

## Semi-invariants of Finite Reflection Groups

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Let  $G$  be a finite group of complex  $n \times n$  unitary matrices generated by reflections acting on  $\mathbb{C}^n$ . Let  $R$  be the ring of invariant polynomials, and let  $\chi$  be a multiplicative character of  $G$ . Let  $\Omega^\chi$  be the  $R$ -module of  $\chi$ -invariant differential forms. We define a multiplication in  $\Omega^\chi$  and show that under this multiplication  $\Omega^\chi$  has an *exterior algebra* structure. We also show how to extend the results to vector fields, and exhibit a relationship between  $\chi$ -invariant forms and logarithmic forms. © 1999 Academic Press

### 1. INTRODUCTION

In 1989, Doyle and McMullen [2] solved the fifth degree polynomial with a highly symmetrical dynamical system which preserved the Galois group  $A_5$ . In 1997, Crass and Doyle [1] solved the sixth degree polynomial by again finding a dynamical system with special symmetry—this time  $A_6$  symmetry. Each dynamical system was formed by iterating a map that was equivariant under the projective action of the group. Such maps correspond naturally to semi-invariant differential forms. Because almost nothing was known about these forms, constructing the necessary dynamical systems was a difficult step in both cases.

We introduce here a general theory of semi-invariants. Specifically, we show that for any finite unitary reflection group  $G$  and multiplicative character  $\chi$  of  $G$ , the module of  $\chi$ -invariant differential forms has a natural multiplication which turns the module into an *exterior algebra*. This exterior algebra structure allows us to understand completely the forms that give rise to highly symmetrical dynamical systems and gives us tools to compute these forms explicitly. We also show how to extend these results



to vector fields (or *derivations*), and observe the relationship between semi-invariants and logarithmic forms.

The theory presented here builds on work by Stanley, who characterized the module of  $\chi$ -invariant polynomials in 1977 [8]. It also builds on more recent work by Orlik, Saito, Solomon, Terao, and others on invariant derivations and the theory of hyperplane arrangements (see [3, Chapter 6]). Note that det-invariant forms have received attention under the name of *anti-invariant forms* in the context of Coxeter groups (see, e.g., [7]).

## 2. NOTATION

Let  $G$  be a finite group of complex  $n \times n$  unitary matrices generated by reflections acting on  $V := \mathbb{C}^n$ . Recall that a unitary matrix is a reflection if it has finite order and fixes a hyperplane pointwise in  $V$ . Let  $S := \mathbb{C}[x_1, \dots, x_n]$  be the ring of polynomials of  $V$ . Let  $f_1, \dots, f_n \in S$  be basic invariants, and let  $R = \mathbb{C}[f_1, \dots, f_n]$  be the ring of invariant polynomials. Let  $\chi$  be a multiplicative character of  $G$ . Denote the module of differential  $p$ -forms on  $V$  by

$$\begin{aligned} \Omega^p &:= \bigoplus_{1 \leq i_1 < \dots < i_p \leq n} S dx_{i_1} \wedge \dots \wedge dx_{i_p} \\ &\simeq S \otimes \wedge^p V^*. \end{aligned}$$

The group  $G$  acts contragradiently on  $V^*$  and  $S$ , and  $\Omega^p$  is a  $\mathbb{C}[G]$ -module. Define the  $R$ -module of  $\chi$ -invariant differential  $p$ -forms

$$(\Omega^p)^\chi := \{ \omega \in \Omega^p : g\omega = \chi(g)\omega \text{ for all } g \in G \}.$$

Let

$$\Omega^\chi := \bigoplus_{0 \leq p} (\Omega^p)^\chi.$$

It is convenient to define  $\mathcal{S}^p$  as the set of multi-indices of  $\{1, \dots, n\}$  of length  $p$ :

$$\mathcal{S}^p := \{ I = \{I_1, \dots, I_p\} : 1 \leq I_1 < \dots < I_p \leq n \}.$$

For a multi-index  $I$ , let  $I^c$  denote the complementary index. Denote the volume form on  $V$  by  $\text{vol} := dx_1 \wedge \dots \wedge dx_n$ . If  $f$  and  $g$  are differential forms, we write  $f \doteq g$  if  $f = cg$  for some  $c \in \mathbb{C}^*$ .

We recall some facts and notation from “Arrangements of Hyperplanes” [3, p. 228]. Let  $\mathcal{A}$  be the hyperplane arrangement defined by  $G$ . For each

$H \in \mathcal{A}$ , define  $\alpha_H \in S$  by  $\ker(\alpha_H) = H$ . Fix some  $H \in \mathcal{A}$ , and let  $G_H$  be the cyclic subgroup of elements in  $G$  that fix  $H$  pointwise. Let  $s_H$  be a generator of  $G_H$  and let  $o(s_H)$  be the order of  $s_H$ . Define  $a_H(\chi)$  as the least integer satisfying  $0 \leq a_H(\chi) < o(s_H)$  and  $\chi(s_H) = \det(s_H)^{-a_H(\chi)}$ . Let

$$Q_\chi = \prod_{H \in \mathcal{A}} \alpha_H^{a_H(\chi)}.$$

The polynomial  $Q_\chi$  is uniquely determined by the group  $G$  up to a nonzero scalar multiple.

Stanley [8] proved that  $(\Omega^0)^\chi = RQ_\chi$ , and since  $\text{vol}$  is  $(\det^{-1})$ -invariant, it follows that

$$(\Omega^n)^\chi = RQ_{\chi \cdot \det} \text{vol}. \tag{*}$$

Steinberg [9] proved that  $Q_{\det} = \prod_{H \in \mathcal{A}} \alpha_H^{o(s_H)-1}$  is the determinant of the Jacobian matrix  $\{(\partial/\partial x_i)f_j\}$ , up to a nonzero scalar multiple. Note also that  $Q_{\det^{-1}} = \prod_{H \in \mathcal{A}} \alpha_H$  [3, p. 229].

### 3. $\chi$ -WEDGING

The next lemma will be used to show that  $Q_\chi$  divides the exterior product of any two  $\chi$ -invariant forms.

LEMMA 1. *Suppose that  $\mu$  is a  $\chi$ -invariant  $p$ -form. Fix a hyperplane  $H \in \mathcal{A}$ , and let  $a = a_H(\chi)$ . Choose coordinates in which  $x_1 = \alpha_H$  and  $s_H$  is diagonal. If*

$$\mu = \sum_{I \in \mathcal{I}^p} \mu_I dx_{I_1} \wedge \cdots \wedge dx_{I_p}$$

*in these coordinates, then  $x_1^{a-1}$  divides  $\mu_I$  whenever  $I_1 = 1$  and  $x_1^a$  divides  $\mu_I$  whenever  $I_1 \neq 1$ , for each  $I = \{I_1, \dots, I_p\} \in \mathcal{I}^p$ .*

*Proof.* Let  $s = s_H$  and let  $\rho$  be the determinant of  $s$ . Then

$$s = \begin{pmatrix} \rho & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

and  $s^{-1} dx_1 = \rho dx_1, s^{-1} dx_2 = dx_2, \dots, s^{-1} dx_n = dx_n$ .

Let  $I = \{I_1, I_2, \dots, I_p\} \in \mathcal{I}^p$ . If  $I_1 = 1$ , then

$$\begin{aligned} s^{-1}(\mu_I dx_{I_1} \wedge \dots \wedge dx_{I_p}) &= s^{-1}\mu_I s^{-1} dx_{I_1} \wedge \dots \wedge s^{-1} dx_{I_p} \\ &= \mu_I \circ s \rho dx_{I_1} \wedge \dots \wedge dx_{I_p}. \end{aligned}$$

If  $I_1 \neq 1$ , then

$$\begin{aligned} s^{-1}(\mu_I dx_{I_1} \wedge \dots \wedge dx_{I_p}) &= s^{-1}\mu_I s^{-1} dx_{I_1} \wedge \dots \wedge s^{-1} dx_{I_p} \\ &= \mu_I \circ s dx_{I_1} \wedge \dots \wedge dx_{I_p}. \end{aligned}$$

But  $\mu$  is  $\chi$ -invariant, so  $\rho^a \mu = \det(s)^a \mu = \chi^{-1}(s)\mu = s^{-1}\mu$ . Hence if  $I_1 = 1$ , then  $\rho^a \mu_I = \rho \mu_I \circ s$ , i.e.,  $\rho^{a-1} \mu_I = \mu_I \circ s$ . Thus  $x_1^{a-1}$  divides  $\mu_I$ . Similarly, if  $I_1 \neq 1$ , then  $\rho^a \mu_I = \mu_I \circ s$  and  $x_1^a$  divides  $\mu_I$ . ■

LEMMA 2.  $Q_\chi$  divides the exterior product of any two  $\chi$ -invariant differential forms.

*Proof.* Let  $\mu$  be a  $\chi$ -invariant  $p$ -form and let  $\omega$  be a  $\chi$ -invariant  $q$ -form. Fix  $H \in \mathcal{A}$ . Let  $s = s_H$  and  $a = a_H(\chi)$ . Assume that  $a \neq 0$ . We show that  $\alpha_H^a$  divides  $\mu \wedge \omega$  by choosing coordinates from Lemma 1 in which  $\alpha_H = x_1$ . Let

$$\begin{aligned} \mu &= \sum_{I \in \mathcal{I}^p} \mu_I dx_{I_1} \wedge \dots \wedge dx_{I_p}, \\ \omega &= \sum_{J \in \mathcal{I}^q} \omega_J dx_{J_1} \wedge \dots \wedge dx_{J_q}, \end{aligned}$$

and

$$\mu \wedge \omega = \sum_{K \in \mathcal{I}^{p+q}} \gamma_K dx_{K_1} \wedge \dots \wedge dx_{K_{p+q}}$$

in these coordinates. Then  $x_1^a$  divides  $\mu_I$  whenever  $I_1 \neq 1$  and  $x_1^a$  divides  $\omega_J$  whenever  $J_1 \neq 1$ .

Hence, for  $I \in \mathcal{I}^p$  and  $J \in \mathcal{I}^q$ , the polynomial  $\mu_I \omega_J$  is divisible by  $x_1^a$  given that not both  $I_1$  and  $J_1$  are 1. Since each  $\gamma_K$  is either zero or a sum of terms of the form  $\pm \mu_I \omega_J$ , where the multi-indices  $I$  and  $J$  are disjoint,  $x_1^a$  divides each  $\gamma_K$  and hence  $\mu \wedge \omega$ . Thus,  $\mu \wedge \omega$  is divisible by  $\alpha_H^a = \alpha_H^{a_H(\chi)}$ . Since  $H$  was arbitrary,  $Q_\chi$  divides  $\mu \wedge \omega$ . ■

Lemma 2 prompts us to define the following multiplication in  $\Omega^\chi$ : For differential forms  $\mu$  and  $\omega$ , define the  $\chi$ -wedge of  $\mu$  and  $\omega$  as

$$\mu \wedge \omega := \frac{\mu \wedge \omega}{Q_\chi}.$$

If  $\mu$  and  $\omega$  are  $\chi$ -invariant forms,  $\mu \wedge \omega$  is again  $\chi$ -invariant. Thus, Lemma 2 implies

COROLLARY 1. *The  $R$ -module  $\Omega^\chi$  is closed under  $\chi$ -wedging.*

The following proposition gives a condition (similar to Saito’s criterion) for  $n$  1-forms to generate  $\Omega^\chi$ . The proof is similar to Solomon’s original argument [6] that  $df_1, \dots, df_n$  generate the module of invariant differential forms.

PROPOSITION 1. *Let  $\omega_1, \dots, \omega_n$  be  $\chi$ -invariant 1-forms. The forms  $\omega_{I_1} \wedge \dots \wedge \omega_{I_p}$ , for  $I \in \mathcal{I}^p$  and  $p \geq 0$ , generate  $\Omega^\chi$  over  $R$  if and only if*

$$\omega_1 \wedge \dots \wedge \omega_n \doteq Q_{\chi\text{-det}} \text{ vol.}$$

*Proof.* Assume that  $\omega_1 \wedge \dots \wedge \omega_n \doteq Q_{\chi\text{-det}} \text{ vol.}$  The  $p$ -forms  $\omega_{I_1} \wedge \dots \wedge \omega_{I_p}$ ,  $I \in \mathcal{I}^p$ , are  $\chi$ -invariant by Corollary 1.

Since  $\omega_1 \wedge \dots \wedge \omega_n \neq 0$ ,  $\omega_1 \wedge \dots \wedge \omega_n \neq 0$ , and the forms  $\omega_{I_1} \wedge \dots \wedge \omega_{I_p}$ ,  $I \in \mathcal{I}^p$ , are linearly independent over  $F := \mathbb{C}(x_1, \dots, x_n)$ . If not, there exist rational functions  $r_I$  with

$$0 = \sum_{I \in \mathcal{I}^p} r_I \omega_{I_1} \wedge \dots \wedge \omega_{I_p}.$$

Fix  $J \in \mathcal{I}^p$  and  $J^c \in \mathcal{I}^{n-p}$ . Then

$$\begin{aligned} 0 &= \left( \sum_{I \in \mathcal{I}^p} r_I \omega_{I_1} \wedge \dots \wedge \omega_{I_p} \right) \wedge \omega_{J^c_1} \wedge \dots \wedge \omega_{J^c_{n-p}} \\ &= \pm r_J \omega_1 \wedge \dots \wedge \omega_n, \end{aligned}$$

and  $r_J$  must be zero. Hence the forms

$$\omega_{I_1} \wedge \dots \wedge \omega_{I_p} = (Q_\chi)^{1-p} \omega_{I_1} \wedge \dots \wedge \omega_{I_p}, \quad I \in \mathcal{I}^p,$$

are also linearly independent over  $F$ , and thus span

$$\Omega^p(V) := \bigoplus_{I \in \mathcal{I}^p} F dx_{I_1} \wedge \dots \wedge dx_{I_p}$$

since  $\Omega^p(V)$  has dimension  $\binom{n}{p}$ .

Choose an arbitrary  $\chi$ -invariant  $p$ -form  $\mu$ . Then there exist rational functions  $t_I \in F$  with

$$\mu = \sum_{I \in \mathcal{I}^p} t_I \omega_{I_1} \wedge \dots \wedge \omega_{I_p}.$$

Fix  $J \in \mathcal{I}^p$  and its complementary index  $J^c$ . We will show that  $t_J \in R$ .

By Corollary 1, the  $n$ -form  $(\omega_{J_1^c} \wedge \cdots \wedge \omega_{J_{n-p}^c}) \wedge \mu$  is  $\chi$ -invariant. Thus by Eq. (\*) above, there exists a polynomial  $f \in R$  with

$$(\omega_{J_1^c} \wedge \cdots \wedge \omega_{J_{n-p}^c}) \wedge \mu = f Q_{\chi \cdot \det} \text{vol.}$$

On the other hand,

$$\begin{aligned} & (\omega_{J_1^c} \wedge \cdots \wedge \omega_{J_{n-p}^c}) \wedge \mu \\ &= (\omega_{J_1^c} \wedge \cdots \wedge \omega_{J_{n-p}^c}) \wedge \sum_{I \in \mathcal{I}^p} t_I \omega_{I_1} \wedge \cdots \wedge \omega_{I_p} \\ &= (Q_{\chi}^{1-n}) (\omega_{J_1^c} \wedge \cdots \wedge \omega_{J_{n-p}^c}) \wedge \sum_{I \in \mathcal{I}^p} t_I \omega_{I_1} \wedge \cdots \wedge \omega_{I_p} \\ &= (Q_{\chi}^{1-n}) \pm t_J \omega_1 \wedge \cdots \wedge \omega_n \\ &= \pm t_J \omega_1 \wedge \cdots \wedge \omega_n \\ &\doteq \pm t_J Q_{\chi \cdot \det} \text{vol.} \end{aligned}$$

Thus  $f Q_{\chi \cdot \det} \doteq t_J Q_{\chi \cdot \det}$ . Hence,  $t_J \in R$ . Since  $J$  was arbitrary,  $\mu$  is in the  $R$ -span of  $\{\omega_{I_1} \wedge \cdots \wedge \omega_{I_p}, I_p \in \mathcal{I}^p\}$ .

The converse follows from Eq. (\*) above. ■

#### 4. CONDITION SATISFIED

Since  $\Omega^p$  has rank  $\binom{n}{p}$ , the  $R$ -module  $(\Omega^p)^\chi$  is also free of rank  $\binom{n}{p}$  (this follows from Lemma 6.45 of [3, p. 232]). We will show that the generators of  $(\Omega^1)^\chi$  satisfy the condition given in Proposition 1, but we must first gather some preliminary facts.

We recall some results about invariant vector fields. There exist  $n$  invariant vector fields, called *basic derivations*, that generate the module of invariant vector fields over  $R$  (see [3, Sect. 6.3]). Using Saito's criterion, Terao showed that the coefficient matrix of the basic derivations has determinant  $Q_{\det^{-1}}$  up to a nonzero scalar multiple (see [3, p. 238]). Using the minors of this coefficient matrix, we construct  $(\det^{-1})$ -invariant 1-forms,  $\mu_1, \dots, \mu_n$ , that satisfy

$$\mu_1 \wedge \cdots \wedge \mu_n = Q_{\det^{-1}}^{n-1} \text{vol.}$$

The forms  $\mu_1, \dots, \mu_n$  thus generate  $\Omega^{\det^{-1}}$  over  $R$  by Proposition 1. We will use these forms to give an argument for arbitrary  $\chi$ .

We also note the relationship between  $Q_{\chi \cdot \det}$  and  $Q_\chi$ : Fix  $H \in \mathcal{A}$  with  $a_H(\chi) \neq 0$ . The exponent  $a_H(\chi \cdot \det)$  is the least nonnegative integer satisfying

$$\begin{aligned} \det(s_H)^{-a_H(\chi \cdot \det)} &= (\chi \cdot \det)(s_H) \\ &= \chi(s_H) \det(s_H) \\ &= \det(s_H)^{-a_H(\chi)} \det(s_H) \\ &= \det(s_H)^{-(a_H(\chi)-1)}. \end{aligned}$$

Hence,  $a_H(\chi \cdot \det) = a_H(\chi) - 1$ . Now fix  $H \in \mathcal{A}$  with  $a_H(\chi) = 0$ . Then

$$\begin{aligned} \det(s_H)^{-a_H(\chi \cdot \det)} &= (\chi \cdot \det)(s_H) \\ &= \chi(s_H) \det(s_H) \\ &= \det(s_H) \\ &= \det(s_H)^{-(o(s_H)-1)}, \end{aligned}$$

and  $a_H(\chi \cdot \det) = o(s_H) - 1$ . Thus,

$$\begin{aligned} Q_{\chi \cdot \det} &= \prod_{H \in \mathcal{A}} \alpha_H^{a_H(\chi \cdot \det)} \\ &= \prod_{\substack{H \in \mathcal{A} \\ \chi(s_H) \neq 1}} \alpha_H^{a_H(\chi)-1} \prod_{\substack{H \in \mathcal{A} \\ \chi(s_H) = 1}} \alpha_H^{o(s_H)-1}. \end{aligned}$$

PROPOSITION 2. If  $\omega_1, \dots, \omega_n$  generate  $(\Omega^1)^\chi$  over  $R$ , then

$$\omega_1 \wedge \cdots \wedge \omega_n \doteq Q_{\chi \cdot \det} \text{ vol.}$$

*Proof.* Let  $M$  be the coefficient matrix of  $\omega_1, \dots, \omega_n$ , i.e.,  $\omega_1 \wedge \cdots \wedge \omega_n = \det M \text{ vol}$ . Suppose that  $\det M = 0$ . Then one row of  $M$  is a linear combination of the other rows over  $F = \mathbb{C}(x_1, \dots, x_n)$ . Multiplying by a least common multiple yields a relation over  $S$ :  $\sum_{i=1}^n s_i \omega_i = 0$ . To get a relation over  $R$ , apply a group element  $g$ , multiply by  $\chi^{-1}(g)$ , and then sum over  $G$ :

$$\begin{aligned} 0 &= \sum_{g \in G} \sum_{i=1}^n \chi^{-1}(g) g s_i g \omega_i \\ &= \sum_{i=1}^n \sum_{g \in G} \chi^{-1}(g) g s_i \chi(g) \omega_i \\ &= \sum_{i=1}^n \left( \sum_{g \in G} g s_i \right) \omega_i. \end{aligned}$$

This contradicts the fact that  $(\Omega^1)^\chi$  is free over  $R$  with basis  $\omega_1, \dots, \omega_n$ . Hence,  $\det M \neq 0$ .

By Corollary 1,  $\omega_1 \wedge \dots \wedge \omega_n$  is a  $\chi$ -invariant  $n$ -form. Thus (from Eq. (\*)) there exists a nonzero  $f \in R$  with

$$(Q_\chi)^{1-n} \det M \text{ vol} = (Q_\chi)^{1-n} \omega_1 \wedge \dots \wedge \omega_n = \omega_1 \wedge \dots \wedge \omega_n = f Q_{\chi \cdot \det} \text{ vol}.$$

Hence,  $\det M = f Q_{\chi \cdot \det} (Q_\chi)^{n-1}$ .

We show that  $f$  is constant by finding two polynomials that are relatively prime, yet are each divisible by  $f$ . Since each  $df_i$  is invariant, each  $Q_\chi df_i$  is  $\chi$ -invariant and hence a combination of  $\omega_1, \dots, \omega_n$  over  $R$ . There exists a matrix of coefficients,  $N$ , with entries in  $S$ , such that

$$\begin{aligned} Q_\chi df_1 \wedge \dots \wedge Q_\chi df_n &= \det M \det N \text{ vol} \\ &= f Q_{\chi \cdot \det} (Q_\chi)^{n-1} \det N \text{ vol}. \end{aligned}$$

But,  $df_1 \wedge \dots \wedge df_n \doteq Q_{\det}$ , so

$$Q_\chi df_1 \wedge \dots \wedge Q_\chi df_n \doteq (Q_\chi)^n Q_{\det} \text{ vol}.$$

Hence,

$$f Q_{\chi \cdot \det} \det(N) \doteq Q_\chi Q_{\det}$$

and, since  $\det N \in S$ ,  $f$  divides  $Q_\chi Q_{\det} (Q_{\chi \cdot \det})^{-1}$ .

Since each  $\mu_i$  (introduced above) is  $(\det^{-1})$ -invariant, each  $Q_{\chi \cdot \det} \mu_i$  is  $\chi$ -invariant, and thus an  $R$ -combination of  $\omega_1, \dots, \omega_n$ . There exists a matrix of coefficients,  $N'$ , with coefficients in  $S$ , such that

$$\begin{aligned} Q_{\chi \cdot \det} \mu_1 \wedge \dots \wedge Q_{\chi \cdot \det} \mu_n &= \det M \det N' \text{ vol} \\ &= f Q_{\chi \cdot \det} (Q_\chi)^{n-1} \det N' \text{ vol}. \end{aligned}$$

But we choose the  $\mu_i$  so that

$$Q_{\chi \cdot \det} \mu_1 \wedge \dots \wedge Q_{\chi \cdot \det} \mu_n = (Q_{\chi \cdot \det})^n (Q_{\det^{-1}})^{n-1} \text{ vol}.$$

Hence,

$$f Q_{\chi \cdot \det} (Q_\chi)^{n-1} \det N' = (Q_{\chi \cdot \det})^n (Q_{\det^{-1}})^{n-1}$$

and, since  $\det N' \in S$ ,  $(Q_{\chi \cdot \det} Q_{\det^{-1}})^{n-1} (Q_\chi)^{1-n}$  is divisible by  $f$ .



We show that the two polynomials

$$Q_\chi Q_{\det}(Q_{\chi \cdot \det})^{-1} \quad \text{and} \quad (Q_{\chi \cdot \det} Q_{\det^{-1}})^{n-1} (Q_\chi)^{1-n}$$

have no common factors by writing them both in terms of the  $\alpha_H$ . We expand the factors:

$$\begin{aligned} Q_\chi &= \prod_{\substack{H \in \mathcal{A} \\ \chi(s_H) \neq 1}} \alpha_H^{a_H(\chi)}, \\ Q_{\det} &= \prod_{\substack{H \in \mathcal{A} \\ \chi(s_H) \neq 1}} \alpha_H^{o(s_H)-1} \prod_{\substack{H \in \mathcal{A} \\ \chi(s_H) = 1}} \alpha_H^{o(s_H)-1}, \\ Q_{\chi \cdot \det} &= \prod_{\substack{H \in \mathcal{A} \\ \chi(s_H) \neq 1}} \alpha_H^{a_H(\chi)-1} \prod_{\substack{H \in \mathcal{A} \\ \chi(s_H) = 1}} \alpha_H^{o(s_H)-1}, \\ Q_{\det^{-1}} &= \sum_{\substack{H \in \mathcal{A} \\ \chi(s_H) \neq 1}} \alpha_H \prod_{\substack{H \in \mathcal{A} \\ \chi(s_H) = 1}} \alpha_H. \end{aligned}$$

The first polynomial,  $Q_\chi Q_{\det}(Q_{\chi \cdot \det})^{-1}$ , simplifies to

$$\prod_{\substack{H \in \mathcal{A} \\ \chi(s_H) \neq 1}} \alpha_H^{o(s_H)},$$

and the second polynomial,  $(Q_{\chi \cdot \det} Q_{\det^{-1}})^{n-1} (Q_\chi)^{1-n}$ , simplifies to

$$\left( \prod_{\substack{H \in \mathcal{A} \\ \chi(s_H) = 1}} \alpha_H^{o(s_H)} \right)^{n-1}.$$

Since  $f$  divides both polynomials,  $f$  must be constant. Thus,  $\omega_1, \dots, \omega_n$  satisfy the criterion of Proposition 1. ■

**COROLLARY 2.** *There exist  $n$  1-forms  $\omega_1, \dots, \omega_n$  such that  $\Omega^\chi$  is generated over  $R$  by the forms  $\omega_{I_1} \wedge \dots \wedge \omega_{I_p}$ ,  $I \in \mathcal{I}^p$ ,  $p \geq 0$ . Thus  $\Omega^\chi$  has the structure of an exterior algebra.*

## 5. EXAMPLE: $G_{26}$

For an example, let us take a three-dimensional complex reflection group,  $G_{26}$ . This group is the symmetry group of a regular complex polyhedron and is number 26 in Shephard and Todd's enumeration of finite irreducible unitary groups generated by reflections [4]. The group

$G_{26}$  consists of 1,296 complex  $3 \times 3$  matrices and is generated by reflections of orders 2 and 3. The associated collineation group (which results from moding out by the scalar matrices) is the Hessian group of order 216.

The group is generated by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}, \quad \text{and} \quad \frac{i}{\sqrt{3}} \begin{pmatrix} \alpha & \alpha^2 & \alpha^2 \\ \alpha^2 & \alpha & \alpha^2 \\ \alpha^2 & \alpha^2 & \alpha \end{pmatrix},$$

where  $\alpha$  is a primitive cube root of unity.

The character table for this group reveals six multiplicative characters, each a power of the determinant character. Consider  $\chi = \det^3$ . Note that

$$Q_{\det^3} = (x^3 - y^3)(x^3 - z^3)(y^3 - z^3),$$

and

$$Q_{\det^4} = x^2 y^2 z^2 (x^9 + 3x^6(y^3 + z^3) + (y^3 + z^3)^3 + 3x^3(y^6 - 7y^3 z^3 + xz^6))^2.$$

The following 1-forms are  $\det^3$ -invariant:

$$\begin{aligned} \omega_1 &= x^2(y-z)(y^2+yz+z^2)(2x^3-y^3-z^3) dx \\ &\quad - y^2(x-z)(x^2+xz+z^2)(-x^3+2y^3-z^3) dy \\ &\quad - z^2(x-y)(x^2+xy+y^2)(x^3+y^3-2x^3) dz, \\ \omega_2 &= x^2(x^3-y^3)(x^3-z^3)(y^3-z^3)(x^3-5y^3-5z^3) dz \\ &\quad + y^2(x^3-y^3)(x^3-z^3)(y^3-z^3)(-5x^3+y^3-5z^3) dy \\ &\quad + z^2(x^3-y^3)(x^3-z^3)(y^3-z^3)(-5x^3-5y^3+z^3) dz. \\ \omega_3 &= x^2(x^3-y^3)(x^3-z^3)(y^3-z^3)(x^9+3y^9+61y^6z^3+61y^3z^6+3z^9 \\ &\quad +9x^6(y^3+z^3)+x^3(-13y^6+122y^3z^3-13z^6)) dx \\ &\quad +y^2(x^3-y^3)(x^3-z^3)(y^3-z^3)(3x^9+y^9+9y^6z^3-13y^3z^6+3z^9 \\ &\quad +x^6(-13y^3+61z^3)+x^3(9y^6+122y^3z^3+61z^6)) dy \\ &\quad +z^2(x^3-y^3)(x^3-z^3)(y^3-z^3)(3x^9+3y^9-13y^6z^3+9y^3z^6+z^9 \\ &\quad +x^6(61y^3-13z^3)+x^3(61y^6+122y^3z^3+9z^6)) dz. \end{aligned}$$

The polynomial  $Q_{\det^3}$  divides  $\omega_1 \wedge \omega_2$ ,  $\omega_2 \wedge \omega_3$ , and  $\omega_1 \wedge \omega_3$ . The determinant of the coefficient matrix of  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$  is  $(-16)Q_{\det^4}Q_{\det^3}^2$ ,

hence  $\omega_1, \omega_2$ , and  $\omega_3 \det^3$ -wedge to a multiple of  $Q_{\chi \cdot \det} = Q_{\det^4}$ . Proposition 1 then implies that  $\omega_1, \omega_2$ , and  $\omega_3$  generate the entire module of  $\det^3$ -invariants over the ring of invariants via  $\det^3$ -wedging.

### 6. LOGARITHMIC FORMS

We have so far only discussed regular differential forms; we now consider rational differential forms. The  $S$ -module of *logarithmic  $p$ -forms with poles along  $\mathcal{A}$*  (see also [3, p. 124]) is defined as

$$\Omega^p(\mathcal{A}) := \left\{ \frac{\omega}{Q_{\det^{-1}}} : \omega \in \Omega^p \text{ and } \omega \wedge d\alpha_H \in \alpha_H \Omega^{p+1} \text{ for all } H \in \mathcal{A} \right\}.$$

Ziegler [10] extends this definition to *multiarrangements of hyperplanes*, hyperplane arrangements in which each hyperplane has a positive integer multiplicity. We apply his definitions to our context of reflection groups and semi-invariants: Let  $\mathcal{A}_\chi$  be the multiarrangement consisting of hyperplanes  $H \in \mathcal{A}$  each with multiplicity  $\alpha_H(\chi)$ , i.e., the multiarrangement defined by  $Q_\chi$ . We define (as in [10]) the module of *logarithmic  $p$ -forms of  $\mathcal{A}_\chi$* :

$$\Omega^p(\mathcal{A}_\chi) := \left\{ \frac{\omega}{Q_\chi} : \omega \in \Omega^p \text{ and } \omega \wedge d\alpha_H \in \alpha_H^{a_H(\chi)} \Omega^{p+1} \text{ for all } H \in \mathcal{A} \right\}.$$

Let

$$\Omega(\mathcal{A}_\chi) := \bigoplus_{p \geq 0} \Omega^p(\mathcal{A}_\chi).$$

COROLLARY 3.

$$\Omega^\chi \subset Q_\chi \Omega(\mathcal{A}_\chi).$$

*Proof.* Chose  $\omega$  in  $(\Omega^p)^\chi$  and fix  $H \in \mathcal{A}$ . Using Lemma 1, choose coordinates in which  $x_1 = \alpha_H$ ,  $\omega = \sum_{I \in \mathcal{J}^p} \omega_I dx_{I_1} \wedge \cdots \wedge dx_{I_p}$ , and  $x_1^{a_H(\chi)}$  divides  $\omega_I$  if  $1 \notin I$ . Then  $d\alpha_H = dx_1$ , and  $\omega \wedge d\alpha_H = \omega \wedge dx_1 = \sum_{I, 1 \notin I} \omega_I \wedge dx_1$ , which is divisible by  $x_1^{a_H(\chi)}$ . Hence,  $\omega \wedge d\alpha_H \in \alpha_H^{a_H(\chi)} \Omega^{p+1}$ . As  $H$  was arbitrary,  $\omega/Q_\chi \in \Omega(\mathcal{A}_\chi)$ . ■

This relationship is stronger when  $\chi = \det^{-1}$ . In this case, the forms that generate  $\Omega^\chi$  via  $\chi$ -wedging over  $R$  also generate  $\Omega(\mathcal{A})$  over  $S$  (see [5] for a different proof).

On a similar note, we have

PROPOSITION 3.  $\Omega(\mathcal{A}_\chi)$  is closed under the exterior product.

*Proof.* Let  $\omega/Q_\chi$  and  $\mu/Q_\chi$  be in  $\Omega(\mathcal{A}_\chi)$ . Fix  $H$  in  $\mathcal{A}$  and let  $a_H(\chi) = a$ . Choose coordinates such that  $x_1 = \alpha_H$ , and write  $\omega = \sum_{I \in \mathcal{I}^p} \omega_I dx_{I_1} \wedge \cdots \wedge dx_{I_p}$  and  $\mu = \sum_{J \in \mathcal{J}^q} \mu_J dx_{J_1} \wedge \cdots \wedge dx_{J_q}$  in these coordinates. Since  $\omega \wedge dx_1 = \omega \wedge d\alpha_H \in \alpha_H^a \Omega^{p+1} = x_1^a \Omega^{p+1}$  (for some  $p$ ),  $\omega_I$  is divisible by  $x_1^a$  as long as  $1 \notin I$ . Similarly,  $\mu_J$  is divisible by  $x_1^a$  whenever  $1 \notin J$ . As in the proof of Lemma 2, it follows that  $Q_\chi$  divides  $\omega \wedge \mu$ . Whenever  $1 \notin I$  and  $1 \notin J$ ,  $x_1^{2a}$  divides  $\omega_I \mu_J$ , and thus

$$\frac{\omega \wedge \mu}{Q_\chi} \wedge dx_1$$

is also divisible by  $x_1^a$ . Hence  $\alpha_H^a$  divides  $(1/Q_\chi)\omega \wedge \mu \wedge d\alpha_H$ , and, as  $H$  was arbitrary,  $(\omega/Q_\chi) \wedge (\mu/Q_\chi)$  is in  $\Omega(\mathcal{A}_\chi)$ . ■

## 7. REMARKS

Analogous results hold for vector fields, or *derivations*. Let  $\mathbb{T}^\chi$  be the module of  $\chi$ -invariants in the exterior algebra of derivations. Because the group action differs here, Lemma 1 is slightly different, with  $a + 1$  taking the place of  $a - 1$  when  $I_1 = 1$ . The case where  $I_1 \neq 1$  is the same as in the original lemma, and hence  $Q_\chi$  also divides the exterior product of two elements in  $\mathbb{T}^\chi$  (the proof is analogous to the case of  $\Omega^\chi$ ). The criterion for  $n$  derivations to generate  $\mathbb{T}^\chi$  via  $\chi$ -wedging is also slightly different: they must  $\chi$ -wedge to

$$Q_{\chi \cdot \det^{-1}} \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}$$

instead of  $Q_{\chi \cdot \det} dx_1 \wedge \cdots \wedge dx_n$ . This follows from the fact that  $dx_1 \wedge \cdots \wedge dx_n$  is  $(\det^{-1})$ -invariant while  $\partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_n$  is  $\det$ -invariant. Finally, we note that the correspondence between differential  $p$ -forms (in  $\Omega^p$ ) and  $(n - p)$ -forms in  $\mathbb{T}$  (the exterior algebra of derivations) induces a module isomorphism between  $\Omega^\chi$  and  $\mathbb{T}^{\chi \cdot \det}$ .

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