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J. Differential Equations 206 (2004) 265–279
Journal of Differential Equations
http://www.elsevier.com/locate/jde

Stable closed characteristics on partially symmetric convex hypersurfaces

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Received January 4, 2004

Abstract

Let n be a positive integer and $P = \mathrm{diag}(-I_{n-\kappa}, I_\kappa, -I_{n-\kappa}, I_\kappa)$ for some integer $\kappa \in [0, n]$. In this paper, we prove that for any convex compact smooth hypersurface Σ in $\mathbf{R}^{2n}$ with $n \geqslant 2$ there always exists at least one closed characteristic on Σ which possesses at least $2n - 4\kappa$ Floquet multipliers on the unit circle of the complex plane, provided Σ is P-symmetric, i.e., $x \in \Sigma$ implies $Px \in \Sigma$.

MSC: 58E05; 70H12; 34D08; 34C25

Keywords: Hamiltonian systems; Convex energy surface; P-symmetry; Closed orbit; Ellipticity

1. The main result

As in Chapter 15 of [8], let Σ be a C^2 compact hypersurface in $\mathbf{R}^{2n}$ bounding a convex compact set C with non-empty interior, and possess a non-vanishing Gaussian curvature. Without loss of generality we assume $0 \in C$. We denote the set of all such hypersurfaces in $\mathbf{R}^{2n}$ by $\mathscr{H}(2n)$. For any $x \in \Sigma$, let $N_\Sigma(x)$ be the outward

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[1] Partially supported by the National Natural Science Foundation of China(10251001), the Educational Committee Foundation of Jiangsu, the Natural Science Foundation of Jiangsu(BK2002023), and the PD Foundation of China.
[2] Partially supported by the 973 Program of STM, NSFC, Cheung Kong Scholars Programme, MCME, RFDP, LPMC of EM of China, S. S. Chern Foundation, and Nankai University.

0022-0396/$ - see front matter
doi:10.1016/j.jde.2004.03.004

normal unit vector at x of Σ. We consider the given energy problem of finding $\tau > 0$ and a C^1 curve $x : [0, \tau] \to \mathbf{R}^{2n}$ such that

$$\dot{x}(t) = JN_{\Sigma}(x(t)), \quad x(t) \in \Sigma, \quad \forall t \in \mathbf{R}, \tag{1.1}$$

$$x(\tau) = x(0), \tag{1.2}$$

where $J = J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, and I_n is the identity matrix on $\mathbf{R}^n$. A solution (τ, x) of problem (1.1)–(1.2) is called a closed characteristic on Σ. We denote by $\mathscr{J}(\Sigma)$ the set of all closed characteristics (τ, x) on Σ with τ being the minimal period of x. Two closed characteristics (τ, x) and $(\sigma, y) \in \mathscr{J}(\Sigma)$ are geometrically distinct, if $x(\mathbf{R}) \neq y(\mathbf{R})$. We denote by $\tilde{\mathscr{J}}(\Sigma)$ the set of all geometrically distinct closed characteristics (τ, x) on Σ.

Problem (1.1)–(1.2) can be put into a Hamiltonian version. Let $j : \mathbf{R}^{2n} \to [0, +\infty)$ be the gauge function of Σ defined by

$$j(0) = 0, \quad \text{and} \quad j(x) = \inf\left\{\lambda > 0 \,\middle|\, \frac{x}{\lambda} \in C\right\}, \quad \forall x \neq 0.$$

Fix a constant α with $1 < \alpha < 2$ in this paper, we define $H : \mathbf{R}^{2n} \to [0, +\infty)$ by

$$H(x) = j(x)^{\alpha}, \quad \forall x \in \mathbf{R}^{2n}. \tag{1.3}$$

Then $H \in C^1(\mathbf{R}^{2n}, \mathbf{R}) \cap C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R})$ is convex and $\Sigma = H^{-1}(1)$. It is well known that the problem (1.1)(1.2) is equivalent to the following problem:

$$\dot{x}(t) = JH'(x(t)), \quad H(x(t)) = 1, \quad \forall t \in \mathbf{R}, \tag{1.4}$$

$$x(\tau) = x(0). \tag{1.5}$$

Denote by $\mathscr{J}(\Sigma, \alpha)$ the set of all distinct solutions (τ, x) of (1.4)–(1.5) with τ being the minimal period of x. Note that elements in $\mathscr{J}(\Sigma)$ and $\mathscr{J}(\Sigma, \alpha)$ are one to one correspondent to each other.

Let $(\tau, x) \in \mathscr{J}(\Sigma, \alpha)$. As usual we call the fundamental solution $\gamma_x : [0, +\infty) \to \mathrm{Sp}(2n)$ with $\gamma_x(0) = I_{2n}$ of the linearized Hamiltonian system

$$\dot{y}(t) = JA(t)y(t), \quad \forall t \in \mathbf{R}, \tag{1.6}$$

where $A(t) = H''(x(t))$, the *associated symplectic path* of (τ, x). The eigenvalues of $\gamma_x(\tau)$ are called *Floquet multipliers* of (τ, x). It is well-known that the Floquet multipliers with their multiplicity and Krein type numbers of $(\tau, x) \in \mathscr{J}(\Sigma, \alpha)$ do not depend on the particular choice of the Hamiltonian function H in (1.4). As in [9] and Chapter 15 of [8], for any symplectic matrix M, we define the *elliptic height* $e(M)$ of M by the total algebraic multiplicity of all eigenvalues of M on the unit circle $\mathbf{U} = \{z \in \mathbf{C} \mid |z| = 1\}$ of the complex plane. And for any $(\tau, x) \in \mathscr{J}(\Sigma, \alpha)$ we define

$e(m\tau, x) = e(\gamma_x(m\tau))$, and call (τ, x) *elliptic* or *hyperbolic* if $e(\tau, x) = 2n$ or $e(\tau, x) = 2$, respectively.

Note that $e(M) = e(M^k)$ holds for any integer $k \geqslant 1$ if M is a symplectic matrix.

A long-standing conjecture is mentioned on p. 235 of [4]: for every $\Sigma \in \mathscr{H}(2n)$, there exists a $(\tau, x) \in \mathscr{J}(\Sigma)$ such that $e(\tau, x) = 2n$. Ekeland [3] proved the conjecture if Σ is $\sqrt{2}$-pinched. Long [6] studied the existence of non-hyperbolic closed characteristics if there exist only finitely many hyperbolic ones on $\Sigma \in \mathscr{H}(2n)$. A similar result was proved for star-shaped hypersurfaces in [5]. If $\Sigma \in \mathscr{H}(2n)$ is symmetric with respect to the origin 0 of $\mathbf{R}^{2n}$, i.e., $x \in \Sigma$ implies $-x \in \Sigma$, the conjecture was proved by Dell'Antonio et al. [1]. In [7], Long proved that both the two closed characteristics are elliptic if there are precisely two geometrical distinct ones on a $\Sigma \in \mathscr{H}(4)$. Long and Zhu [9], further proved the existence of at least one elliptic closed characteristic on Σ if the number of elements in $\mathscr{J}(\Sigma)$ is finite, as well as a result on the existence of at least two elliptic closed characteristics on Σ under certain conditions.

In this paper we study the stability of closed characteristics on partially symmetric hyper-surface. Fixing an integer κ with $0 \leqslant \kappa \leqslant n$, let $P = \mathrm{diag}(-I_{n-\kappa}, I_\kappa, -I_{n-\kappa}, I_\kappa)$ and $\mathscr{H}_\kappa(2n) = \{\Sigma \in \mathscr{H}(2n) \mid x \in \Sigma \text{ implies } Px \in \Sigma\}$. Recall that a lower bound estimate on $^{\#}\tilde{\mathscr{J}}(\Sigma)$ was established for any $\Sigma \in \mathscr{H}_\kappa(2n)$ by the authors in the recent [2]. The following is the main result of this paper:

Theorem 1. *For any $\Sigma \in \mathscr{H}_\kappa(2n)$, there exists $(\tau, x) \in \mathscr{J}(\Sigma, \alpha)$ such that*

$$e(\tau, x) \geqslant 2n - 4\kappa.$$

In the following Section 2, we prove Theorem 1. Then in Section 3 we compute the $(P, 1)$-index of a minimal solution of the functional corresponding to the problem (1.4)–(1.5) which is used in Section 2.

2. The proof of Theorem 1

In order to prove Theorem 1 we need some results about (P, ω)-index theory introduced in [2]. As usual we define

$$\mathrm{Sp}(2n) = \{M \in GL(\mathbf{R}^{2n}) | M^T J M = J\},$$

$$\mathscr{P}_\tau(2n) = \{\gamma \in C([0, \tau], \mathrm{Sp}(2n)) \mid \gamma(0) = I_{2n}\},$$

where M^T denotes the transpose of M and $\tau > 0$ is a constant.

For every $\gamma \in \mathscr{P}_\tau(2n)$ and $\omega \in \mathbf{U}$ a pair of integers $(i_{P,\omega}(\gamma), \nu_{P,\omega}(\gamma)) \in \mathbf{Z} \times \{0, 1, \ldots, 2n\}$ was defined by the authors in [2]. The nullity has a simple expression:

$$\nu_{P,\omega}(\gamma) = \nu_{P,\omega}(\gamma(\tau)) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(\gamma(\tau) - \omega P). \tag{2.1}$$

We refer to [2] for the definition of $i_{P,\omega}(\gamma)$. The splitting numbers of M at (P,ω) were also defined by:

$$S_M^{\pm}(P,\omega)=\lim_{\varepsilon\to 0^+} i_{P,\exp(\pm\sqrt{-1}\varepsilon)\omega}(\gamma)-i_{P,\omega}(\gamma).$$

We shall need the following results from Definition I.2.8 and Lemma I.2.9 of [4], Propositions 3.8 and 3.12, and Theorem 4.1 of [2]:

Lemma 2. (i) *For any $M\in \mathrm{Sp}(2n)$ and $\omega\in\mathbf{U}$ being an eigenvalue of M, denote by $m_\omega(M)$ and $(p_\omega(M),q_\omega(M))$ the algebraic multiplicity of the eigenvalue ω and the Krein type numbers of M at ω, and denote by $\bar\omega$ the complex conjugate of ω, then $\bar\omega$ is an eigenvalue of M and*

$$p_\omega(M)+q_\omega(M)=m_\omega(M),\ \ m_\omega=m_{\bar\omega}.$$

Moreover, we have

$$p_1(M)=q_1(M)=\frac{1}{2}m_1(M).$$

(ii) *For any $M\in\mathrm{Sp}(2n)$ and $\omega\in\mathbf{U}$ being an eigenvalue of M, we have*

$$0\leqslant S_M^+(P,\omega)\leqslant p_\omega(MP),\quad 0\leqslant S_M^-(P,\omega)\leqslant q_\omega(MP).$$

(iii) *For $\gamma\in\mathscr{P}_\tau(2n)$ and $M=\gamma(\tau)$, we have*

$$i_{P,-1}(\gamma)=i_{P,1}(\gamma)+\sum_{0\leqslant\theta<\pi} S_M^+\left(P,e^{\sqrt{-1}\theta}\right)-\sum_{0<\theta\leqslant\pi} S_M^-\left(P,e^{\sqrt{-1}\theta}\right).$$

(iv) *Let $A(t)$ be a symmetric, positive definite $2n\times 2n$ real matrix function and continuous in $t\in[0,\tau]$. Denote the fundamental solution of* (1.6) *satisfying $\gamma_A(0)=I_{2n}$ by $\gamma=\gamma_A(t)$. Then*

$$i_{P,1}(\gamma_A)=\kappa+\sum_{0<s<\tau}\nu_{P,1}(\gamma_A(s)).$$

Now we give

Proof of Theorem 1. Define two function spaces $W_P=\{x\in W^{1,2}([0,1],\mathbf{R}^{2n})\mid x(1)=Px(0)\}$ and $L^2=L^2((0,1),\mathbf{R}^{2n})$. Define $\Lambda: W_P\subset L^2\to L^2$ by $(\Lambda x)(t)=\dot x(t)$. Then we have an orthogonal decomposition $L^2=\mathrm{Im}(\Lambda)\oplus\ker(\Lambda)$. We denote by $(x_1,y_1)\diamondsuit(x_2,y_2)=(x_1,x_2,y_1,y_2)$ for any $x_i,\ y_i\in\mathbf{R}^{m_i}$ with some integer m_i for $i=1,2$. Simple computations yield $\ker(\Lambda)=\{0\diamondsuit\xi\mid\xi\in\mathbf{R}^{2\kappa}\}$, where 0 is the origin of $\mathbf{R}^{2n-2\kappa}$ and

$$\mathrm{Im}(\Lambda)=\{u_1\diamondsuit u_2\mid u_1\in L^2((0,1),\mathbf{R}^{2n-2\kappa}),\int_0^1 u_2(t)\,\mathrm{d}t=0,u_2\in L^2((0,1);\mathbf{R}^{2\kappa})\}.$$

Denote by $L^2_\kappa = \mathrm{Im}(\Lambda)$. Then $\Lambda_0 = \Lambda|_{L^2_\kappa \cap W_P}$ is invertible, and for any $u = u_1 \diamond u_2 \in \mathrm{Im}(\Lambda)$ we have

$$(\Lambda_0^{-1}u)(t) = x_1(t) \diamond x_2(t), \tag{2.2}$$

$$x_1(t) = \int_0^t u_1(\tau)\,\mathrm{d}\tau - \frac{1}{2}\int_0^1 u_1(\tau)\,\mathrm{d}\tau, \tag{2.3}$$

$$x_2(t) = \int_0^t u_2(\tau)\,\mathrm{d}\tau - \int_0^1 \mathrm{d}t \int_0^t u_2(\tau)\,\mathrm{d}\tau. \tag{2.4}$$

So $J\Lambda_0^{-1} = \Lambda_0^{-1}J$ and $J\Lambda_0^{-1} : L^2_\kappa \to L^2_\kappa$ is self-adjoint and compact.

Define $\beta > 0$ by $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and set $L^\beta_\kappa = L^2_\kappa \cap L^\beta((0,1); \mathbf{R}^{2n})$. Consider the functional

$$\psi(u) = \int_0^1 \left[\frac{1}{2}(Ju, \Lambda_0^{-1}u) + H^*(-Ju)\right]\mathrm{d}t, \quad \forall u \in L^\beta_\kappa,$$

where $H^*(x) = \sup_{y \in \mathbf{R}^{2n}} \{(x,y) - H(y)\}$ is the Fenchel dual of H, It is well-known that the global minimum of ψ on L^β_κ is reached. Denote by $\bar{u}$ one of its global minimal point. Then we have

$$\psi(\bar{u}) = \min_{u \in L^\beta_\kappa} \psi(u) < 0. \tag{2.5}$$

Then it is also well-known (cf. [4]) that $\bar{u} \not\equiv$ constant holds and that the minimal period of $\bar{u}$ must be 1 if it is periodic from (2.5), although it may not be periodic at all in our case. We also have $\psi'(\bar{u}) = 0$. Note that for any $u \in L^\beta_\kappa$, we have $\psi'(u) \in L^\alpha_\kappa((0,1); \mathbf{R}^{2n})$ is a linear functional on L^β_κ:

$$\psi'(u)(v) = \int_0^1 [(Ju, \Lambda_0^{-1}v) + (H^{*'}(-Ju), -Jv)]\,\mathrm{d}t, \quad \forall u, v \in L^\beta_\kappa. \tag{2.6}$$

We obtain

$$H^{*'}(-J\bar{u}) = \Lambda_0^{-1}\bar{u} + \xi_{\bar{u}}, \tag{2.7}$$

for some $\xi_{\bar{u}} \in \ker(\Lambda)$. By the Legendre reciprocity formula of Proposition II.1.15 of [4], $\bar{x} \equiv \Lambda_0^{-1}\bar{u} + \xi_{\bar{u}} \neq 0$ is a solution of the boundary value problem:

$$\dot{x} = JH'(x), \quad \forall t \in (0,1), \quad x(1) = Px(0). \tag{2.8}$$

From (2.6) we have

$$(\psi''(\bar{u})v, v) = \int_0^1 [(Jv, \Lambda_0^{-1}v) + (H^{*''}(-J(\bar{u}))Jv, Jv)]\,\mathrm{d}t, \tag{2.9}$$

for every $v \in L_\kappa^\beta$. By Proposition II.2.10 of [4] and (2.7), we have

$$H^{*''}(-J\bar{u}(t)) = (H''(\bar{x}(t))^{-1}. \tag{2.10}$$

It follows that $\psi''(\bar{u})$ in (2.9) can be defined on $v \in L_\kappa^2$. From (2.5), $\bar{u}$ is a minimal point, we know that the Morse index of $\psi''(\bar{u})$ defined on L_κ^2 is zero. That is

$$i_P^E(A_{\bar{x}}) = 0, \tag{2.11}$$

where $A_{\bar{x}}(t) = H''(\bar{x}(t))$ for $t \in [0,1]$. We postpone the definition of $i_P^E(A_{\bar{x}})$ to the next section. Note that $\bar{x}$ is defined on $[0,1]$. Let

$$\bar{x}(t+1) = P\bar{x}(t), \quad \forall t \in [0,1].$$

By (2.7), we have $\lim_{\varepsilon\to 0+} \bar{x}(1+\varepsilon) = P\bar{x}(0) = \bar{x}(1)$. So $\bar{x} \in C([0,2], \mathbf{R}^{2n})$, and $\bar{x}(2) = P\bar{x}(1) = P^2\bar{x}(0) = \bar{x}(0)$. By definition, we have

$$H(Py) = H(y), \quad \forall\, y \in \mathbf{R}^{2n}.$$

Thus there hold

$$PH'(Py) = H'(y), \quad PH''(Py)P = H''(y), \quad \forall\, y \in \mathbf{R}^{2n}. \tag{2.12}$$

We have from (2.12)

$$\begin{aligned}
\lim_{\varepsilon\to 0+} \dot{\bar{x}}(1+\varepsilon) &= \lim_{\varepsilon\to 0+} P\dot{\bar{x}}(0+\varepsilon) = \lim_{\varepsilon\to 0+} PJH'(\bar{x}(\varepsilon)) \\
&= PJH'(\bar{x}(0)) = JPH'(\bar{x}(0)) = JH'(P\bar{x}(0)) \\
&= JH'(\bar{x}(1)) = \lim_{\varepsilon\to 0+} JH'(\bar{x}(1-\varepsilon)) \\
&= \lim_{\varepsilon\to 0+} \dot{\bar{x}}(1-\varepsilon).
\end{aligned}$$

So $\bar{x} \in C^1([0,2], \mathbf{R}^{2n})$, and $x = \bar{x}(t)$ satisfies

$$\dot{x} = JH'(\bar{x}), \quad \forall t \in (0,2), \quad \bar{x}(2) = \bar{x}(0). \tag{2.13}$$

Let $\gamma = \gamma_{\bar{x}}(t)$ be the fundamental solution of (1.6) with $A(t) = H''(\bar{x}(t))$ for $t \in [0,2]$ satisfying $\gamma(0) = I_{2n}$. By (2.12), we have

$$H''(\bar{x}(t+1)) = H''(P\bar{x}(t)) = PH''(\bar{x}(t))P.$$

Direct calculations give

$$\begin{aligned}
\frac{d}{dt}(P\gamma(t)P\gamma(1)) &= P\dot{\gamma}(t)P\gamma(1) \\
&= PJH''(\bar{x}(t))\gamma(t)P\gamma(1) = JH''(\bar{x}(t+1))P\gamma(t)P\gamma(1).
\end{aligned}$$

Since the fundamental solution of (1.6) is unique, γ satisfies

$$\gamma(t+1) = P\gamma(t)P\gamma(1), \quad \forall t \in [0,1].$$

Specially

$$\gamma(2) = (P\gamma(1))^2. \tag{2.14}$$

We claim that (2.11) implies

$$i_{P,1}(\gamma|_{[0,1]}) = \kappa, \tag{2.15}$$

and postpone its proof to the next section.

By Theorem 4.1 of [2], we also have

$$i_{P,-1}(\gamma|_{[0,1]}) \geqslant n - \kappa. \tag{2.16}$$

By (i)–(iii) of Lemma 2, (2.15), and (2.16) we obtain

$$\sum_{0<\theta<\pi} m_{e^{\sqrt{-1}\theta}}(P\gamma(1)) + \frac{1}{2} m_1(P\gamma(1)) \geqslant \sum_{0\leqslant\theta<\pi} p_{e^{\sqrt{-1}\theta}}(P\gamma(1)) \geqslant n - 2\kappa.$$

Thus by (i) of Lemma 2 and (2.14) we get

$$e(P\gamma(1)) \geqslant 2n - 4\kappa \quad \text{and} \quad e(\gamma(2)) = e((P\gamma(1))^2) = e(P\gamma(1)) \geqslant 2n - 4\kappa.$$

Note that by our above study the minimal period $\bar{\tau}$ of $\bar{x}$ is either 1 or 2. Let $h = H(\bar{x}(t))$.

If $h = 1$, then $(\bar{\tau}, \bar{x}) \in \mathcal{J}(\Sigma, \alpha)$ from (2.13), and

$$e(\bar{\tau}, \bar{x})) = e(2, \bar{x}) = e((P\gamma(1))^2) \geqslant 2n - 4\kappa.$$

If $h \neq 1$, viewing $\bar{x}$ as a 2-periodic solution, let

$$x_h = h^{-\frac{1}{\alpha}} \bar{x}(h^{\frac{2}{\alpha}-1} t), \quad \tau_h = 2h^{1-\frac{2}{\alpha}}.$$

Then $H(x_h(t)) = 1$ for all $t \in [0,2]$, the minimal period $\bar{\tau}_h$ of x_h is either $\tau_h/2$ or τ_h, and $(\bar{\tau}_h, x_h) \in \mathcal{J}(\Sigma)$. Because $H(\cdot)$ is α-homogenous, $H''(\cdot)$ is $(\alpha-2)$-homogenous, we get

$$H''(\bar{x}_h(t)) = h^{\frac{2}{\alpha}-1} H''(\bar{x}(h^{\frac{2}{\alpha}-1} t)).$$

Let γ_h be the fundamental solution of (1.6) with $A(t) = H''(\bar{x}_h(t))$ for $t \in (0, \tau_h)$ satisfying $\gamma_h(0) = I_{2n}$. Then we have

$$\gamma_h(t) = \gamma(h^{\frac{2}{\alpha}-1} t).$$

Thus $\gamma_h(\tau_h) = \gamma(2)$. Then we have $(\bar{\tau}, \bar{x}_h) \in \mathscr{J}(\Sigma)$ and

$$e(\bar{\tau}_h, \bar{x}_h) = e(\bar{\tau}_h, \bar{x}_h) = e(\gamma_h(\tau_h)) = e(\gamma(2)) \geqslant 2n - 4\kappa.$$

This completes the proof of Theorem 1. $\square$

3. The proof of the computation (2.15)

In this section we will define $i_P^E(A)$, which appeared in (2.11), and prove (2.15). In order to do this we will give a slight generalization of the contents of Section I.4 in [4] of Ekeland first.

For any symmetric and positive definite $2n \times 2n$ real matrix $A(t)$ continuous in $t \in [0, +\infty)$, let $B(t) = A(t)^{-1}$ and consider the following quadratic form:

$$q_{s,\kappa}(u, u) = \frac{1}{2} \int_0^s [(Ju, \Pi_{s,\kappa} u) + (B(t)Ju, Ju)]\, \mathrm{d}t, \quad \forall u \in L_\kappa^2(0, s), \tag{3.1}$$

where $L_\kappa^2(0, s) = \{u_1 \diamond u_2 \mid u_1 \in L^2((0, s), \mathbf{R}^{2n-2\kappa})$ and $u_2 \in L^2((0, s); \mathbf{R}^{2\kappa})$ with $\int_0^s u_2(t)\, \mathrm{d}t = 0\}$ and $\Pi_{s,\kappa} : L_\kappa^2(0, s) \to L_\kappa^2(0, s)$ is defined by

$$(\Pi_{s,\kappa} u)(t) = x_1(t) \diamond x_2(t), \tag{3.2}$$

$$x_1(t) = \int_0^t (u_1(\tau))\, \mathrm{d}\tau - \frac{1}{2} \int_0^s u_1(\tau)\, \mathrm{d}\tau, \tag{3.3}$$

$$x_2(t) = \int_0^t u_2(\tau)\, \mathrm{d}\tau - \frac{1}{s} \int_0^s \mathrm{d}t \int_0^t u_2(\tau)\, \mathrm{d}\tau, \tag{3.4}$$

for any $u = u_1 \diamond u_2 \in L_\kappa^2(0, s)$. Note that we have $\Pi_{1,\kappa} = \Lambda_0^{-1}$ and $L_\kappa^2(0, 1) = L_\kappa^2$.

Lemma 3. *For any symmetric and positive definite $2n \times 2n$ real matrix $A(t)$ continuous in $t \in [0, +\infty)$, there is a $q_{s,\kappa}$-orthogonal decomposition*

$$L_\kappa^2(0, s) = E_\kappa^+(A) \oplus E_\kappa^0(A) \oplus E_\kappa^-(A)$$

such that $q_{s,\kappa}$ is positive definite, null and negative definite on $E_\kappa^+(A)$, $E_\kappa^0(A)$ and $E_\kappa^-(A)$, respectively. Moreover, the dimensions of $E_\kappa^0(A)$ and $E_\kappa^-(A)$ are finite.

Proof. Define $\bar{B}_\kappa : L_\kappa^2 \to L_\kappa^2$ by

$$(\bar{B}_\kappa u, v) = \int_0^s (B(t)Ju(t), Jv(t))\, \mathrm{d}t, \quad \forall u, v \in L_\kappa^2(0, s).$$

From the Lax–Milgram theorem, $\bar{B}_\kappa$ is an isomorphism and $L^2_\kappa(0,s)$ is a Hilbert space under the inner product $(\bar{B}_\kappa u, v)$. Because

$$(\bar{B}_\kappa \bar{B}_\kappa^{-1} J\Pi_{s,\kappa} u, v) = (J\Pi_{s,\kappa} u, v) = (u, J\Pi_{s,\kappa} v) = (\bar{B}_\kappa u, \bar{B}_\kappa^{-1} J\Pi_{s,\kappa} v), \quad \forall u, v \in L^2_\kappa(0,s),$$

the map $\bar{B}_\kappa^{-1} J\Pi_{s,\kappa} : L^2_\kappa(0,s) \to L^2_\kappa(0,s)$ is self-adjoint. From the spectral theory of compact self-adjoint operators on a Hilbert space, there exist a basis $\{e_j\}_{j\in\mathbf{N}}$ of $L^2_\kappa(0,s)$, and a sequence $\lambda_j \to 0$ in $\mathbf{R}$ as $j \to +\infty$ such that

$$(\bar{B}_\kappa e_i, e_j) = \delta_{ij},$$

$$\bar{B}_\kappa^{-1} J\Pi_{s,\kappa} e_j = \lambda_j e_j.$$

And hence, for any $u = \sum_{j=1}^{\infty} c_j e_j \in L^2_\kappa(0,s)$, we have

$$\begin{aligned} q_{s,\kappa}(u,u) &= -\frac{1}{2}(J\Pi_{s,\kappa} u, u) + \frac{1}{2}(\bar{B}_\kappa u, u) \\ &= \frac{1}{2}\sum_{j=1}^{\infty} (1-\lambda_j) c_j^2. \end{aligned}$$

Define

$$E^+_\kappa(A) = \left\{ \sum c_j e_j \mid c_j = 0 \text{ if } 1-\lambda_j \leqslant 0 \right\},$$

$$E^0_\kappa(A) = \left\{ \sum c_j e_j \mid c_j = 0 \text{ if } 1-\lambda_j \neq 0 \right\},$$

$$E^-_\kappa(A) = \left\{ \sum c_j e_j \mid c_j = 0 \text{ if } 1-\lambda_j \geqslant 0 \right\},$$

Then the claim of the lemma follows because $\lambda_j \to 0$ as $j \to +\infty$. □

Definition 4. For any symmetric positive definite continuous $2n \times 2n$ real matrix function $A(t)$ in $t \in [0,s]$, we define

$$\nu^E_P(A) = \dim E^0_\kappa(A), \quad i^E_P(A) = \dim E^-_\kappa(A).$$

Proposition 5. *For any symmetric positive definite continuous $2n \times 2n$ real matrix function $A(t)$ in $t \in [0,s]$, there hold*

$$\nu^E_P(A) = \nu_{P,1}(\gamma_A), \quad i_{P,1}(\gamma_A) = \kappa + i^E_P(A),$$

where $\gamma = \gamma_A(t)$ is the fundamental solution of (1.6) *with $\gamma(0) = I_{2n}$.*

Proof. The proof is carried out in 6 steps.

Step 1. The proof of the first equality in Proposition 5.

For any $u \in E^0_\kappa(A)$, by definition and Lemma 3, we have

$$q_{s,\kappa}(u,v) = \frac{1}{2}\int_0^s (\Pi_{s,\kappa}u + B(t)Ju, Jv)\,\mathrm{d}t = 0, \quad \forall v \in L^2_\kappa(0,s).$$

Thus there exists $\xi_u \in \ker(\Lambda)$ such that

$$\Pi_{s,\kappa}u + B(t)Ju = \xi_u.$$

Denote by $x = \Pi_{s,\kappa}u - \xi_u$. We obtain $u = \dot{x}$ and

$$\dot{x} = JA(t)x \quad \text{for } t \in (0,s), \quad x(s) = Px(0).$$

Therefore $x(t) = \gamma_A(t)c$, where $c \in \mathbf{R}^{2n}$ satisfies

$$\gamma_A(s)c = P\gamma_A(0)c = Pc.$$

That is,

$$(\gamma_A(s) - P)c = 0. \tag{3.5}$$

Hence we obtain

$$E^0_\kappa(A) \cong \{c \in \mathbf{R}^{2n} | (\gamma_A(s) - P)c = 0\} = \ker(\gamma_A(s) - P),$$

and from (2.1)

$$v^E_P(A) = \dim E^0_\kappa(A) = \dim\ker(\gamma_A(s) - P) = v_{P,1}(\gamma_A).$$

This yields the first equality claimed by Proposition 5.

In the next 5 steps we prove the second equality claimed in Proposition 5. By (iv) of Lemma 2 and the first equality in Proposition 5, it suffices to prove

$$i^E_P(A) = \sum_{0<\sigma<s} v^E_P(A_\sigma), \tag{3.6}$$

where $A_\sigma = A|_{[0,\sigma]}$.

Step 2. Proof for $i^E_P(A_\sigma) = 0$ with $\sigma > 0$ sufficiently small.

In fact, by Definition (3.2)–(3.4) we have

$$\begin{aligned} |x_1(t)| &\leqslant 2\int_0^\sigma |u_1(t)|\,\mathrm{d}t \leqslant 2\sigma^{\frac{1}{2}}||u_1||, \\ ||x_1|| &\leqslant 2\sigma||u_1||, \\ ||x_2|| &\leqslant (1+\sigma)||u_2||, \end{aligned}$$

where $||\cdot||$ is the usual norm in $L^2((0,\tau);\mathbf{R}^{2n})$. Therefore we have

$$||\Pi_{\sigma,\kappa}u|| \leqslant (2+\sigma)\sigma||u||, \quad \forall u \in L^2_\kappa(0,\sigma).$$

Since $A(t)$ is symmetric, positive definite, and continuous, we have

$$(B(t)x,x) \geqslant b(x,x), \quad \forall x \in \mathbf{R}^{2n}, t \in [0,s],$$

where $b>0$ is a constant. It follows that

$$\begin{aligned} q_{\sigma,\kappa}(u,u) &= \frac{1}{2}\int_0^1 [(-J\Pi_{s,\kappa}u,u) + (B(t)Ju,Ju)]\,\mathrm{d}t \\ &\geqslant \frac{1}{2}(-||u||\cdot||\Pi_{\sigma,\kappa}u|| + b||u||^2) \\ &\geqslant \frac{1}{2}(b-(2+\sigma)\sigma)||u||^2. \end{aligned}$$

Therefore $q_{\sigma,\kappa}$ is positive definite when $\sigma < \min\{1, \frac{b}{3}\}$.

Step 3. We claim that there exist only finitely many points $\sigma \in [0,1]$ with $\nu_P^E(A_\sigma) \neq 0$.

In fact, if not, by (3.5) there exist $s_j \in [0,1]$ and $\xi_j \in \mathbf{R}^{2n} \backslash \{0\}$ with $|\xi_j| = 1$ such that

$$\gamma_A(s_j)\xi_j = P\xi_j, \quad \text{for } j = 1,2,\ldots. \tag{3.7}$$

Without loss of generality, we assume $s_j \to s$ and $\xi_j \to \xi$ as $j \to +\infty$. Then we have

$$\gamma_A(s)\xi = P\xi, \tag{3.8}$$

$$(\gamma_A(s_j) - P)(\xi_j - \xi) = (\gamma_A(s) - \gamma_A(s_j))\xi. \tag{3.9}$$

Since $\gamma_A(s_j)$ is symplectic, we have $\gamma_A(s_j)^T J = J\gamma_A(s_j)^{-1}$. By (3.7) we have

$$\begin{aligned} (\gamma_A(s_j)(\xi_j - \xi), JP\xi_j) &= (\xi_j - \xi, \gamma_A(s_j)^T JP\xi_j) \\ &= (\xi_j - \xi, J\gamma_A(s_j)^{-1}P\xi_j) \\ &= (\xi_j - \xi, J\xi_j) \\ &= (P(\xi_j - \xi), JP\xi_j). \end{aligned}$$

Thus $((\gamma_A(s_j) - P)(\xi_j - \xi), JP\xi_j) = 0$, and by (3.9) we have

$$((\gamma_A(s) - \gamma_A(s_j))\xi, JP\xi_j) = 0.$$

Multiplying the left hand side of the above equality by $(s-s_j)^{-1}$ and taking the limit as $j\to+\infty$, from (3.8) and $\dot{\gamma}_A(s)=JA(s)\gamma_A(s)$ we have

$$0=(\dot{\gamma}_A(s)\xi,JP\xi)=(JA(s)P\xi,JP\xi)=(A(s)P\xi,P\xi).$$

This contradiction proves our claim.

Step 4. If $\sigma_1<\sigma_2$, there hold

$$i_P^E(A_{\sigma_1})\leqslant i_P^E(A_{\sigma_2}), \tag{3.10}$$

$$i_P^E(A_{\sigma_1})+\nu_P^E(A_{\sigma_1})\leqslant i_P^E(A_{\sigma_2}). \tag{3.11}$$

In fact, we define a map $r:L_\kappa^2(0,\sigma_1)\to L_\kappa^2(0,\sigma_2)$ by

$$(ru)(t)=\begin{cases} u(t) & \text{if } 0\leqslant t\leqslant\sigma_1,\\ 0 & \text{if } \sigma_1<t\leqslant\sigma_2.\end{cases}$$

Then for any $u\in L_\kappa^2(0,\sigma_1)$ we have

$$q_{\sigma_2,\kappa}(ru,ru)=q_{\sigma_1,\kappa}(u,u).$$

And hence,

$$q_{\sigma_2,\kappa}(u,u)<0,\quad \forall u\in r(E_\kappa^-(A_{\sigma_1}))\setminus\{0\}.$$

This yields

$$i_P^E(A_{\sigma_2})\geqslant\dim(r(E_\kappa^-(A_{\sigma_1})))=i_P^E(A_{\sigma_1}),$$

i.e., (3.10) holds. In a similar way we obtain

$$i_P^E(A_{\sigma_1})+\nu_P^E(A_{\sigma_1})\leqslant i_P^E(A_{\sigma_2})+\nu_P^E(A_{\sigma_2}).$$

From Step 3, it follows that $\nu_P^E(A_{\sigma_2})=0$ as $\sigma_2\to\sigma_1^+$, and hence (3.11).

Step 5. $i_P^E(A_s)$ is left continuous with respect to s.

In fact, from (3.1)–(3.4) we obtain

$$\begin{aligned} x_1(st)&=s\left[\int_0^t u_1(s\tau)\,\mathrm{d}\tau-\frac{1}{2}\int_0^1 u_1(s\tau)\,\mathrm{d}\tau\right],\\ x_2(st)&=s\left[\int_0^t u_2(s\tau)\,\mathrm{d}\tau-\int_0^1\mathrm{d}t\int_0^t u_2(s\tau)\,\mathrm{d}\tau\right],\end{aligned}$$

and $q_{s,\kappa}(u,u)=sq_{s,\kappa}^1(pu,pu)$, where $(pu)(t)=u(st)$ for $t\in[0,1]$ and

$$q_{s,\kappa}^1(u,u)=\frac{1}{2}\int_0^1[s(Ju,\Pi_{1,\kappa}u)+(B(st)Ju,Ju)]\,\mathrm{d}t,\quad \forall u\in L_\kappa^2(0,1). \tag{3.12}$$

For any fixed s_0, let $E_1 := p(E_\kappa^-(A_{s_0}))$, then $i_P^E(A_{s_0}) = \dim E_1$, and

$$q^1_{s_0,\kappa}(u,u) < 0, \quad \forall u \in E_1 \backslash \{0\}.$$

From the continuity of $q^1_{s,\kappa}$ with respect to s in (3.12), we also have

$$q^1_{s,\kappa}(u,u) < 0, \quad \forall u \in E_1 \backslash \{0\},$$

as $s \to s_0$. Then we get

$$i_P^E(A_s) \geqslant i_P^E(A_{s_0})$$

as $s \to s_0$. Together with (3.10), the claim is proved.

Step 6. $i_P^E(A_s)$ is continuous at the point $s \in (0,1)$ with $v_P^E(A_s) = 0$ and for any $s \in [0,1)$, there holds

$$\lim_{\varepsilon \to 0+} i_P(A_{s+\varepsilon}) = i_P^E(A_s) + v_P^E(A_s). \tag{3.13}$$

In fact, denote by

$$(B^1_\kappa(s)u,u) = \int_0^1 (B(st)Ju, Ju)\,\mathrm{d}t, \quad \forall u \in L^2_\kappa(0,1).$$

Then $L^2_\kappa(0,1)$ is a Hilbert space under the inner product $(B^1_\kappa(s)u,u)$. $B^1_\kappa(s)^{-1} J\Pi_{1,\kappa}$ is a self-adjoint compact operator, so there exist a basis $\{e_j^s \mid j \in \mathbf{N}\}$ of $L^2_\kappa(0,1)$ and a sequence λ_j^s in $\mathbf{R}$ with $\lambda_j^s \to 0$, such that

$$\begin{aligned} &(B^1_\kappa(s)e_j^s, e_j^s) = \delta_{ij}, \\ &(J\Pi_{1,\kappa} e_j^s, u) = \lambda_j^s (B^1_\kappa(s)e_j^s, u), \quad \forall u \in L^2_\kappa(0,1). \end{aligned}$$

For any $u = \sum_{j=1}^\infty \xi_j e_j^s$, we have

$$q^1_{s,\kappa}(u,u) = \frac{1}{2} \sum_{j=1}^\infty (1 - s\lambda_j^s)\xi_j^2.$$

Fix a $\sigma > 0$ and denote by $K = \lim_{\varepsilon \to 0+} i_P^E(A_{\sigma+\varepsilon})$. There is a $\sigma' > \sigma$ such that $i_P^E(A_s) = K$ for $s \in (\sigma, \sigma')$. So for any $s \in (\sigma, \sigma')$ we have

$$1 - s\lambda_j^s < 0 \quad \text{for } 1 \leqslant j \leqslant K.$$

Fix $j \leqslant K$. Then $\lambda_j^s = (J\Pi_{1,\kappa} e_j^s, e_j^s)$ is bounded and $\lambda_j^s > \frac{1}{s} \geqslant \frac{1}{\sigma'}$. There exist $\{e_j^{s(l)}\}$ and $\{\lambda_j^{s(l)}\}$ such that $e_j^{s(l)} \rightharpoonup e_j$ in $L^2_\kappa(0,1)$ and $\lambda_j^{s(l)} \to \lambda_j, s(l) \to \sigma$ in $\mathbf{R}$ as $l \to \infty$.

So we have

$$(B^1_\kappa(\sigma)e_j, e_j) = \delta_{ij}, i,j = 1,2,\dots,K,$$

$$(J\Pi_{1,\kappa}e_j, u) = \lambda_j(B^1_\kappa(\sigma)e_j, u), \quad \forall u \in L^2_\kappa(0,1), j = 1,2,\dots,K,$$

$$1 - \sigma\lambda_j \leqslant 0, \quad \text{for } j = 1,2,\dots,K.$$

Therefore, for any $u = \sum_{j=1}^K \xi_j e_j$, we have

$$q^1_{\sigma,\kappa}(u,u) = \frac{1}{2}\sum_{j=1}^K (1-\sigma\lambda_j)\xi_j^2 \leqslant 0.$$

Hence,

$$K \leqslant i^E_P(A_\sigma) + v^E_P(A_\sigma).$$

Combining (3.11) we obtain (3.13). If $v^E_P(A_s) = 0$, then $\lim_{\varepsilon\to 0+} i_P(A_{s+\varepsilon}) = i^E_P(A_s)$ from (3.13), and $i^E_P(A_t)$ is continuous at $t = s$. □

Proof of (2.15). By (2.10) we know that the Morse index of $(\psi''(\bar{u})v, v)$ defined by (2.9) on L^2_κ is $i^E_P(A_{\bar{x}})$, which yields (2.11). Since $\gamma = \gamma_A$ and $A(t) = H''(\bar{x}(t))$ for $t \in [0,1]$, we obtain (2.15) from the second equality of Proposition 5. □

At the end of this section we give an example for readers on computing the index $i^E_P(cI_{2n})$ for $c > 0$. Let $E[a] = \max\{k \in \mathbf{Z} \mid k < a\}$.

Example 6. For any $c > 0$, we have

$$i_P(cI_{2n}|_{[0,s]}) = \begin{cases} 2\kappa\, E\left[\dfrac{cs}{2\pi}\right] & \text{if } c \leqslant \frac{3}{2}\pi, \\ 2\kappa\, E\left[\dfrac{cs}{2\pi}\right] + 2(n-\kappa)E\left[\dfrac{cs - \frac{3}{2}\pi}{2\pi}\right] & \text{if } c > \frac{3}{2}\pi. \end{cases} \tag{3.14}$$

In fact, denoting by $\gamma(t)$ the fundamental solution of (1.6) with $A(t) = cI_{2n}$, by definition we have $\gamma(t) = e^{ctJ}$ and that $\ker(\gamma(t) - P) \neq \{0\}$ if and only if $t = t_k = \frac{2k\pi}{c}$ with $\dim\ker(\gamma(t_k) - P)) = 2\kappa$, or $t = s_k = (2k\pi + \frac{3}{2}\pi)/c$ with $\dim\ker(\gamma(s_k) - P)) = 2(n-\kappa)$. Then (3.14) follows from (3.6) and the first equality of Proposition 5.

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