{P}$?

His answer ([17, Theorem 5.3]) is that if the identity element e is not an isolated point of H, then $\mathbb{N}_0(\mathscr{P}) = \hat{J}(\alpha, \beta)$ for some $(\alpha, \beta) \in E_J$. Theorem 1 has the effect of removing the topological restriction to yield:

Theorem 4. *If $\mathbb{N}_0(\mathscr{P})$ is a discrete polynomial hypergroup, and if there is a continuous polynomial hypergroup $(H, *)$ with H an infinite subset of $\mathbb{R}$ and $\hat{H} \supset \mathscr{P}$, then $\mathbb{N}_0(\mathscr{P}) = \hat{J}(\alpha, \beta)$ for some $(\alpha, \beta) \in E_J$.*

In other words, the only discrete polynomial hypergroups which have dual hypergroups in any sense of the word are $\hat{J}(\alpha, \beta)$ for (α, β) belonging to E_J.

4. Proof of Theorems 1–3

We begin with a technical lemma.

Lemma 2. *Suppose the family of polynomials $\mathscr{P}$ has a positive product formula on $H \subset \mathbb{R}$ with identity e, and suppose that corresponding to every $t \in H$ and $\varepsilon > 0$ there is $\eta > 0$ such that*

$$\operatorname{supp}(\mu_{s,t}) \subset N_\varepsilon(t) \quad (s \in N_\eta(e) \cap H), \tag{8}$$

then for every $n > 2$ and $t \in H$

$$\lim_{s \to e,\, s \in H} \frac{1}{s - e} \int_H (r - t)^n \, \mathrm{d}\mu_{s,t}(r) = 0.$$

Proof. Let

$$M_n(s,t) = \int_H (r-t)^n \, d\mu_{s,t}(r),$$

and suppose that $n > 0$, then $M_n(e,t) = 0$. Now, since we can write

$$(r-t)^n = \sum_{k=0}^{n} c_k(t) p_k(r),$$

we use the fact that each p_k is a character to see that

$$M_n(s,t) = \sum_{k=0}^{n} c_k(t) p_k(s) p_k(t)$$

is a polynomial which satisfies $M_n(e,t) = 0$. Thus

$$M_n(s,t) = (s-e) Q_n(s,t)$$

for some polynomial Q. Now if $n > 2$ and $\varepsilon > 0$, we have

$$M_n(s,t) = \int_H (r-t)^{n-2}(r-t)^2 \, d\mu_{s,t}(r).$$

The hypotheses imply that there is $\eta > 0$ such that $|s-e| < \eta$ yields

$$|M_n(s,t)| \leqslant \varepsilon^{n-2} M_2(s,t) = \varepsilon^{n-2} |(s-e) Q_2(s,t)|,$$

and the lemma follows. $\square$

Proof of Theorem 1. Let $(H, *)$ be a hypergroup, and let $\mathscr{P}$ be an algebraically complete family of polynomials contained in $\hat{H}$. We begin with the assumption that H is an infinite compact subset of $\mathbb{R}$. The first part of the proof is to show that the hypotheses of Theorem A are satisfied with $v(t) = t$. This will require three steps: (1) $\mathscr{P}$ is an orthogonal family, (2) e is an accumulation point of H, and (3) the limit relation in condition (ii) of Theorem A holds.

Step 1: Since $(H, *)$ is compact, it has a Haar measure m and $\operatorname{supp}(m) = H$ [11, Theorem 7.2A]. We now show that m is the orthogonality measure for $\mathscr{P}$. The following relation holds for f and g in $C(H)$ (this is a simple extension of [11, Theorem 5.1D]):

$$\int_H \int_H f(z) \, d(\delta_x * \delta_y)(z) \bar{g}(y) \, dm(y) = \int_H f(y) \int_H \bar{g}(z) \, d(\delta_x^{\vee} * \delta_y)(z) \, dm(y).$$

Suppose that f and g are distinct Hermitian characters so that $\bar{g}$ is also a Hermitian character and so

$$\int_H f(x) f(y) \bar{g}(y) \, dm(y) = \int_H f(y) \bar{g}(x^{\vee}) \bar{g}(y) \, dm(y),$$

whence

$$[f(x) - g(x)] \int_H f(y) \bar{g}(y) \, dm(y) = 0$$

and the orthogonality of $\mathscr{P}$ follows.

Step 2: If e is isolated, then H is discrete ([11, Theorem 7.1B]), but this is impossible since H is an infinite compact set.

Step 3: If $\varepsilon > 0$, then if $U = N_\varepsilon(t) \cap H$

$$\mathscr{C}_U(U) = \{C\colon C \text{ is a compact set in } H \text{ such that } C \subset N_\varepsilon(t)\}$$

is a neighborhood of $\{t\}$ in the topology of compact subsets of H [11, Section 2.5]. Thus there is $\eta > 0$ such that (8) holds and so the limit relation in (A) follows from Lemma 2. Thus we invoke Theorem A to conclude that $\mathscr{P} \cong \mathscr{R}(\alpha, \beta)$.

We must still show that $(H, *)$ and $J(\alpha, \beta)$ are equivalent. Since $\mathscr{P} \cong \mathscr{R}(\alpha, \beta)$, there will be no loss of generality in assuming that $\mathscr{P} = \mathscr{R}(\alpha, \beta)$. The family $\mathscr{R}(\alpha, \beta)$ is orthogonal with respect to the Haar measure of both $(H, *)$ and $J(\alpha, \beta)$, hence $H = [-1, 1]$. Now, with $\circ$ denoting the convolution for $J(\alpha, \beta)$ we have for all $p \in \mathscr{P}$ that $\int p\,\mathrm{d}(\delta_x * \delta_y) = p(x)p(y) = \int p\,\mathrm{d}(\delta_x \circ \delta_y)$, so $* = \circ$ and $(H, *) = J(\alpha, \beta)$.

Instead of asserting H to be compact, let us assume that $\mathscr{P}$ is orthogonal with respect to some $\mu \in M(H)$. Since $(H, *)$ has polynomial characters and any character must be a bounded function, the only way that a polynomial can be bounded on H is for H to be a bounded set. Thus μ has compact support. Hence for each $x \in H$, $\mu * \delta_x$ has compact support (see [11, (3.2G)]), and for each $n > 0$

$$\int p_n\,\mathrm{d}(\mu * \delta_x) = \left(\int p_n\,\mathrm{d}\mu\right) p_n(x) = 0 = \int p_n\,\mathrm{d}\mu,$$

while for $n = 0$ (since $p_0 \equiv 1$)

$$\int p_0\,\mathrm{d}(\mu * \delta_x) = \left(\int p_0\,\mathrm{d}\mu\right) p_0(x) = \int p_0\,\mathrm{d}\mu$$

thus $\mu * \delta_x = \mu$ by the Stone–Weierstrass Theorem, and similarly $\delta_x * \mu = \mu$, so μ is Haar measure and it is finite. Since Haar measure is finite, H is compact ([11, Theorem 7.2B]), and the result follows by Steps 2 and 3. □

Proof of Theorem 3. Suppose that (B) holds. Theorem A yields the conclusion under the hypothesis that the element e is an accumulation point of H. Theorem 3 will be proved by showing that this is the case.

The Banach algebra $(M(H), *)$ associated with the product formula is not immediately a hypergroup but it does satisfy the axioms of the more general type of measure algebra which Dunkl calls a hypergroup in [6]. It follows from [6, Theorem 1.12] that there is a positive measure m of unit total variation which satisfies $m * \delta_t = m$ for every $t \in H$, and that $K = \operatorname{supp}(m)$ is the smallest compact subset of H such that $K \cdot H \subset H$. Thus (ii) implies that $\operatorname{supp}(m) = H$. We claim m is also the spectral measure for $\mathscr{P}$.

For $\nu \in M(\mathbb{R})$ define

$$\hat{\nu}(n) = \int p_n\,\mathrm{d}\nu \quad (n \in \mathbb{N}_0).$$

Let μ be the spectral measure of $\mathscr{P}$ normalized to unit total variation. Then $\mu\hat{}(0) = 1$ and $\mu\hat{}(n) = 0$ for $n > 1$. Observe

$$m\hat{}(n) = \int_H p_n \,\mathrm{d}m = \int_H p_n \,\mathrm{d}(m * \delta_t)$$
$$= \left(\int_H p_n \,\mathrm{d}m\right)\left(\int_H p_n \,\mathrm{d}(\delta_t)\right) = m\hat{}(n) p_n(t),$$

so that $m\hat{}(n) = \mu\hat{}(n)$ for every $n \in \mathbb{N}_0$, thus $m = \mu$.

Now assume by way of contradiction that e is an isolated point of H. Since e is isolated in $H = \operatorname{supp}(m)$, it follows that $0 < \gamma = m(\{e\}) < 1$. Let $H' = H - \{e\}$ and define f by setting $f(e) = 0$ and $f(x) = 1$ if $x \in H'$. Then the Plancherel Theorem yields:

$$\int_H |f|^2 \,\mathrm{d}m = \sum_{n \in \mathbb{N}_0} |f\hat{}(n)|^2 h_n, \tag{9}$$

where

$$f\hat{}(n) = \int_H f \bar{p}_n \,\mathrm{d}m \quad \text{and} \quad h_n = \left(\int_H |p_n|^2 \,\mathrm{d}m\right)^{-1}.$$

Now observe that since $\|\mu_{s,t}\| = 1$, the product formula yields $\|p_n\|_\infty^2 \leqslant \|p_n\|_\infty$ so that $\|p_n\|_\infty \leqslant 1$ for all $n \in \mathbb{N}_0$. Thus $h_n \geqslant 1$. Now for $n > 0$, we have

$$0 = \int_H p_n \,\mathrm{d}m = \gamma + \int_{H'} p_n \,\mathrm{d}m = \gamma + f\hat{}(n),$$

so $f\hat{}(n) = -\gamma$ and the required contradiction is obtained because the left-hand side of (9) is finite but right-hand side diverges. □

Proof of Theorem 2. We will show that conditions (B) of Theorem 2 imply conditions (B) of Theorem 3. Lemma 2 shows that condition (i) of Theorem 2 implies (i) of Theorem 3.

To establish (ii) of Theorem 3, assume condition (ii) of Theorem 2 holds, and that $K \cdot H = K$ for some K with $\emptyset \neq K \subset H$. Let $t \in K$, then $e \in \operatorname{supp}(\mu_{t,t})$ so $e \in K \cdot H$, hence $e \in K$, and so $K = K \cdot H = H$. □

References

[1] R. Askey and M.E.H. Ismail, A generalization of ultraspherical polynomials, in: P. Erdős, Ed., *Studies in Pure Mathematics* (Birkhäuser, Boston, MA, 1983) 55–78.

[2] Y.M. Berezanskii and A.A. Kalyuzhnyi, *Harmonic Analysis in Hypercomplex Systems* (Academia Nauk Ukranii, Institut Matematekii, Kiev Naukova Dumka, Kiev, 1992, in Russian).

[3] W.C. Connett, C. Markett and A.L. Schwartz, Jacobi polynomials and related hypergroup structures, in: H. Heyer, Ed., *Probability Measures on Groups X* (Plenum Press, New York, 1991) 45–81.

[4] W.C. Connett and A.L. Schwartz, Product formulas, hypergroups, and the Jacobi polynomials, *Bull. Amer. Math. Soc.* **22** (1990) 91–96.

[5] W.C. Connett and A.L. Schwartz, Continuous 2-variable polynomial hypergroups, in: *Applications of Hypergroups and Related Measure Algebras*, Contemp. Math. **183** (Amer. Math. Soc., Providence, RI, 1995) 89–109.
[6] C. Dunkl, The measure algebra of a locally compact hypergroup, *Trans. Amer. Math. Soc.* **179** (1973) 331–348.
[7] G.K. Eagleson, A characterization theorem for positive definite sequences on the Krawtchouk polynomials, *Austral. J. Statist.* **11** (1) (1969) 29–38.
[8] G. Gasper, Linearization of the product of Jacobi polynomials. II, *Can. J. Math.* **32** (1970) 582–593.
[9] G. Gasper, Banach algebras for Jacobi series and positivity of a kernel, *Ann. Math.* **95** (1972) 261–280.
[10] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications (Cambridge University Press, Cambridge, 1990).
[11] R.I. Jewett, Spaces with an abstract convolution of measure, *Adv. Math.* **18** (1975) 1–101.
[12] T.P. Laine, The product formula and convolution structure for the generalized Chebyshev polynomials, *SIAM J. Math. Anal.* **11** (1980) 133–146.
[13] E. Michael, Topologies on spaces of subsets, *Trans. Amer. Math. Soc.* **71** (1951) 152–182.
[14] R. Spector, Measures invariantes sur les hypergroupes, *Trans. Amer. Math. Soc.* **239** (1978) 147–165.
[15] R. Szwarc, Orthogonal polynomials and a discrete boundary value problem I, *SIAM J. Math. Anal.* **23** (1992) 959–964.
[16] R. Szwarc, Orthogonal polynomials and a discrete boundary value problem II, *SIAM J. Math. Anal.* **23** (1992) 965–969.
[17] H. Zeuner, Duality of commutative hypergroups, in: H. Heyer, Ed., *Probability Measures on Groups X* (Plenum Press, New York, 1991) 467–488.