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Algebraic Bethe ansatz for the Temperley–Lieb spin-1 chain

Rafael I. Nepomechie [a,*], Rodrigo A. Pimenta [a,b]

[a] *Physics Department, P.O. Box 248046, University of Miami, Coral Gables, FL 33124 USA*

[b] *Departamento de Física, Universidade Federal de São Carlos, Caixa Postal 676, CEP 13565-905, São Carlos, Brazil*

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Abstract

We use the algebraic Bethe ansatz to obtain the eigenvalues and eigenvectors of the spin-1 Temperley–Lieb open quantum chain with "free" boundary conditions. We exploit the associated reflection algebra in order to prove the off-shell equation satisfied by the Bethe vectors.

1. Introduction

The algebraic Bethe ansatz is a powerful method for solving both closed [1] and open [2] integrable quantum spin chains. The primary objective is to solve the spectral problem associated with the transfer matrix. The ultimate goals are to also compute scalar products and correlation functions [3].

There are models, however, that have resisted solution by means of the algebraic Bethe ansatz, despite being integrable. One such example is the open quantum spin-s chain with "free" boundary conditions constructed from the Temperley–Lieb (TL) algebra TL_N [4], for spin $s > \frac{1}{2}$. This is a unital algebra over the complex numbers $\mathbb{C}$ with $N-1$ generators $\{X_{(1)}, \dots, X_{(N-1)}\}$ satisfying

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* Corresponding author.

E-mail addresses: nepomechie@physics.miami.edu (R.I. Nepomechie), pimenta@df.ufscar.br (R.A. Pimenta).

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$$\begin{aligned} X_{(i)}^2 &= c X_{(i)} \,, \\ X_{(i)} X_{(i\pm 1)} X_{(i)} &= X_{(i)} \,, \\ X_{(i)} X_{(j)} &= X_{(i)} X_{(j)} \,, \qquad |i-j|>1 \,, \end{aligned} \tag{1.1}$$

where $c = -(q+q^{-1})$ and q is an arbitrary parameter. The associated spin chain Hamiltonian is given by

$$H = \sum_{i=1}^{N-1} X_{(i)} \,. \tag{1.2}$$

The operator $X_{(i)}$, which acts on $\left(\mathbb{C}^{(2s+1)}\right)^{\otimes N}$, is defined by

$$X_{(i)} = X_{i,i+1} = \mathbb{I}^{\otimes(i-1)} \otimes X \otimes \mathbb{I}^{\otimes(N-i-1)} \,, \tag{1.3}$$

where X is a $(2s+1)^2$ by $(2s+1)^2$ matrix with the following matrix elements [5]

$$\langle m_1, m_2 | X | m_1', m_2' \rangle = (-1)^{m_1 - m_1'} Q^{m_1 + m_1'} \delta_{m_1+m_2,0} \delta_{m_1'+m_2',0} \,, \tag{1.4}$$

where $m_1, m_2, m_1', m_2' = -s, -s+1, \ldots, s$; and $\mathbb{I}$ is the identity operator on $\mathbb{C}^{2s+1}$. The parameter Q is related to q by

$$c = -(q+q^{-1}) = [2s+1]_Q = \frac{Q^{2s+1} - Q^{-2s-1}}{Q - Q^{-1}} = \sum_{k=-s}^{s} Q^{2k} \,. \tag{1.5}$$

The integrability of the Hamiltonian (1.2), as well as the possibility of solving it by algebraic Bethe ansatz, is based on the fact that the TL algebra gives rise to solutions of the Yang–Baxter equation by means of a procedure known as Baxterization [6]. However, the R-matrix associated with this Hamiltonian (see Eq. (2.1) below) leads to very unusual exchange relations for the generators of the Yang–Baxter and reflection algebras. This seems to be the main difficulty that has obstructed the use of the algebraic Bethe ansatz for R-matrices from the TL algebra.

In a previous paper [7], we have proposed a number of results related to the spectrum of (1.2). In particular, we have conjectured the Bethe states and the off-shell equations that they satisfy. Interestingly, such off-shell equations have a universal character, in the sense that they are independent of the value of the spin s. The aim of this note is to present a proof of this conjecture for $s = 1$, in which case X (1.4) is the following 9×9 matrix

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q^{-2} & 0 & -Q^{-1} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -Q^{-1} & 0 & 1 & 0 & -Q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -Q & 0 & Q^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \tag{1.6}$$

and

$$1 + Q^2 + Q^{-2} = -\left(q + q^{-1}\right). \tag{1.7}$$

For $Q = 1$, this is just the pure biquadratic spin-1 chain

$$X\Big|_{Q=1} = \left(\vec{S} \dot{\otimes} \vec{S}\right)^2 - \mathbb{I} \otimes \mathbb{I}, \tag{1.8}$$

while the general-Q case corresponds to the $U_Q sl(2)$-deformation [5]. Let us mention that the model (1.2), as well as its closed version with periodic boundary conditions, has been previously studied by many authors using alternative approaches to the algebraic Bethe ansatz, see *e.g.* [8–16] and references therein.

We briefly review the construction of the transfer matrix corresponding to the Hamiltonian (1.2) in section 2. We then recall in section 3 the basic objects of the quantum inverse scattering method and present the main results of this paper, given in Propositions 1–4, which follow from a careful analysis of the reflection algebra. We also briefly consider the scalar product between an on-shell Bethe vector and its off-shell dual. We discuss our results and some further directions of investigation in section 4. Some functions introduced in the main text are collected in Appendix A. We present some details of the proof of Proposition 1 in Appendix B.

2. Transfer matrix

Integrable quantum spin chains are characterized by a set of commuting conserved quantities (among them the Hamiltonian), whose generating function is the transfer matrix. We briefly review here the construction of the transfer matrix corresponding to the Hamiltonian (1.2), (1.6). The main ingredient is the R-matrix, which acts on the vector space $\mathbb{C}^3 \otimes \mathbb{C}^3$, and is given (using the notation of [15]) by [6]

$$R(u) = \omega(qu)\mathcal{P} + \omega(u)\mathcal{P}X\,, \qquad \omega(u) = u - u^{-1}\,, \tag{2.1}$$

where X is given by (1.6), and

$$\mathcal{P} = \sum_{a,b=1}^{3} e_{ab} \otimes e_{ba}\,, \qquad (e_{ab})_{ij} = \delta_{a,i}\delta_{b,j}\,, \tag{2.2}$$

is the permutation matrix. As a consequence of the TL algebra (1.1), the R-matrix satisfies the Yang–Baxter equation

$$R_{12}(u/v)\, R_{13}(u)\, R_{23}(v) = R_{23}(v)\, R_{13}(u)\, R_{12}(u/v). \tag{2.3}$$

It also has the unitarity property

$$R_{12}(u) R_{21}(u^{-1}) = \zeta(u)\, \mathbb{I}^{\otimes 2}\,, \qquad \zeta(u) = \omega(uq^{-1})\, \omega(u^{-1}q^{-1})\,, \tag{2.4}$$

where $R_{21} = \mathcal{P}_{12}\, R_{12}\, \mathcal{P}_{12} = R_{12}^{t_1 t_2}$. The R-matrix can be used to construct the single-row monodromy matrices

$$T_0(u) = R_{0N}(u) \dots R_{01}(u)\,, \qquad \hat{T}_0(u) = R_{10}(u) \dots R_{N0}(u)\,, \tag{2.5}$$

where 0 denotes an auxiliary vector space. It follows from the Yang–Baxter equation that T obeys the fundamental relation

$$R_{12}(u/v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u/v)\,, \tag{2.6}$$

and $\hat{T}$ obeys a similar relation. The double-row monodromy matrix [2] is given by

$$U_0(u) = T_0(u)\,\hat{T}_0(u)\,, \tag{2.7}$$

and it obeys the reflection equation

$$R_{12}(u/v)\,U_1(u)\,R_{21}(uv)\,U_2(v) = U_2(v)\,R_{12}(uv)\,U_1(u)\,R_{21}(u/v)\,. \tag{2.8}$$

The double-row transfer matrix is given by [2,7]

$$t(u) = \mathrm{tr}_0\left[M_0\,U_0(u)\right]\,, \qquad M = \mathrm{diag}\left(Q^{-2}, 1, Q^2\right)\,. \tag{2.9}$$

Indeed, it has the fundamental commutativity property

$$[t(u)\,, t(v)] = 0\,, \tag{2.10}$$

and it contains the Hamiltonian (1.2)

$$H = \alpha \frac{d}{du} t(u)\Big|_{u=1} + \beta\, \mathbb{I}^{\otimes N}\,, \tag{2.11}$$

where

$$\alpha = -\left[4\omega(q^2)\,\omega(q)^{2N-2}\right]^{-1}\,, \qquad \beta = \frac{\omega(q)}{\omega(q^2)} - \frac{N}{2}\frac{\omega(q^2)}{\omega(q)}\,. \tag{2.12}$$

The relations (2.10) and (2.11) imply that the model (1.2) is integrable.

3. Algebraic Bethe ansatz

We now use the algebraic Bethe ansatz to solve the spectral problem associated with the transfer matrix (2.9). Let us recall the basic needed steps:

1. Identify suitable operators on the quantum space from auxiliary-space matrix elements of the double-row monodromy matrix (2.7).
2. Identify a reference state with respect to the creation and annihilation operators.
3. Formulate convenient exchange relations from the reflection algebra (2.8).
4. Define a Bethe vector as a product of creation operators acting on the reference state; and use the exchange relations to determine the action of the transfer matrix on an off-shell Bethe vector.

Let us denote the auxiliary-space matrix elements of the single-row monodromy matrices $T_0(u)$ and $\hat{T}_0(u)$ by

$$T_0(u) = \begin{pmatrix} T_{11}(u) & T_{12}(u) & T_{13}(u) \\ T_{21}(u) & T_{22}(u) & T_{23}(u) \\ T_{31}(u) & T_{32}(u) & T_{33}(u) \end{pmatrix}\,,$$
$$\hat{T}_0(u) = \begin{pmatrix} \hat{T}_{11}(u) & \hat{T}_{12}(u) & \hat{T}_{13}(u) \\ \hat{T}_{21}(u) & \hat{T}_{22}(u) & \hat{T}_{23}(u) \\ \hat{T}_{31}(u) & \hat{T}_{32}(u) & \hat{T}_{33}(u) \end{pmatrix}\,, \tag{3.1}$$

where each entry acts on the quantum space $\left(\mathbb{C}^3\right)^{\otimes N}$. It is convenient to denote the auxiliary-space matrix elements of the double-row monodromy matrix (2.7) by

$$U_0(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}_1(u) & \mathcal{B}(u) \\ \mathcal{C}_1(u) & \mathcal{E}(u) + \mathcal{A}(u) & \mathcal{B}_2(u) \\ \mathcal{C}(u) & \mathcal{C}_2(u) & \mathcal{D}(u) + y(u)\mathcal{A}(u) \end{pmatrix}, \tag{3.2}$$

where

$$y(u) = 1 - Q^{-2} d(u)\,, \qquad d(u) = -\frac{\omega(u^2)}{\omega(qu^2)}\,. \tag{3.3}$$

Three-dimensional representations such as (3.1) have been used to solve periodic 19-vertex [17] and nested 15-vertex [18] models; and (3.2) is a generalization for the open case, see for instance [19] and [20].

The operator entries of the double-row monodromy matrix (3.2) are given in terms of single-row monodromy matrix elements T_{ij} and $\hat{T}_{ij}$ by means of (2.7). In terms of the double-row operators, the transfer matrix (2.9) can be written as

$$t(u) = a(u)\,\mathcal{A}(u) + Q^2\,\mathcal{D}(u) + \mathcal{E}(u)\,, \tag{3.4}$$

where

$$a(u) = -\frac{\omega(q^2u^2)}{\omega(qu^2)}\,. \tag{3.5}$$

3.1. Bethe vector

Having defined the operator representation, we need to find a convenient reference state. We note that

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^{\otimes N} \tag{3.6}$$

satisfies the following properties

$$\begin{aligned} &T_{ij}(u)|0\rangle = \hat{T}_{ij}(u)|0\rangle = 0\,, \qquad \text{for} \qquad i > j\,, \\ &T_{11}(u)|0\rangle = \hat{T}_{11}(u)|0\rangle = \omega(qu)^N|0\rangle\,, \\ &T_{22}(u)|0\rangle = \hat{T}_{22}(u)|0\rangle = 0\,, \\ &T_{33}(u)|0\rangle = \hat{T}_{33}(u)|0\rangle = \omega(u)^N|0\rangle\,. \end{aligned} \tag{3.7}$$

Moreover, the fundamental relation (2.6) evaluated at the point $u = v^{-1}$, taking into account (2.4) and (3.7), gives

$$\begin{aligned} &T_{21}(u)\,\hat{T}_{12}(u)|0\rangle = T_{11}(u)\,\hat{T}_{11}(u)|0\rangle\,, \\ &T_{32}(u)\,\hat{T}_{23}(u)|0\rangle = Q^{-2}T_{33}(u)\,\hat{T}_{33}(u)|0\rangle\,, \\ &T_{31}(u)\,\hat{T}_{13}(u)|0\rangle = y(u)\,T_{11}(u)\,\hat{T}_{11}(u)|0\rangle - \left(Q^{-2} + y(u)\right)T_{33}(u)\,\hat{T}_{33}(u)|0\rangle\,, \\ &T_{11}(u)\,\hat{T}_{12}(u)|0\rangle = 0\,. \end{aligned} \tag{3.8}$$

The results (3.7) and (3.8) imply

$$\begin{aligned}
&\mathcal{A}(u)|0\rangle = \Lambda_1(u)|0\rangle\,, \\
&\mathcal{D}(u)|0\rangle = Q^{-2}\, d(u)\, \Lambda_2(u)|0\rangle\,, \\
&\mathcal{E}(u)|0\rangle = 0\,,
\end{aligned} \tag{3.9}$$

where

$$\Lambda_1(u) = \omega(qu)^{2N}\,, \qquad \Lambda_2(u) = \omega(u)^{2N}\,. \tag{3.10}$$

In addition, we also have the following properties for the off-diagonal double-row operators,

$$\mathcal{C}(u)|0\rangle = \mathcal{C}_1(u)|0\rangle = \mathcal{C}_2(u)|0\rangle = 0\,, \tag{3.11}$$

and

$$\mathcal{B}_1(u)|0\rangle = 0\,. \tag{3.12}$$

Due to (3.9), we see that the reference state $|0\rangle$ is an eigenstate of the transfer matrix,

$$t(u)|0\rangle = (a(u)\Lambda_1(u) + d(u)\Lambda_2(u))\,|0\rangle. \tag{3.13}$$

Since the operator $\mathcal{B}_1(u)$ (in addition to $\mathcal{C}(u)$, $\mathcal{C}_1(u)$ and $\mathcal{C}_2(u)$) annihilates the reference state, we are left in principle with two operators to play the role of raising operators, either $\mathcal{B}(u)$ or $\mathcal{B}_2(u)$. However, the reflection algebra (2.8) strongly suggests that $\mathcal{B}(u)$ is the correct choice. Indeed, after some manipulation,[1] we found the following exchange relations from the reflection algebra,

$$\begin{aligned}
\mathcal{A}(u)\mathcal{B}(v) = {}& f(u,v)\mathcal{B}(v)\mathcal{A}(u) + f_1(u,v)\mathcal{B}(u)\mathcal{A}(v) + f_2(u,v)\mathcal{B}(u)\mathcal{D}(v) \\
&+ f_3(u,v)\mathcal{B}(u)\mathcal{E}(v) - \mathcal{B}_1(u)\mathcal{B}_2(v)\,,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\mathcal{D}(u)\mathcal{B}(v) = {}& h(u,v)\mathcal{B}(v)\mathcal{D}(u) + h_1(u,v)\mathcal{B}(u)\mathcal{D}(v) + h_2(u,v)\mathcal{B}(u)\mathcal{A}(v) \\
&+ h_3(u,v)\mathcal{B}(u)\mathcal{E}(v) + Q^{-2}a(u)\mathcal{B}_1(u)\mathcal{B}_2(v) - Q^{-2}\mathcal{E}(u)\mathcal{B}(v)\,,
\end{aligned} \tag{3.15}$$

$$\mathcal{B}(u)\mathcal{B}(v) = \mathcal{B}(v)\mathcal{B}(u)\,, \tag{3.16}$$

$$\mathcal{B}(u)\mathcal{B}_1(v) = \mathcal{B}(v)\mathcal{B}_1(u)\,, \tag{3.17}$$

$$\mathcal{B}(u)\mathcal{E}(v) = \mathcal{B}(v)\mathcal{E}(u)\,, \tag{3.18}$$

where the coefficients are given by

$$\begin{aligned}
f(u,v) &= \frac{\omega(uq^{-1}v^{-1})\omega(uv)}{\omega(uv^{-1})\omega(quv)}\,, \\
f_1(u,v) &= \frac{\omega(v^2)}{\omega(qv^2)\omega(uv^{-1})}\Big(\omega(qvu^{-1}) + Q^{-2}\omega(vu^{-1})\Big)\,, \\
f_2(u,v) &= -1 - \frac{Q^2\omega(uv)}{\omega(quv)}\,, \\
f_3(u,v) &= -\frac{\omega(uv)}{\omega(quv)}\,,
\end{aligned} \tag{3.19}$$

[1] Let us call eq$[i,j]$ the (i,j) entry of equation (2.8) regarded as a 9×9 matrix in the auxiliary space. The exchange relations (3.14)–(3.18) all follow from eq$[1,3]$, eq$[1,7]$, eq$[1,8]$, eq$[1,9]$, eq$[4,6]$ and eq$[7,9]$.

and

$$h(u,v) = \frac{\omega(uqv^{-1})\omega(q^2uv)}{\omega(uv^{-1})\omega(quv)},$$
$$h_1(u,v) = \frac{\omega(q^2u^2)}{Q^2\omega(qu^2)\omega(uv^{-1})}\left(\left(1+Q^{-2}\right)\omega(quv^{-1}) + \omega(q^2uv^{-1})\right),$$
$$h_2(u,v) = \frac{\omega(q^2u^2)\omega(v^2)}{Q^4\omega(qu^2)\omega(quv)\omega(qv^2)}\left(\left(1+Q^{-2}\right)\omega(q^2uv) + \omega(q^3uv)\right),$$
$$h_3(u,v) = \frac{\left(q-q^{-1}\right)\omega(quv^{-1})}{Q^2\omega(qu^2)\omega(quv)}. \tag{3.20}$$

We can observe that the commutation relations (3.14) and (3.15) have a structure similar to those for the six-vertex model [2], although with some extra terms. The relation (3.16) guarantees that the vector $\mathcal{B}(u_1)\ldots\mathcal{B}(u_M)|0\rangle$ is a symmetric quantity in its arguments. Thus, the operator $\mathcal{B}(u)$ is indeed a good raising operator candidate. The relations (3.17) and (3.18) do not have an analogous counterpart in the six-vertex model, but they play a fundamental role in the algebraic Bethe ansatz analysis for the present case.

At this point, it is convenient to introduce a shorthand notation, as follows:

- For a set of M rapidities, we will use the notation $\bar{u} = \{u_1, \ldots, u_M\}$, where the cardinality of $\bar{u}$ is $\#\bar{u} = M$.
- If the ith rapidity is dropped from the set $\bar{u}$, we denote $\bar{u}_i = \{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_M\}$.

In addition, let us introduce the following strings of operators,

$$B^M(\bar{u}) = \prod_{i=1}^{M}\mathcal{B}(u_i),$$
$$B_i^M(\{u,\bar{u}_i\}) = \mathcal{B}(u)\prod_{j\neq i}^{M}\mathcal{B}(u_j),$$
$$\bar{B}_i^M(\{u,\bar{u}\}) = \prod_{j=0}^{i-1}\mathcal{B}(u_j)\mathcal{B}_1(u_i)\mathcal{B}_2(u_{i+1})\prod_{j=i+2}^{M}\mathcal{B}(u_j),$$
$$\tilde{B}_i^M(\{u,\bar{u}\}) = \prod_{j=0}^{i-1}\mathcal{B}(u_j)\mathcal{E}(u_i)\prod_{j=i+1}^{M}\mathcal{B}(u_j), \tag{3.21}$$

for $\#\bar{u} = M$ and where we identify $u_0 \equiv u$. Let us also introduce the vectors

$$|\bar{u}\rangle = B^M(\bar{u})|0\rangle, \tag{3.22}$$

and

$$|\{u,\bar{u}_i\}\rangle = B_i^M(\{u,\bar{u}_i\})|0\rangle. \tag{3.23}$$

Following the previous discussion, we propose that the Bethe vectors are given by (3.22). We therefore need to compute the action of the transfer matrix (3.4) on this vector. The result is obtained as a consequence of the following proposition

Proposition 1. *The action of the operator $\mathcal{A}(u)$ on the string $B^M(\bar{u})$, with $\#\bar{u} = M$, is given by*

$$\mathcal{A}(u)B^M(\bar{u}) = B^M(\bar{u})\mathcal{A}(u)\prod_{i=1}^{M} f(u,u_i) + \sum_{i=1}^{M} \hat{F}_i^M(\{u,\bar{u}\}) + \sum_{i=2}^{M} \hat{Z}_i^M(\{u,\bar{u}\})$$
$$+ \sum_{i=0}^{M-1} r_i \bar{B}_i^M(\{u,\bar{u}\}) + \sum_{i=1}^{M-1} s_i \tilde{B}_i^M(\{u,\bar{u}\}) + \alpha^M(\{u,\bar{u}\})\tilde{B}_M^M(\{u,\bar{u}\}), \tag{3.24}$$

while the action of the operator $\mathcal{D}(u)$ on the string $B^M(\bar{u})$ is given by

$$\mathcal{D}(u)B^M(\bar{u}) = B^M(\bar{u})\mathcal{D}(u)\prod_{i=1}^{M} h(u,u_i) + \sum_{i=1}^{M} \hat{G}_i^M(\{u,\bar{u}\}) - Q^{-2}a(u)\sum_{i=2}^{M} \hat{Z}_i^M(\{u,\bar{u}\})$$
$$- Q^{-2}a(u)\sum_{i=0}^{M-1} r_i \bar{B}_i^M(\{u,\bar{u}\}) - Q^{-2}a(u)\sum_{i=1}^{M-1} s_i \tilde{B}_i^M(\{u,\bar{u}\})$$
$$+ \delta^M(\{u,\bar{u}\})\tilde{B}_M^M(\{u,\bar{u}\}) - Q^{-2}\mathcal{E}(u)B^M(\bar{u}), \tag{3.25}$$

where

$$\hat{F}_i^M(\{u,\bar{u}\}) = B_i^M(\{u,\bar{u}_i\})\mathcal{A}(u_i)f_1(u,u_i)\prod_{j\neq i}^{M} f(u_i,u_j)$$
$$+ B_i^M(\{u,\bar{u}_i\})\mathcal{D}(u_i)f_2(u,u_i)\prod_{j\neq i}^{M} h(u_i,u_j) \tag{3.26}$$

and

$$\hat{G}_i^M(\{u,\bar{u}\}) = B_i^M(\{u,\bar{u}_i\})\mathcal{A}(u_i)h_2(u,u_i)\prod_{j\neq i}^{M} f(u_i,u_j)$$
$$+ B_i^M(\{u,\bar{u}_i\})\mathcal{D}(u_i)h_1(u,u_i)\prod_{j\neq i}^{M} h(u_i,u_j). \tag{3.27}$$

The explicit expressions for the functions $\hat{Z}_i^M$, r_i, s_i, α^M and δ^M are not relevant for the following and are collected in Appendix A.

Proof. The proof is obtained by induction on M and by the use of the commutation relations (3.14), (3.15), (3.16), (3.17) and (3.18). We provide some details in Appendix B. □

We note an intricate structure of the operator relations (3.24) and (3.25), with many "unwanted" terms. However, the final off-shell equation satisfied by the Bethe vector is amazingly simple, since many of terms from (3.24) cancel with those from (3.25). Indeed, we can now easily prove the following proposition

Proposition 2. *The off-shell equation for the transfer matrix (3.4) acting on the Bethe vector (3.22) is given by*

$$t(u)|\bar{u}\rangle = \Lambda(\{u,\bar{u}\})|\bar{u}\rangle + \sum_{i=1}^{M} H(u,u_i)E(\{u_i,\bar{u}_i\})|\{u,\bar{u}_i\}\rangle\,, \tag{3.28}$$

where

$$\Lambda(\{u,\bar{u}\}) = a(u)\Lambda_1(u)\prod_{i=1}^{M} f(u,u_i) + d(u)\Lambda_2(u)\prod_{i=1}^{M} h(u,u_i)\,, \tag{3.29}$$

$$E(\{u_i,\bar{u}_i\}) = \Lambda_1(u_i)\prod_{j\neq i}^{M} f(u_i,u_j) - \Lambda_2(u_i)\prod_{j\neq i}^{M} h(u_i,u_j)\,, \tag{3.30}$$

and

$$H(u,v) = (q-q^{-1})\frac{\omega(q^2u^2)}{\omega(uv^{-1})\omega(quv)}d(v)\,. \tag{3.31}$$

The functions entering equations (3.29) and (3.30) are given by (3.3), (3.5), (3.10), (3.19) and (3.20).

Proof. Applying (3.24) and (3.25) on the reference state (3.6), taking into account (3.9), (3.22) and (3.23), and the identities

$$\begin{aligned} a(u)f_1(u,v) + Q^2h_2(u,v) &= H(u,v)\,, \\ a(u)f_2(u,v) + Q^2h_1(u,v) &= -\frac{Q^2}{d(v)}H(u,v)\,, \end{aligned} \tag{3.32}$$

we obtain (3.28). Note that the terms with α^M and δ^M separately vanish when acting on the reference state since $\tilde{B}_M^M = \prod_{j=0}^{M-1}\mathcal{B}(u_j)\mathcal{E}(u_M)$, and $\mathcal{E}$ annihilates the reference state. □

The result (3.28) has been conjectured for arbitrary values of spin in [7].[2] By imposing $E(\{u_i,\bar{u}_i\}) = 0$ for $i = 1,\dots M$ (the Bethe equations), we obtain the eigenvalues (3.29) and the eigenvectors (3.22) of the transfer matrix (3.4). It is remarkable that the off-shell equation (3.28) has exactly the same form as the analogous relation for the XXZ spin-$\frac{1}{2}$ chain with $U_q sl(2)$ symmetry [21], see *e.g.* equation (A.17) of [22].

3.2. Dual Bethe vector

We can follow a similar procedure to obtain the dual Bethe vectors. Indeed, defining

$$\langle 0| = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^{\otimes N} \tag{3.33}$$

such that $\langle 0|0\rangle = 1$, we can obtain, as before,

$$\begin{aligned} \langle 0|\mathcal{A}(u) &= \langle 0|\Lambda_1(u)\,, \\ \langle 0|\mathcal{D}(u) &= \langle 0|Q^{-2}d(u)\Lambda_2(u)\,, \\ \langle 0|\mathcal{E}(u) &= 0\,, \end{aligned} \tag{3.34}$$

[2] We use here a slightly different notation compared with [7]. In particular, the functions $a(u)$ and $d(u)$ are related as follows: $a_{\text{previous}}(u) = a_{\text{now}}(u)\Lambda_1(u)$ and $d_{\text{previous}}(u) = d_{\text{now}}(u)\Lambda_2(u)$.

where $\Lambda_1(u)$ and $\Lambda_2(u)$ are given by (3.10). We also have

$$\langle 0|\mathcal{B}(u) = \langle 0|\mathcal{B}_1(u) = \langle 0|\mathcal{B}_2(u) = \langle 0|\mathcal{C}_1(u) = 0\,. \tag{3.35}$$

For this case, the needed commutation relations are[3]

$$\begin{aligned}\mathcal{C}(v)\mathcal{A}(u) = f(u,v)\mathcal{A}(u)\mathcal{C}(v) + f_1(u,v)\mathcal{A}(v)\mathcal{C}(u) + f_2(u,v)\mathcal{D}(v)\mathcal{C}(u)\\ + f_3(u,v)\mathcal{E}(v)\mathcal{C}(u) - \mathcal{C}_2(v)\mathcal{C}_1(u)\,,\end{aligned} \tag{3.36}$$

$$\begin{aligned}\mathcal{C}(v)\mathcal{D}(u) = h(u,v)\mathcal{D}(u)\mathcal{C}(v) + h_1(u,v)\mathcal{D}(v)\mathcal{C}(u) + h_2(u,v)\mathcal{A}(v)\mathcal{C}(u)\\ + h_3(u,v)\mathcal{E}(v)\mathcal{C}(u) + Q^{-2}a(u)\mathcal{C}_2(v)\mathcal{C}_1(u) - Q^{-2}\mathcal{C}(v)\mathcal{E}(u)\,,\end{aligned} \tag{3.37}$$

$$\mathcal{C}(v)\mathcal{C}(u) = \mathcal{C}(u)\mathcal{C}(v)\,, \tag{3.38}$$

$$\mathcal{C}_1(v)\mathcal{C}(u) = \mathcal{C}_1(u)\mathcal{C}(v)\,, \tag{3.39}$$

$$\mathcal{E}(v)\mathcal{C}(u) = \mathcal{E}(u)\mathcal{C}(v)\,. \tag{3.40}$$

Let us also introduce

$$\begin{aligned}C^M(\bar{u}) &= \prod_{i=M}^{1} \mathcal{C}(u_i)\,,\\ C_i^M(\{u,\bar{u}_i\}) &= \prod_{j=M,j\neq i}^{1} \mathcal{C}(u_j)\mathcal{C}(u)\,,\\ \bar{C}_i^M(\{u,\bar{u}\}) &= \prod_{j=M}^{i+2} \mathcal{C}(u_j)\mathcal{C}_2(u_{i+1})\mathcal{C}_1(u_i) \prod_{j=i-1}^{0} \mathcal{C}(u_j)\,,\\ \tilde{C}_i^M(\{u,\bar{u}\}) &= \prod_{j=M}^{i+1} \mathcal{C}(u_j)\mathcal{E}(u_i) \prod_{j=i-1}^{0} \mathcal{C}(u_j)\,,\end{aligned} \tag{3.41}$$

for $\#\bar{u} = M$ and where we identify again $u_0 \equiv u$. In the above definitions, the product indices run backwards.[4] Let us define

$$\langle \bar{u}| = \langle 0|C^M(\bar{u})\,, \tag{3.42}$$

and

$$\langle \{u,\bar{u}_i\}\,| = \langle 0|C_i^M(\{u,\bar{u}_i\})\,. \tag{3.43}$$

We use (3.42) as the dual Bethe vector, and compute the action of the transfer matrix (3.4) on it. The result is

[3] The commutation relations now follow from eq[3, 1], eq[7, 1], eq[8, 1], eq[9, 1], eq[6, 4] and eq[9, 7].

[4] Throughout this subsection, we use the following ordering of the rapidities: $\{u_M, \ldots, u_1\}$. While for the products of operators C^M and C_i^M this ordering is irrelevant thanks to (3.38), it is important to maintain this ordering for the auxiliary products $\bar{C}_i^M$ and $\tilde{C}_i^M$, when compared with $\bar{B}_i^M$ and $\tilde{B}_i^M$.

Proposition 3. *The (left) action of the operator $\mathcal{A}(u)$ on the string $C^M(\bar{u})$, with $\#\bar{u} = M$, is given by*

$$C^M(\bar{u})\mathcal{A}(u) = \mathcal{A}(u)C^M(\bar{u})\prod_{i=1}^{M} f(u,u_i) + \sum_{i=1}^{M} \check{F}_i^M(\{u,\bar{u}\}) + \sum_{i=2}^{M} \check{Z}_i^M(\{u,\bar{u}\})$$
$$+ \sum_{i=0}^{M-1} r_i \bar{C}_i^M(\{u,\bar{u}\}) + \sum_{i=1}^{M-1} s_i \tilde{C}_i^M(\{u,\bar{u}\}) + \alpha^M(\{u,\bar{u}\})\tilde{C}_M^M(\{u,\bar{u}\}), \tag{3.44}$$

while the (left) action of the operator $\mathcal{D}(u)$ on the string $C^M(\bar{u})$ is given by

$$C^M(\bar{u})\mathcal{D}(u) = \mathcal{D}(u)C^M(\bar{u})\prod_{i=1}^{M} h(u,u_i) + \sum_{i=1}^{M} \check{G}_i^M(\{u,\bar{u}\}) - Q^{-2}a(u)\sum_{i=2}^{M} \check{Z}_i^M(\{u,\bar{u}\})$$
$$- Q^{-2}a(u)\sum_{i=0}^{M-1} r_i \bar{C}_i^M(\{u,\bar{u}\}) - Q^{-2}a(u)\sum_{i=1}^{M-1} s_i \tilde{C}_i^M(\{u,\bar{u}\})$$
$$+ \delta^M(\{u,\bar{u}\})\tilde{C}_M^M(\{u,\bar{u}\}) - Q^{-2}C^M(\bar{u})\mathcal{E}(u), \tag{3.45}$$

where

$$\check{F}_i^M(\{u,\bar{u}\}) = \mathcal{A}(u_i)C_i^M(\{u,\bar{u}_i\}) f_1(u,u_i)\prod_{j\neq i}^{M} f(u_i,u_j)$$
$$+ \mathcal{D}(u_i)C_i^M(\{u,\bar{u}_i\}) f_2(u,u_i)\prod_{j\neq i}^{M} h(u_i,u_j), \tag{3.46}$$

and

$$\check{G}_i^M(\{u,\bar{u}\}) = \mathcal{A}(u_i)C_i^M(\{u,\bar{u}_i\})h_2(u,u_i)\prod_{j\neq i}^{M} f(u_i,u_j)$$
$$+ \mathcal{D}(u_i)C_i^M(\{u,\bar{u}_i\})h_1(u,u_i)\prod_{j\neq i}^{M} h(u_i,u_j). \tag{3.47}$$

The explicit expressions for the functions $\check{Z}_i^M$, r_i, s_i, α^M and δ^M are not relevant for the following and are given in Appendix A.

Proof. The proof is similar to the proof of Proposition 1; the needed commutation relations are now (3.36), (3.37), (3.38), (3.39) and (3.40). □

Using the previous proposition, we obtain

Proposition 4. *The (left) off-shell equation for the transfer matrix (3.4) acting on the Bethe vector (3.42) is given by*

$$\langle\bar{u}|t(u) = \langle\bar{u}|\Lambda(\{u,\bar{u}\}) + \sum_{i=1}^{M}\langle\{u,\bar{u}_i\}|H(u,u_i)E(\{u_i,\bar{u}_i\})\,, \tag{3.48}$$

with the same functions as in (3.28).

Proof. The proof is similar to the proof of Proposition 2. □

Again, the result (3.48) has been conjectured for arbitrary values of spin in [7]. The (left) eigenvalues and eigenvectors of the transfer matrix are obtained by imposing $E(\{u_i,\bar{u}_i\}) = 0$ for $i = 1,\dots,M$.

3.3. Scalar product

Let us now briefly consider the scalar product between the Bethe vector (3.22) and the dual Bethe vector (3.42). In the paper [7], we have proposed that the scalar product between an on-shell state $\langle\bar{u}|$ and an arbitrary off-shell state $|\bar{v}\rangle$ is given by

$$\langle\bar{u}|\bar{v}\rangle = \left(\frac{1}{2Q^{2s}}\right)^M \prod_{i=1}^{M}\frac{\omega(u_i)^{2N}u_i\,\omega(u_i^2)}{\omega(u_i^2 q)\omega(v_i^2 q^2)}\prod_{j<i}^{M}\frac{\omega(u_iu_jq^2)}{\omega(u_iu_j)}\,\frac{\mathrm{Det}_M\left(\frac{\partial}{\partial u_i}\Lambda(\{v_j,\bar{u}\})\right)}{\mathrm{Det}_M\left(\frac{1}{\omega(v_iu_j^{-1})\omega(v_iu_jq)}\right)}\,, \tag{3.49}$$

where $\#\bar{u} = \#\bar{v} = M$ and the set $\bar{u}$ is a solution of the Bethe equations, *i.e.*, $E(\{u_i,\bar{u}_i\}) = 0$ for $i = 1,\dots,M$. Here, we have $s = 1$. A formula of the type (3.49) is generally known as a Slavnov formula [23], while its limit $v_k \to u_k$ (the square of the norm) is known as a Gaudin–Korepin formula [24–26]. For the $s = \frac{1}{2}$ chain with (diagonal) open boundary conditions, the formula (3.49) was obtained in [27] (see also [28] for the XXX chain), using a method different from the one in [23].

The proof of the formula (3.49) remains an open problem for $s > \frac{1}{2}$. We now briefly comment on the obstacles that we have encountered for $s = 1$, which come up already in the simplest $M = 1$ case. The scalar product (3.49) for $M = 1$ can be in principle obtained from the commutation relation between the operators $\mathcal{C}(u_1)$ and $\mathcal{B}(v_1)$, which is given by[5]

$$\begin{aligned}\mathcal{C}(u_1)\mathcal{B}(v_1) = {}&\mathcal{B}(v_1)\mathcal{C}(u_1)\\ &+ x_1(u_1,v_1)\mathcal{A}(u_1)\mathcal{A}(v_1) + x_2(u_1,v_1)\mathcal{A}(v_1)\mathcal{A}(u_1) + x_3(u_1,v_1)\mathcal{D}(u_1)\mathcal{A}(v_1)\\ &+ x_4(u_1,v_1)\mathcal{A}(u_1)\mathcal{D}(v_1) + x_5(u_1,v_1)\mathcal{A}(v_1)\mathcal{D}(u_1) + x_6(u_1,v_1)\mathcal{D}(u_1)\mathcal{D}(v_1)\\ &+ y_1(u_1,v_1)\mathcal{A}(v_1)\mathcal{E}(u_1) + y_2(u_1,v_1)\mathcal{E}(u_1)\mathcal{A}(v_1) + y_3(u_1,v_1)\mathcal{E}(u_1)\mathcal{D}(v_1)\\ &+ \mathcal{B}_1(v_1)\mathcal{C}_1(u_1) - \mathcal{C}_2(u_1)\mathcal{B}_2(v_1)\,,\end{aligned} \tag{3.50}$$

where the coefficients are given in Appendix A. Note that the first three lines of (3.50) are similar to the analogous relation in the six-vertex model; all the other terms are new. Applying (3.50) on the reference state $|0\rangle$, taking into account (3.9) and (3.11), and projecting the result on $\langle 0|$, we obtain

[5] This commutation relation follows from eq[7, 3] of the reflection algebra.

$$\langle u_1|v_1\rangle = [x_1(u_1,v_1)+x_2(u_1,v_1)]\,\Lambda_1(u_1)\Lambda_1(v_1) - \frac{\omega(v_1^2)x_4(u_1,v_1)}{Q^2\omega(qv_1^2)}\Lambda_1(u_1)\Lambda_2(v_1)$$
$$- \frac{\omega(u_1^2)\,[x_3(u_1,v_1)+x_5(u_1,v_1)]}{Q^2\omega(qu_1^2)}\Lambda_2(u_1)\Lambda_1(v_1)$$
$$+ \frac{\omega(u_1^2)\omega(v_1^2)x_6(u_1,v_1)}{Q^4\omega(qu_1^2)\omega(qv_1^2)}\Lambda_2(u_1)\Lambda_2(v_1) - \langle 0|\mathcal{C}_2(u_1)\mathcal{B}_2(v_1)|0\rangle\,, \tag{3.51}$$

where we observe that most of the extra terms in (3.50) do not contribute to the scalar product, except for $\mathcal{C}_2(u_1)\mathcal{B}_2(v_1)$. We now suppose that the variable u_1 is a Bethe root. Under this condition, we have numerically checked (up to $N=6$) that

$$\langle 0|\mathcal{C}_2(u_1) = 0\,. \tag{3.52}$$

Then, using in (3.51) the fact that $\Lambda_1(u_1)=\Lambda_2(u_1)$ when u_1 is a Bethe root, we obtain by explicit computation the right-hand side of (3.49) for $M=1$. For $M>1$, new terms (when compared to the analogous relations in the six-vertex model) appear in the off-shell/off-shell scalar product $\langle\bar{u}|\bar{v}\rangle$. All these terms, however, presumably disappear when $\langle\bar{u}|$ is on-shell; the proof of this fact, which would be a first step towards proving the formula (3.49), has so far eluded us.

4. Discussion

We have considered the quantum spin-1 chain with "free" boundary conditions constructed from the TL algebra in the algebraic Bethe ansatz framework. The main result of this note is the proof of the off-shell equations satisfied by the Bethe vector, see Proposition 2, and by the dual Bethe vector, see Proposition 4. The complexity of the proof originates from the unusual exchange relations (3.14)–(3.18); and we believe it is quite remarkable that they lead to such simple off-shell equations (3.28).

We note that despite of the fact that the auxiliary space is 3-dimensional, the off-shell equations have the same form of those of the quantum-group-invariant XXZ spin-$\frac{1}{2}$ chain (which has a 2-dimensional auxiliary space). This is a step towards a proof of the more general conjecture in [7], which states that the off-shell equation of TL spin chains with "free" boundary conditions associated with the spin-s representation of $U_Q sl(2)$ is actually universal, *i.e.*, it is independent of the value of the spin. We hope that the results presented here can be further developed in order to prove the formula (3.49) for the scalar product between the off-shell Bethe vector and its on-shell dual, which is also independent (up to a constant factor) of the value of the spin.

According to conventional wisdom and experience, closed chains should be simpler than corresponding open chains. However, we have seen that this is not the case for the TL model. Nevertheless, the method presented here may also shed some light on the algebraic Bethe ansatz formulation for the closed TL spin-1 chain with periodic boundary conditions. Interestingly, the Yang–Baxter algebra (2.6) seems subtler than the associated reflection algebra (2.8). Indeed, the spectrum of the closed chain is characterized by a "dynamically" generated twist, see [13,16,29,30,7]. It would be interesting to obtain the Bethe vectors and the associated Bethe equations from the Yang–Baxter algebra.

As a further direction of investigation, it may be worth to consider the algebraic Bethe ansatz formulation of the spin-1 TL chain with more complicated boundary interactions. Both diagonal and non-diagonal reflection matrices are available [31,32]. For the former, the procedure described in this paper can probably be applied without significant changes. For the latter, one would have to extend the modified algebraic Bethe ansatz, see [33–35] for the XXZ spin-$\frac{1}{2}$ chain with non-diagonal boundaries, or to extend the construction of the on-shell Bethe states from the off-diagonal Bethe ansatz [36].

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Appendix A. Functions

We list here some functions used in the main text

$$\hat{Z}_i^M(\{u,\bar{u}\}) = (q+q^{-1})\sum_{k=2}^{i} r_{k-2}\left(-Q^{-2}d(u_i)B_i^M(\{u,\bar{u}_i\})\mathcal{A}(u_i)\prod_{j=k,j\neq i}^{M} f(u_i,u_j)\right.$$
$$\left. + B_i^M(\{u,\bar{u}_i\})\mathcal{D}(u_i)\prod_{j=k,j\neq i}^{M} h(u_i,u_j)\right), \tag{A.1}$$

$$\check{Z}_i^M(\{u,\bar{u}\}) = (q+q^{-1})\sum_{k=2}^{i} r_{k-2}\left(-Q^{-2}d(u_i)\mathcal{A}(u_i)C_i^M(\{u,\bar{u}_i\})\prod_{j=k,j\neq i}^{M} f(u_i,u_j)\right.$$
$$\left. + \mathcal{D}(u_i)C_i^M(\{u,\bar{u}_i\})\prod_{j=k,j\neq i}^{M} h(u_i,u_j)\right), \tag{A.2}$$

$$\alpha^M(\{u,\bar{u}\}) = f_3(u,u_M)\prod_{i=1}^{M-1} f(u,u_i)$$
$$+\sum_{i=1}^{M-1}\left\{ f_1(u,u_i)f_3(u_i,u_M)\prod_{j\neq i}^{M-1} f(u_i,u_j) + f_2(u,u_i)h_3(u_i,u_M)\prod_{j\neq i}^{M-1} h(u_i,u_j)\right.$$
$$+(q+q^{-1})\sum_{k=2}^{i} r_{k-2}\left(-Q^{-2}d(u_i)f_3(u_i,u_M)\prod_{j=k,j\neq i}^{M-1} f(u_i,u_j)\right.$$
$$\left.\left. + h_3(u_i,u_M)\prod_{j=k,j\neq i}^{M-1} h(u_i,u_j)\right)\right\}, \tag{A.3}$$

$$\delta^M(\{u,\bar{u}\}) = h_3(u,u_M)\prod_{i=1}^{M-1} h(u,u_i)$$
$$+\sum_{i=1}^{M-1}\left\{ h_1(u,u_i)h_3(u_i,u_M)\prod_{j\neq i}^{M-1} h(u_i,u_j) + h_2(u,u_i)f_3(u_i,u_M)\prod_{j\neq i}^{M-1} f(u_i,u_j)\right.$$

$$
- (q+q^{-1})Q^{-2}a(u)\sum_{k=2}^{i} r_{k-2}\left(-Q^{-2}d(u_i)f_3(u_i,u_M)\prod_{j=k,j\neq i}^{M-1} f(u_i,u_j) + h_3(u_i,u_M)\prod_{j=k,j\neq i}^{M-1} h(u_i,u_j)\right)\Bigg\}, \tag{A.4}
$$

$$
r_i = -\left(\frac{1+Q^2}{Q^4}\right)^i, \tag{A.5}
$$

$$
s_i = \frac{1}{Q^2}\left(\frac{1+Q^2}{Q^4}\right)^{i-1}. \tag{A.6}
$$

The coefficients of the commutation relation (3.50) are given by

$$
\begin{aligned}
x_1(u,v) &= \frac{\omega(u^2)\left(Q^2\omega(qv^2)+\omega(v^2)\right)\left(\omega(quv^{-1})+Q^{-2}\omega(uv^{-1})\right)}{Q^2\omega(qu^2)\omega(qv^2)\omega(vu^{-1})},\\
x_2(u,v) &= \frac{\omega(u^2)\omega(quv^{-1})\left(\omega(quv)+Q^{-2}\omega(uv)\right)}{\omega(qu^2)\omega(uv^{-1})\omega(quv)},\\
x_3(u,v) &= -\frac{\left(Q^2\omega(qv^2)+\omega(v^2)\right)\left(\omega(quv)+Q^2\omega(uv)\right)}{Q^2\omega(qv^2)\omega(quv)},\\
x_4(u,v) &= \frac{\omega(u^2)\left(\omega(quv^{-1})+Q^{-2}\omega(uv^{-1})\right)}{\omega(qu^2)\omega(vu^{-1})},\\
x_5(u,v) &= \frac{\omega(uv)\left(\omega(quv^{-1})+Q^2\omega(uv^{-1})\right)}{\omega(uv^{-1})\omega(quv)},\\
x_6(u,v) &= -\frac{\omega(quv)+Q^2\omega(uv)}{\omega(quv)},\\
y_1(u,v) &= \frac{\omega(uv)}{\omega(quv)},\\
y_2(u,v) &= -\frac{\omega(uv)\left(Q^2\omega(qv^2)+\omega(v^2)\right)}{Q^2\omega(qv^2)\omega(quv)},\\
y_3(u,v) &= -\frac{\omega(uv)}{\omega(quv)}.
\end{aligned} \tag{A.7}
$$

Appendix B. Proof of Proposition 1

In this appendix we prove Proposition 1 by induction.

B.1. Proof of (3.24)

Let us consider the relation (3.24). Its validity for $M = 1$ follows directly from the commutation relations (3.14) and (3.15). Let us suppose that (3.24) is valid for arbitrary M, and compute the action

$$\mathcal{A}(u)B^{M+1}(\bar{\bar{u}}) = \mathcal{A}(u)B^{M}(\bar{u})\mathcal{B}(u_{M+1}) \tag{B.1}$$

where $\bar{\bar{u}} = \{\bar{u}, u_{M+1}\}$ with $\#\bar{u} = M$. Using the induction hypothesis (3.24) in (B.1) we obtain

$$\begin{aligned}\mathcal{A}(u)B^{M+1}(\bar{\bar{u}}) &= B^{M}(\bar{u})\mathcal{A}(u)\mathcal{B}(u_{M+1})\prod_{i=1}^{M} f(u,u_i) + \sum_{i=1}^{M} \hat{F}_i^{M}(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) \\ &+ \sum_{i=2}^{M} \hat{Z}_i^{M}(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) + \sum_{i=0}^{M-1} r_i \bar{B}_i^{M}(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) \\ &+ \sum_{i=1}^{M-1} s_i \tilde{B}_i^{M}(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) + \alpha^{M}(\{u,\bar{u}\})\tilde{B}_M^{M}(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) .\end{aligned} \tag{B.2}$$

The next step consists of using the commutation relations (3.14), (3.15), (3.16), (3.17) and (3.18) in (B.2). Let us consider each term in the right-hand side of (B.2) separately. We have

$$\begin{aligned}&B^{M}(\bar{u})\underbrace{\mathcal{A}(u)\mathcal{B}(u_{M+1})}_{\text{Eq. (3.14)}}\prod_{i=1}^{M} f(u,u_i) \\ &= B^{M+1}(\bar{\bar{u}})\mathcal{A}(u)\prod_{i=1}^{M+1} f(u,u_i) + \gamma_a^{M}\, B^{M}(\bar{u})\mathcal{B}(u)\mathcal{A}(u_{M+1}) \\ &\quad + \gamma_d^{M}\, B^{M}(\bar{u})\mathcal{B}(u)\mathcal{D}(u_{M+1}) + \gamma_e^{M}\, B^{M}(\bar{u})\mathcal{B}(u)\mathcal{E}(u_{M+1}) \\ &\quad + \gamma_{b_2}^{M}\, B^{M}(\bar{u})\mathcal{B}_1(u)\mathcal{B}_2(u_{M+1}) ,\end{aligned} \tag{B.3}$$

where

$$\gamma_a^{M} = f_1(u,u_{M+1})\prod_{i=1}^{M} f(u,u_i) , \tag{B.4}$$

$$\gamma_d^{M} = f_2(u,u_{M+1})\prod_{i=1}^{M} f(u,u_i) , \tag{B.5}$$

$$\gamma_e^{M} = f_3(u,u_{M+1})\prod_{i=1}^{M} f(u,u_i) , \tag{B.6}$$

$$\gamma_{b_2}^{M} = -\prod_{i=1}^{M} f(u,u_i) , \tag{B.7}$$

are auxiliary quantities introduced for convenience.[6] The next term is given by

[6] Here, and in the auxiliary functions defined hereafter, we omit the functional dependency on the rapidities in order to lighten the notation.

$$
\begin{aligned}
&\sum_{i=1}^{M} \hat{F}_i^M(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) \\
&\quad = \sum_{i=1}^{M} \Bigg\{ f_1(u,u_i) B_i^M(\{u,\bar{u}_i\}) \underbrace{\mathcal{A}(u_i)\mathcal{B}(u_{M+1})}_{\text{Eq. (3.14)}} \prod_{j\neq i}^{M} f(u_i,u_j) \\
&\qquad + f_2(u,u_i) B_i^M(\{u,\bar{u}_i\}) \underbrace{\mathcal{D}(u_i)\mathcal{B}(u_{M+1})}_{\text{Eq. (3.15)}} \prod_{j\neq i}^{M} h(u_i,u_j) \Bigg\} \\
&\quad = \sum_{i=1}^{M} \Bigg\{ f_1(u,u_i) B_i^{M+1}(\{u,\bar{\bar{u}}_i\}) \mathcal{A}(u_i) \prod_{j\neq i}^{M+1} f(u_i,u_j) \\
&\qquad + f_2(u,u_i) B_i^{M+1}(\{u,\bar{\bar{u}}_i\}) \mathcal{D}(u_i) \prod_{j\neq i}^{M+1} h(u_i,u_j) \Bigg\} \\
&\quad + \sum_{i=1}^{M} \Bigg\{ \Bigg(f_1(u,u_i) f_1(u_i,u_{M+1}) \prod_{j\neq i}^{M} f(u_i,u_j) \\
&\qquad\qquad + f_2(u,u_i) h_2(u_i,u_{M+1}) \prod_{j\neq i}^{M} h(u_i,u_j) \Bigg) \underbrace{B_i^M(\{u,\bar{u}_i\})\mathcal{B}(u_i)}_{\text{Eq. (3.16)}} \mathcal{A}(u_{M+1}) \\
&\qquad + \Bigg(f_1(u,u_i) f_2(u_i,u_{M+1}) \prod_{j\neq i}^{M} f(u_i,u_j) \\
&\qquad\qquad + f_2(u,u_i) h_1(u_i,u_{M+1}) \prod_{j\neq i}^{M} h(u_i,u_j) \Bigg) \underbrace{B_i^M(\{u,\bar{u}_i\})\mathcal{B}(u_i)}_{\text{Eq. (3.16)}} \mathcal{D}(u_{M+1}) \\
&\qquad + \Bigg(f_1(u,u_i) f_3(u_i,u_{M+1}) \prod_{j\neq i}^{M} f(u_i,u_j) \\
&\qquad + f_2(u,u_i) h_3(u_i,u_{M+1}) \prod_{j\neq i}^{M} h(u_i,u_j) \Bigg) \underbrace{B_i^M(\{u,\bar{u}_i\})\mathcal{B}(u_i)}_{\text{Eq. (3.16)}} \mathcal{E}(u_{M+1}) \\
&\qquad - \Bigg(f_1(u,u_i) \prod_{j\neq i}^{M} f(u_i,u_j) \\
&\qquad - Q^{-2} f_2(u,u_i) a(u_i) \prod_{j\neq i}^{M} h(u_i,u_j) \Bigg) \underbrace{B_i^M(\{u,\bar{u}_i\})\mathcal{B}_1(u_i)}_{\text{Eq. (3.17)}} \mathcal{B}_2(u_{M+1}) \\
&\qquad - \Bigg(Q^{-2} f_2(u,u_i) \prod_{j\neq i}^{M} h(u_i,u_j) \Bigg) \underbrace{B_i^M(\{u,\bar{u}_i\})\mathcal{E}(u_i)}_{\text{Eq. (3.18)}} \mathcal{B}(u_{M+1}) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{M+1} \hat{F}_i^{M+1}(\{u,\bar{\bar{u}}\}) + \theta_a^M\, B^M(\bar{u})\mathcal{B}(u)\mathcal{A}(u_{M+1}) \\
&\quad + \theta_d^M\, B^M(\bar{u})\mathcal{B}(u)\mathcal{D}(u_{M+1}) + \theta_e^M\, B^M(\bar{u})\mathcal{B}(u)\mathcal{E}(u_{M+1}) \\
&\quad + \theta_{b_2}^M\, B^M(\bar{u})\mathcal{B}_1(u)\mathcal{B}_2(u_{M+1}) + \theta_b^M\, B^M(\bar{u})\mathcal{E}(u)\mathcal{B}(u_{M+1})\,,
\end{aligned} \tag{B.8}
$$

where we identified $B_i^M(\{u,\bar{u}_i\})\mathcal{B}(u_{M+1}) = B_i^{M+1}(\{u,\bar{\bar{u}}_i\})$ and introduced the auxiliary functions

$$
\begin{aligned}
\theta_a^M &= -f_1(u,u_{M+1})\prod_{j=1}^{M} f(u_{M+1},u_j) + \sum_{i=1}^{M}\Bigg(f_1(u,u_i)f_1(u_i,u_{M+1})\prod_{j\neq i}^{M} f(u_i,u_j) \\
&\quad + f_2(u,u_i)h_2(u_i,u_{M+1})\prod_{j\neq i}^{M} h(u_i,u_j)\Bigg),
\end{aligned} \tag{B.9}
$$

$$
\begin{aligned}
\theta_d^M &= -f_2(u,u_{M+1})\prod_{j=1}^{M} h(u_{M+1},u_j) + \sum_{i=1}^{M}\Bigg(f_1(u,u_i)f_2(u_i,u_{M+1})\prod_{j\neq i}^{M} f(u_i,u_j) \\
&\quad + f_2(u,u_i)h_1(u_i,u_{M+1})\prod_{j\neq i}^{M} h(u_i,u_j)\Bigg),
\end{aligned} \tag{B.10}
$$

$$
\theta_e^M = \sum_{i=1}^{M}\Bigg(f_1(u,u_i)f_3(u_i,u_{M+1})\prod_{j\neq i}^{M} f(u_i,u_j) + f_2(u,u_i)h_3(u_i,u_{M+1})\prod_{j\neq i}^{M} h(u_i,u_j)\Bigg), \tag{B.11}
$$

$$
\theta_{b_2}^M = -\sum_{i=1}^{M}\Bigg(f_1(u,u_i)\prod_{j\neq i}^{M} f(u_i,u_j) - Q^{-2} f_2(u,u_i)a(u_i)\prod_{j\neq i}^{M} h(u_i,u_j)\Bigg), \tag{B.12}
$$

$$
\theta_b^M = -\sum_{i=1}^{M} Q^{-2} f_2(u,u_i)\prod_{j\neq i}^{M} h(u_i,u_j)\,. \tag{B.13}
$$

We proceed in a similar way for the next term, namely,

$$
\begin{aligned}
&\sum_{i=2}^{M} \hat{Z}_i^M(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) \\
&\quad = \sum_{i=2}^{M}(q+q^{-1})\sum_{k=2}^{i} r_{k-2} \\
&\qquad \times \Bigg\{ -Q^{-2} d(u_i) B_i^M(\{u,\bar{u}_i\}) \underbrace{\mathcal{A}(u_i)\mathcal{B}(u_{M+1})}_{\text{Eq. (3.14)}} \prod_{j=k,\, j\neq i}^{M} f(u_i,u_j)
\end{aligned}
$$

$$+ B_i^M(\{u, \bar{u}_i\}) \underbrace{\mathcal{D}(u_i)\mathcal{B}(u_{M+1})}_{\text{Eq. (3.15)}} \prod_{j=k, j\neq i}^{M} h(u_i, u_j) \Bigg\}$$

$$= \sum_{i=2}^{M+1} \hat{Z}_i^{M+1}(\{u, \bar{\bar{u}}\}) + \tau_a^M B^M(\bar{u})\mathcal{B}(u)\mathcal{A}(u_{M+1})$$
$$+ \tau_d^M B^M(\bar{u})\mathcal{B}(u)\mathcal{D}(u_{M+1}) + \tau_e^M B^M(\bar{u})\mathcal{B}(u)\mathcal{E}(u_{M+1})$$
$$+ \tau_{b_2}^M B^M(\bar{u})\mathcal{B}_1(u)\mathcal{B}_2(u_{M+1}) + \tau_b^M B^M(\bar{u})\mathcal{E}(u)\mathcal{B}(u_{M+1}), \tag{B.14}$$

where we used the relations (3.16), (3.17) and (3.18) to rewrite, respectively, the terms $B_i^M(\{u, \bar{u}_i\})\mathcal{B}(u_i)$, $B_i^M(\{u, \bar{u}_i\})\mathcal{B}_1(u_i)$ and $B_i^M(\{u, \bar{u}_i\})\mathcal{E}(u_i)$. The auxiliary functions τ_i are given by

$$\tau_a^M = (q+q^{-1})Q^{-2}d(u_{M+1}) \sum_{k=2}^{M+1} r_{k-2} \prod_{j=k}^{M} f(u_{M+1}, u_j)$$
$$- \sum_{i=2}^{M}(q+q^{-1}) \sum_{k=2}^{i} r_{k-2}\Bigg(Q^{-2}d(u_i) f_1(u_i, u_{M+1}) \prod_{j=k, j\neq i}^{M} f(u_i, u_j)$$
$$- h_2(u_i, u_{M+1}) \prod_{j=k, j\neq i}^{M} h(u_i, u_j)\Bigg), \tag{B.15}$$

$$\tau_d^M = -(q+q^{-1}) \sum_{k=2}^{M+1} r_{k-2} \prod_{j=k}^{M} h(u_{M+1}, u_j)$$
$$- \sum_{i=2}^{M}(q+q^{-1}) \sum_{k=2}^{i} r_{k-2}\Bigg(Q^{-2}d(u_i) f_2(u_i, u_{M+1}) \prod_{j=k, j\neq i}^{M} f(u_i, u_j)$$
$$- h_1(u_i, u_{M+1}) \prod_{j=k, j\neq i}^{M} h(u_i, u_j)\Bigg), \tag{B.16}$$

$$\tau_e^M = - \sum_{i=2}^{M}(q+q^{-1}) \sum_{k=2}^{i} r_{k-2}\Bigg(Q^{-2}d(u_i) f_3(u_i, u_{M+1}) \prod_{j=k, j\neq i}^{M} f(u_i, u_j)$$
$$- h_3(u_i, u_{M+1}) \prod_{j=k, j\neq i}^{M} h(u_i, u_j)\Bigg), \tag{B.17}$$

$$\tau_{b_2}^M = \sum_{i=2}^{M}(q+q^{-1}) \sum_{k=2}^{i} r_{k-2} Q^{-2}\Bigg(d(u_i) \prod_{j=k, j\neq i}^{M} f(u_i, u_j)$$
$$+ a(u_i) \prod_{j=k, j\neq i}^{M} h(u_i, u_j)\Bigg), \tag{B.18}$$

$$\tau_b^M = -\sum_{i=2}^{M}(q+q^{-1})\sum_{k=2}^{i} r_{k-2} Q^{-2} \prod_{j=k,\, j\neq i}^{M} h(u_i, u_j)\,. \tag{B.19}$$

The last terms are

$$\begin{aligned}
&\sum_{i=0}^{M-1} r_i \bar{B}_i^M(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) \\
&\quad = \sum_{i=0}^{M} r_i \bar{B}_i^{M+1}(\{u,\bar{\bar{u}}\}) - r_M \underbrace{\bar{B}_M^{M+1}(\{u,\bar{\bar{u}}\})}_{\text{Eq. (3.17)}} \\
&\quad = \sum_{i=0}^{M} r_i \bar{B}_i^{M+1}(\{u,\bar{\bar{u}}\}) - r_M B^M(\bar{u})\mathcal{B}_1(u)\mathcal{B}_2(u_{M+1})\,,
\end{aligned} \tag{B.20}$$

$$\begin{aligned}
&\sum_{i=1}^{M-1} s_i \tilde{B}_i^M(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) \\
&\quad = \sum_{i=1}^{M} s_i \tilde{B}_i^{M+1}(\{u,\bar{\bar{u}}\}) - s_M \underbrace{\tilde{B}_M^{M+1}(\{u,\bar{\bar{u}}\})}_{\text{Eq. (3.18)}} \\
&\quad = \sum_{i=1}^{M} s_i \tilde{B}_i^{M+1}(\{u,\bar{\bar{u}}\}) - s_M B^M(\bar{u})\mathcal{E}(u)\mathcal{B}(u_{M+1})\,,
\end{aligned} \tag{B.21}$$

$$\alpha^M(\{u,\bar{u}\}) \underbrace{\tilde{B}_M^M(\{u,\bar{u}\})\mathcal{B}(u_{M+1})}_{\text{Eq. (3.18)}} = \alpha^M(\{u,\bar{u}\}) B^M(\bar{u})\mathcal{E}(u)\mathcal{B}(u_{M+1})\,. \tag{B.22}$$

We observe the following identities

$$\gamma_a^M + \theta_a^M + \tau_a^M = 0\,, \tag{B.23}$$

$$\gamma_d^M + \theta_d^M + \tau_d^M = 0\,, \tag{B.24}$$

$$\gamma_e^M + \theta_e^M + \tau_e^M = \alpha^{M+1}(\{u,\bar{\bar{u}}\})\,, \tag{B.25}$$

$$\gamma_{b_2}^M + \theta_{b_2}^M + \tau_{b_2}^M = r_M\,, \tag{B.26}$$

$$\theta_b^M + \tau_b^M + \alpha^M(\{u,\bar{u}\}) = s_M\,, \tag{B.27}$$

which are typical in algebraic Bethe ansatz analyses, see *e.g.* equations (A.8) and (A.9) in [35]. As an example here, let us show the validity of the simplest relation (B.26), using analytical arguments. We start by calculating the residues of the left-hand side of (B.26); we note that

$$\operatorname{Res}\left(\gamma_{b_2}^M, u = u_{\text{pole}}\right) = -\operatorname{Res}\left(\theta_{b_2}^M, u = u_{\text{pole}}\right) \tag{B.28}$$

and

$$\text{Res}\left(\tau_{b_2}^M, u = u_{\text{pole}}\right) = 0 \tag{B.29}$$

where $u_{\text{pole}} = u_k, -u_k, q^{-1}u_k^{-1}, -q^{-1}u_k^{-1}$ for $k = 1, \dots, M$. This shows that the residue of the left-hand side of the functional relation is zero; therefore it is holomorphic on the entire complex plane, and thus equals a constant. The constant can be determined by taking the limit:

$$\lim_{u\to\infty} \gamma_{b_2}^M + \theta_{b_2}^M + \tau_{b_2}^M = r_M \,. \tag{B.30}$$

The other functional relations (except for (B.25), which is trivial since it is basically the definition of α^M) can be analyzed in the same way.

Finally, using the results (B.3), (B.8), (B.14), (B.20), (B.21) and (B.22) in (B.2), the functional identities (B.23)–(B.27), as well as noticing that $B^M(\bar{u})\mathcal{B}(u)\mathcal{E}(u_{M+1}) = \tilde{B}_{M+1}^{M+1}(\{u, \bar{\bar{u}}\})$, we obtain

$$\begin{aligned}
\mathcal{A}(u)B^{M+1}(\bar{\bar{u}}) &= B^{M+1}(\bar{\bar{u}})\mathcal{A}(u)\prod_{i=1}^{M+1} f(u,u_i) + \sum_{i=1}^{M+1} \hat{F}_i^{M+1}(\{u,\bar{\bar{u}}\}) + \sum_{i=2}^{M+1} \hat{Z}_i^{M+1}(\{u,\bar{\bar{u}}\}) \\
&\quad + \sum_{i=0}^{M} r_i \bar{B}_i^{M+1}(\{u,\bar{\bar{u}}\}) + \sum_{i=1}^{M} s_i \tilde{B}_i^{M+1}(\{u,\bar{\bar{u}}\}) + \alpha^{M+1}(\{u,\bar{\bar{u}}\})\tilde{B}_{M+1}^{M+1}(\{u,\bar{\bar{u}}\}) \,,
\end{aligned} \tag{B.31}$$

which ends the proof.

B.2. Proof of (3.25)

Let us now consider the relation (3.25). Its validity for $M = 1$ follows directly from the commutation relations (3.14) and (3.15). Let us suppose that (3.25) is valid for arbitrary M, and compute the action

$$\mathcal{D}(u)B^{M+1}(\bar{\bar{u}}) = \mathcal{D}(u)B^M(\bar{u})\mathcal{B}(u_{M+1}) \,, \tag{B.32}$$

where $\bar{\bar{u}} = \{\bar{u}, u_{M+1}\}$ with $\#\bar{u} = M$. Using the induction hypothesis (3.25) in (B.32) we obtain

$$\begin{aligned}
\mathcal{D}(u)B^{M+1}(\bar{\bar{u}}) &= B^M(\bar{u})\mathcal{D}(u)\mathcal{B}(u_{M+1})\prod_{i=1}^{M} h(u,u_i) + \sum_{i=1}^{M} \hat{G}_i^M(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) \\
&\quad - Q^{-2}a(u)\sum_{i=2}^{M} \hat{Z}_i^M(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) - Q^{-2}a(u)\sum_{i=0}^{M-1} r_i \bar{B}_i^M(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) \\
&\quad - Q^{-2}a(u)\sum_{i=1}^{M-1} s_i \tilde{B}_i^M(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) + \delta^M(\{u,\bar{u}\})\tilde{B}_M^M(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) \\
&\quad - Q^{-2}\mathcal{E}(u)B^M(\bar{u})\mathcal{B}(u_{M+1}) \,.
\end{aligned} \tag{B.33}$$

The next step consists of using the commutation relations (3.14), (3.15), (3.16), (3.17) and (3.18) in (B.33). Most of the terms have already been computed in the previous subsection; thus, we need to compute here only the first two and the last two terms in the right-hand side of (B.33). We have

$$B^M(\bar{u})\underbrace{\mathcal{D}(u)\mathcal{B}(u_{M+1})}_{\text{Eq. (3.15)}}\prod_{i=1}^{M}h(u,u_i)$$
$$= B^{M+1}(\bar{\bar{u}})\mathcal{D}(u)\prod_{i=1}^{M+1}h(u,u_i) + \bar{\gamma}_d^M B^M(\bar{u})\mathcal{B}(u)\mathcal{D}(u_{M+1})$$
$$+ \bar{\gamma}_a^M B^M(\bar{u})\mathcal{B}(u)\mathcal{A}(u_{M+1}) + \bar{\gamma}_e^M B^M(\bar{u})\mathcal{B}(u)\mathcal{E}(u_{M+1})$$
$$+ \bar{\gamma}_{b_2}^M B^M(\bar{u})\mathcal{B}_1(u)\mathcal{B}_2(u_{M+1}) + \bar{\gamma}_b^M B^M(\bar{u})\mathcal{E}(u)\mathcal{B}(u_{M+1}), \tag{B.34}$$

where

$$\bar{\gamma}_a^M = h_2(u,u_{M+1})\prod_{i=1}^{M}h(u,u_i), \tag{B.35}$$

$$\bar{\gamma}_d^M = h_1(u,u_{M+1})\prod_{i=1}^{M}h(u,u_i), \tag{B.36}$$

$$\bar{\gamma}_e^M = h_3(u,u_{M+1})\prod_{i=1}^{M}h(u,u_i), \tag{B.37}$$

$$\bar{\gamma}_{b_2}^M = Q^{-2}a(u)\prod_{i=1}^{M}h(u,u_i), \tag{B.38}$$

$$\bar{\gamma}_b^M = -Q^{-2}\prod_{i=1}^{M}h(u,u_i). \tag{B.39}$$

The next term is computed in a similar way as (B.8). The result is given by

$$\sum_{i=1}^{M}\hat{G}_i^M(\{u,\bar{u}\})\mathcal{B}(u_{M+1}) = \sum_{i=1}^{M+1}\hat{G}_i^{M+1}(\{u,\bar{\bar{u}}\}) + \bar{\theta}_a^M B^M(\bar{u})\mathcal{B}(u)\mathcal{A}(u_{M+1})$$
$$+ \bar{\theta}_d^M B^M(\bar{u})\mathcal{B}(u)\mathcal{D}(u_{M+1}) + \bar{\theta}_e^M B^M(\bar{u})\mathcal{B}(u)\mathcal{E}(u_{M+1})$$
$$+ \bar{\theta}_{b_2}^M B^M(\bar{u})\mathcal{B}_1(u)\mathcal{B}_2(u_{M+1}) + \bar{\theta}_b^M B^M(\bar{u})\mathcal{E}(u)\mathcal{B}(u_{M+1}), \tag{B.40}$$

where we introduced the auxiliary functions

$$\bar{\theta}_a^M = -h_2(u,u_{M+1})\prod_{j=1}^{M}f(u_{M+1},u_j) + \sum_{i=1}^{M}\Bigg(h_2(u,u_i)f_1(u_i,u_{M+1})\prod_{j\neq i}^{M}f(u_i,u_j)$$
$$+ h_1(u,u_i)h_2(u_i,u_{M+1})\prod_{j\neq i}^{M}h(u_i,u_j)\Bigg), \tag{B.41}$$

$$\bar{\theta}_d^M = -h_1(u,u_{M+1})\prod_{j=1}^{M}h(u_{M+1},u_j) + \sum_{i=1}^{M}\Bigg(h_2(u,u_i)f_2(u_i,u_{M+1})\prod_{j\neq i}^{M}f(u_i,u_j)$$
$$+ h_1(u,u_i)h_1(u_i,u_{M+1})\prod_{j\neq i}^{M}h(u_i,u_j)\Bigg), \tag{B.42}$$

$$\bar{\theta}_e^M = \sum_{i=1}^{M} \left(h_2(u, u_i) f_3(u_i, u_{M+1}) \prod_{j \neq i}^{M} f(u_i, u_j) + h_1(u, u_i) h_3(u_i, u_{M+1}) \prod_{j \neq i}^{M} h(u_i, u_j) \right), \tag{B.43}$$

$$\bar{\theta}_{b_2}^M = -\sum_{i=1}^{M} \left(h_2(u, u_i) \prod_{j \neq i}^{M} f(u_i, u_j) - Q^{-2} h_1(u, u_i) a(u_i) \prod_{j \neq i}^{M} h(u_i, u_j) \right), \tag{B.44}$$

$$\bar{\theta}_b^M = -\sum_{i=1}^{M} Q^{-2} h_1(u, u_i) \prod_{j \neq i}^{M} h(u_i, u_j) . \tag{B.45}$$

The last terms are easily evaluated

$$\delta^M(\{u, \bar{u}\}) \underbrace{\tilde{B}_M^M(\{u, \bar{u}\}) \mathcal{B}(u_{M+1})}_{\text{Eq. (3.18)}} = \delta^M(\{u, \bar{u}\}) B^M(\bar{u}) \mathcal{E}(u) \mathcal{B}(u_{M+1}) , \tag{B.46}$$

$$-Q^{-2} \mathcal{E}(u) B^M(\bar{u}) \mathcal{B}(u_{M+1}) = -Q^{-2} \mathcal{E}(u) B^{M+1}(\bar{\bar{u}}) . \tag{B.47}$$

We now observe identities that are analogous to (B.23)–(B.27), namely,

$$\bar{\gamma}_a^M + \bar{\theta}_a^M - Q^{-2} a(u) \tau_a^M = 0 , \tag{B.48}$$

$$\bar{\gamma}_d^M + \bar{\theta}_d^M - Q^{-2} a(u) \tau_d^M = 0 , \tag{B.49}$$

$$\bar{\gamma}_e^M + \bar{\theta}_e^M - Q^{-2} a(u) \tau_e^M = \delta^{M+1}(\{u, \bar{\bar{u}}\}) , \tag{B.50}$$

$$\bar{\gamma}_{b_2}^M + \bar{\theta}_{b_2}^M - Q^{-2} a(u) \tau_{b_2}^M = -Q^{-2} a(u) r_M , \tag{B.51}$$

$$\bar{\gamma}_b^M + \bar{\theta}_b^M - Q^{-2} a(u) \tau_b^M + \delta^M(\{u, \bar{u}\}) = -Q^{-2} a(u) s_M . \tag{B.52}$$

Finally, using the results (B.34), (B.40), (B.46), (B.47), (B.14), (B.20) and (B.21) in (B.33), the functional identities (B.48)–(B.52), as well as noticing that $B^M(\bar{u}) \mathcal{B}(u) \mathcal{E}(u_{M+1}) = \tilde{B}_{M+1}^{M+1}(\{u, \bar{\bar{u}}\})$, we obtain

$$\begin{aligned} \mathcal{D}(u) B^{M+1}(\bar{\bar{u}}) &= B^{M+1}(\bar{\bar{u}}) \mathcal{D}(u) \prod_{i=1}^{M+1} h(u, u_i) + \sum_{i=1}^{M+1} \hat{G}_i^{M+1}(\{u, \bar{\bar{u}}\}) \\ &- Q^{-2} a(u) \sum_{i=2}^{M+1} \hat{Z}_i^{M+1}(\{u, \bar{\bar{u}}\}) - Q^{-2} a(u) \sum_{i=0}^{M} r_i \bar{B}_i^{M+1}(\{u, \bar{\bar{u}}\}) \\ &- Q^{-2} a(u) \sum_{i=1}^{M} s_i \tilde{B}_i^{M+1}(\{u, \bar{\bar{u}}\}) + \delta^{M+1}(\{u, \bar{u}\}) \tilde{B}_{M+1}^{M+1}(\{u, \bar{\bar{u}}\}) \\ &- Q^{-2} \mathcal{E}(u) B^{M+1}(\bar{\bar{u}}) , \end{aligned} \tag{B.53}$$

which concludes the proof.

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