Formal loops III: Additive functions and the Radon transform

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Abstract

To any algebraic variety X and closed 2-form \( \omega \) on X, we associate the “symplectic action functional” \( T(\omega) \) which is a function on the formal loop space \( LX \) introduced by the authors earlier. The correspondence \( \omega \mapsto T(\omega) \) can be seen as a version of the Radon transform. We give a characterization of the functions of the form \( T(\omega) \) in terms of factorizability (infinitesimal analog of additivity in holomorphic pairs of pants) as well as in terms of vertex operator algebras.

These results will be used in the subsequent paper which will relate the gerbe of chiral differential operators on X (whose lien is the sheaf of closed 2-forms) and the determinantal gerbe of the tangent bundle of \( LX \) (whose lien is the sheaf of invertible functions on \( LX \)). On the level of liens this relation associates to a closed 2-form \( \omega \) the invertible function \( \exp T(\omega) \).

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0. Introduction

(0.1) Let $X$ be a $C^\infty$-manifold and $\eta$ be a smooth 1-form on $X$. Its Radon transform is the function $\tau(\eta)$ on the free loop space $L(X) = C^\infty(S^1, X)$ whose value at $\gamma : S^1 \to X$ is $\int_{S^1} \gamma^* \eta$. This “universal” setting includes many classical instances of integral transforms over families of curves such as the original Radon transform involving integration over straight lines in $\mathbb{R}^3$.

The problem of describing the range of the transform $\tau$ was studied by J.-L. Brylinski [2] who, in the case $X = \mathbb{R}^N$, characterized it by a system of differential equations similar to the A-hypergeometric system [5]. One part of these equations expresses the following obvious property:

(0.1.1) $\tau(\eta)$ is invariant under reparametrizations of $S^1$.

Suppose that $X$ is a complex manifold and $\eta$ is holomorphic. Then $\tau(\eta)$ satisfies two more properties:

(0.1.2) $\tau(\eta) = 0$ on $L^0(X)$, the space of loops extending holomorphically into the unit disk.

(0.1.3) $\tau(\eta)$ is additive in holomorphic pairs of pants (holomorphic maps of sphere minus 3 disks into $X$). Namely, the value on $\tau(\eta)$ on the “waist” circle is equal to the sum of values on the “leg” circles.

(0.2) In the present paper we further assume that $X$ is a smooth algebraic variety over $\mathbb{C}$ and relate the Radon transform to the theory of vertex operator algebras and of related objects, factorization algebras introduced by A. Beilinson and V. Drinfeld [1]. We replace $L^0(X)$ and $L(X)$ by the scheme $L^0(X)$ of formal arcs [4] and the ind-scheme $L(X)$ of formal loops [10]. Informally, they are obtained by replacing the unit disk and the circle by $\text{Spec } \mathbb{C}[[t]]$ and $\text{Spec } \mathbb{C}(t)$.

Then for a regular 1-form $\eta$ we have a function $\tau(\eta)$ on $L(X)$ vanishing on $L^0(X)$. In fact, for any closed 2-form $\omega$ the Radon transform $\tau(d^{-1}(\omega))$ of any analytic primitive $d^{-1}(\omega)$ of $\omega$ is defined algebraically and depends only on $\omega$. If $\omega$ is nondegenerate, then $\tau(d^{-1}(\omega))$ is a version of the symplectic action functional on the space of loops.

In [10] we showed how the space $L(X)$ provides a geometric construction of a particular sheaf of vertex algebras on $X$, the chiral de Rham complex $\Omega^\text{ch}_X$ from [13]. It was realized as a semiinfinite de Rham complex of a particular D-module on $L(X)$ “consisting” of distributions supported on $L^0(X)$. In particular, the sheaf $\mathcal{O}_{L(X)}$ of functions on $L(X)$ acts on $\Omega^\text{ch}_X$ by multiplication.

Our main result is the following.

(0.2.1) Theorem. Let $f$ be a function on $L(X)$ vanishing on $L^0(X)$. Then the following are equivalent:

(i) $f$ has the form $\tau(d^{-1}(\omega))$ for a closed 2-form $\omega$.
(ii) The operator of multiplication by $\exp(f)$ in $\Omega^\text{ch}_X$ is an automorphism of vertex algebras.
(iii) The function $f$ is 2-additive (see below).

(0.3) The condition for $f$ to be 2-factorizing can be seen as an infinitesimal version of the conditions (0.1.1) and (0.1.3). First, we require that $f$ is invariant under reparametrizations of $\text{Spf } \mathbb{C}[[t]]$, see (1.2.8). If $C$ is a smooth curve and $c \in C$ a point, then there is a version $L_{C,c}(X)$ of $L(X)$ involving the punctured formal disk on $C$ near $c$. Reparametrization invariance of $f$ implies that it gives rise to a function $f_{C,c}$ on $L_{C,c}(X)$. 
Now, in [10] we introduced the total space $L^C(X) \xrightarrow{\pi} C$ of all the $L^C,c(X)$ and showed that it has a structure of a factorization monoid. This structure includes, in particular, a family $L^C,c(X) \xrightarrow{\pi} C$ of all the $L^C,c(X)$ while the fiber over $(c_1, c_2)$ with $c_1 \neq c_2$ is $L^C,c_1(X) \times L^C,c_2(X)$ and the fiber over $(c, c)$ is $L^C,c(X)$.

When $c_1, c_2$ merge together to a point $c$, the punctured formal disks around $c_1, c_2$ form an “infinitesimal pair of pants.” A function $f$ is called 2-additive, if there exists a function $f_{C,c_1} \in \mathcal{O}(L^C_2(X))$ whose restriction on $\pi^{-1}(c_1, c_2)$ with $c_1 \neq c_2$ is $f_{C,c_1} + f_{C,c_2}$ while the restriction to $\pi^{-1}(c, c)$ is $f_{C,c}$. So this is an infinitesimal version of (0.1.3). The general concept of additive functions (Definition 1.5.2) involves similar conditions for any number of points merging together and turns out to be equivalent to the 2-additivity.

This paper is a part of a series [10–12] devoted to interpreting, in geometric terms involving $L(X)$, the gerbe of chiral differential operators (CDO) on $X$, see [6,7]. Objects of this gerbe are sheaves of vertex algebras similar to $\Omega^{\text{ch}}_X$ but without the fermionic variables. For any such object $A$ the sheaf $\text{Aut}(A)$ was found in [6] to be canonically isomorphic to $\Omega^{2,\text{cl}}_X$, the sheaf of closed 2-forms. This precisely the domain of definition of the Radon transform $\tau \circ d^{-1}$ from (0.2). In fact, our proof of the equivalence of (i) and (ii) in Theorem 0.2.1 is based on this identification.

In a subsequent paper [12] we will use Theorem 0.2.1 to relate the gerbe of CDO with an appropriate version of the determinantal gerbe of the tangent bundle of $L(X)$. In other words, we will show that the anomaly inherent in construction of CDO is related to the determinantal anomaly for the loop space by the Radon transform. Sheaves of CDO on $X$ form a gerbe with lien $\Omega^{2,\text{cl}}_X$ while the determinantal gerbe of $L(X)$ has the lien $\mathcal{O}^\times_{L(X)}$. On the level of liens the relation between the two gerbes associates to a 2-form $\omega \in \Omega^{2,\text{cl}}_X$ the invertible function $S(\omega) = \exp(\tau d^{-1}(\omega)) \in \mathcal{O}^\times_{L(X)}$.

1. Formal loop spaces and transgressions

(1.1) Conventions. All rings will be assumed commutative and containing the field $\mathbb{C}$ of complex numbers. For a ring $R$ we denote by $R(\{t\})^{\sqrt{\cdot}}$ the subring in the ring $R(\{t\})$ of formal Laurent series formed by series $\sum_{n > -\infty} a_n t^n$ with $a_n$ nilpotent for $n < 0$, see [10]. Let $\text{Sch}$ be the category of schemes over $\mathbb{C}$, and $\text{Isch}$ the category of ind-schemes with countable indexing. Let $\text{Sch}^\text{ft} \subset \text{Sch}$ be the full subcategory of schemes of finite type.

(1.2) Reminder on $L^X$. Let $X$ be a scheme of finite type over $\mathbb{C}$. We have then the scheme $L^0 X$ of formal arcs, and the ind-scheme $L^X$ of formal loops in $X$, see [10]. They represent the following functors

$$\text{Hom}(\text{Spec } R, L^0 X) = \text{Hom}(\text{Spec } R[\{t\}], X), \quad (1.2.1)$$

$$\text{Hom}(\text{Spec } R, L^X) = \text{Hom}(\text{Spec } R(\{t\})^{\sqrt{\cdot}}, X). \quad (1.2.2)$$

There is a diagram

$$X \xleftarrow{p} L^0 X \xrightarrow{i} L^X, \quad (1.2.3)$$

with $p$ affine and $i$ a closed embedding. Moreover, $L^X$ is an inductive limit of schemes $L^e X$ that are nilpotent extensions of $L^0 X$, and so it can be described as a locally ringed space $(L^0 X, \mathcal{O}_{L^X})$.
where

\[ \mathcal{O}_{LX} = \lim_{\leftarrow} \mathcal{O}_{L^r X}. \]

Notice that \( \mathcal{O}_{LX} \) has a natural topology (of the projective limit).

\textbf{(1.2.4) Example.} Let \( X = \mathbb{A}^N \) with coordinates \( b^1, b^2, \ldots, b^N \). Then

\[ L^0 X = \text{Spec} \mathbb{C}[b^i_N; i = 1, \ldots, N, n \geq 0], \]

where the \( b^i_n \)'s can be thought of as the coefficients of \( N \) indeterminate Taylor series

\[ b^i(t) = \sum_{n \geq 0} b^i_n t^n. \]

Further,

\[ \mathcal{O}_{L\mathbb{A}^N} = \mathcal{O}_{L^0 \mathbb{A}^N}[\{b^i_N, i = 1, \ldots, N, n < 0\}] \]

is the sheaf formed by formal power series in infinitely many variables \( b^i_n, n < 0 \). By definition, such a series is a formal sum of monomials, each involving only finitely many \( b^i_n \), and the coefficients at the monomials can be arbitrary. In particular

\[ \Gamma(L^0 \mathbb{A}^N, \mathcal{O}_{L\mathbb{A}^N}) = \mathbb{C}[b^i_n, n \geq 0][[b^i_n, n < 0]]. \]

Let

\[ \mathcal{O}_{LX|L^0 X} = \text{Ker}(\mathcal{O}_{LX} \to \mathcal{O}_{L^0 X}). \]

(1.2.5) This is a sheaf of ideals in \( \mathcal{O}_{LX} \) consisting of topologically nilpotent elements. Let also

\[ \mathcal{O}_{LX|L^0 X}^\times = \text{Ker}(\mathcal{O}_{LX}^\times \to \mathcal{O}_{L^0 X}^\times). \]

(1.2.6) Because of topological nilpotency of \( \mathcal{O}_{LX|L^0 X} \), the exponential series defines an isomorphism

\[ \exp : \mathcal{O}_{LX|L^0 X} \to \mathcal{O}_{LX|L^0 X}^\times. \]

(1.2.7) Let \( K \) be the group scheme \( \text{Aut} \mathbb{C}[[t]] \). Explicitly

\[ K = \text{Spec} \mathbb{C}[a_{\pm 1}, a_2, a_3, \ldots], \]

(1.2.8) where the \( a_i \) can be thought of as coefficients of an indeterminate formal change of the variable

\[ t \mapsto a_1 t + a_2 t^2 + \cdots. \]

The group scheme \( K \) acts on \( LX \) and \( L^0 X \) by the above changes of the variable. We also denote by \( g = \text{Der} \mathbb{C}[[t]] = \mathbb{C}[[t]]d/dt \) the Lie algebra of derivations of \( \mathbb{C}[[t]] \). This algebra acts on \( L^0(X), L(X) \) in a way compatible with the action of \( K \). So we will speak about the action of the Harish-Chandra pair \((g, K)\).
(1.3) The transgression. Let \( R = \lim_{\alpha} R_\alpha \) be a topological algebra represented as a filtering projective limit of discrete algebras \( R_\alpha \). The ring of Laurent series with coefficients in \( R \) is defined by

\[
R((t)) = \lim_{\alpha} R_\alpha((t)).
\]

An element of \( R((t)) \) can be viewed as a formal series \( \sum_{n=-\infty}^{+\infty} a_n t^n, a_n \in R \), possibly infinite in both directions, but with the condition \( \lim_{n \to -\infty} a_n = 0 \).

Recall from [11, Section 6.2] the evaluation map which is a morphism of ringed space

\[
ev = ev_X : (\mathcal{L}^0 X, \mathcal{O}_{\mathcal{L}^0 X}((t))) \to (X, \mathcal{O}_X),
\]

whose underlying morphism of topological spaces is the projection \( p : \mathcal{L}^0 X \to X \).

(1.3.3) Example. Let \( X = \mathbb{A}^N \) as in Example 1.2.4. Then the morphism

\[
H^0(ev) : H^0(\mathbb{A}^N, \mathcal{O}) \to H^0(\mathcal{L}^0 \mathbb{A}^N, \mathcal{O}_{\mathcal{L}^0 \mathbb{A}^N}((t)))
\]

sends \( b^i \) to \( \sum_{n=-\infty}^{+\infty} b^i_n t^n \).

For any commutative algebra \( R \) (topological or not) we denote by \( \Omega^1(R) \) the module of Kähler differentials (defined without taking into account the topology). We also define \( \Omega^i(R) = \bigwedge_R^i \Omega^1(R) \) (the algebraic exterior power). We have then the de Rham differential \( d : \Omega^i(R) \to \Omega^{i+1}(R) \).

For any topological algebra \( R \) as above we have the residue homomorphism

\[
\text{Res} = \text{Res}^p : \Omega^p(R((t))) \to \Omega^{p-1}(R).
\]

To define it, we denote, for each \( a(t) = \sum a_n t^n \in R((t)) \),

\[
d_R a(t) = \sum d(a_n) t^n \in \Omega^1(R)((t)), \quad a'(t) = \sum n a_n t^{n-1} \in R((t)).
\]

Then we put

\[
\text{Res}^p(a_0(t) da_1(t) \ldots da_p(t))
\]

\[
= \left( a_0(t) \sum_{i=1}^p d_R a_1(t) \ldots d_R a_{i-1}(t) a'_i(t) d_R a_{i+1}(t) \ldots d_R a_p(t) \right)_{-1},
\]

where the subscript \((-1)\) means the coefficient at \( t^{-1} \) in a series from \( \Omega^{p-1}(R)((t)) \).

(1.3.5) Proposition. The map \( \text{Res}^p \) is well defined and satisfies the following properties:

(a) If \( \omega \in \Omega^p(R[[t]]) \), then \( \text{Res}^p(\omega) = 0 \).
(b) If \( \xi \in \mathfrak{g} \) and \( L_\xi \) is its action on \( \Omega^P R((t)) \), then \( \text{Res}^p(L_\xi \omega) = 0 \) (invariance of residue).
(c) \( \text{Res}^p(d \omega) = d \text{Res}^p(\omega) \).
The proof is standard. Let

\[ \Omega^p_{LX|L^0X} = \ker \{ \Omega^p_{LX} \to \Omega^p_{L^0X} \}. \]

We now define a morphism of sheaves

\[ \tau = \tau^p : \Omega^p_X \to p_* \Omega^{p-1}_{LX|L^0X} \]  \hspace{1cm} (1.3.6)

called the transgression (or the Radon transform). It is defined as the composition of

\[ \text{Res} : p_* \Omega^p(O_{LX}(t)) \to p_* \Omega^{p-1}(O_{LX}) = p_* \Omega^{p-1}_{LX} \]

and the pullback with respect to the evaluation map

\[ \text{ev}^* : \Omega^p_X = \Omega^p(O_X) \to p_* \Omega^p(O_{LX}(t)). \]

Proposition 1.3.5 implies

(1.3.7) **Proposition.** (a) For any local section \( \omega \) of \( \Omega^p_X \) the form \( \tau(\omega) \) lies in \( p_* \Omega^{p-1}_{LX|L^0X} \).

(b) The form \( \tau(\omega) \) is \( g \)-invariant.

(c) \( \tau(d\omega) = d\tau(\omega) \). In particular, if \( \omega \in \Omega^p_X \) is of the form \( \omega = df \) for a local regular function \( f \), then \( \tau(\omega) = 0 \).

(1.3.8) **Example.** Let

\[ \omega = \sum_{i=1}^N f_i(b, \ldots, b^N) db^i, \quad f_i \in \mathbb{C}[b, \ldots, b^N] \]

be a 1-form regular in \( \mathbb{A}^N \). Then \( \tau(\omega) \) is the element of

\[ \mathbb{C}[b_i^{\geq 0}][[b_i^{< 0}]] = H^0(L^0\mathbb{A}^N, O_{L\mathbb{A}^N}) \]

defined by

\[ \tau(\omega) = \sum_{i=1}^N \text{Res} \left( f_i \left( \sum_{n=-\infty}^{\infty} b_n^1 t^n, \ldots, \sum_{n=-\infty}^{\infty} b_n^N t^n \right) \right) \cdot \sum_{n=-\infty}^{\infty} n^i b_i t^n dt. \]

(1.4) **The map \( \tau d^{-1} \) and the symplectic action functional.** Let \( X \) be a smooth algebraic variety over \( \mathbb{C} \), and \( \omega \in \Omega^p(X) \) be a closed 2-form on \( X \). Then locally on the transcendental topology \( \omega = d\eta \) for a 1-form \( \eta \) with analytic coefficients. In this subsection we observe that \( \tau^1(\eta) \) is in fact an (algebraic) section of \( p_* O_{LX|L^0X} \) (independent on \( \eta \)) and yielding a map

\[ \tau d^{-1} : \Omega^2_{X|L^0X} \to p_* O_{LX|L^0X} \] \hspace{1cm} (1.4.1)

of sheaves on the Zariski topology. We start with a version of the formal Poincaré lemma.
Lemma. Let $Z$ be a scheme, locally in the étale topology isomorphic to $\mathbb{A}^I = \text{Spec}[x_i; i \in I]$ for some possible infinite set $I$. Let $W \supset Z$ be a formal scheme locally isomorphic to $\text{Spf} \mathcal{O}_Z[[y_1, \ldots, y_m]]$. Let $\alpha \in \Omega^{p-1}_{W|Z}$ be a closed $(p-1)$-form on $W$ whose restriction to $Z$ is 0. Then, locally, $\alpha = d\beta$ for some $\beta \in \Omega^{p-2}_{W|Z}$. This $\beta$ is unique, if $p = 2$ and is defined globally, if $\alpha$ is.

Notice that $Z = \mathcal{L}^0 X$ satisfies the conditions of the lemma, while $\mathcal{L} X$ is a union of ind-schemes corresponding to formal schemes $W$ also satisfying the conditions.

Let now $\omega$ be a section of $\Omega^2_{\mathcal{L}}$ as before. Then $\tau^2(\omega)$ is a section of $\Omega^{-1}_{\mathcal{L}X|\mathcal{L}^0 X}$. By Lemma 1.4.2, $\tau^2(\omega) = d\beta$ for a unique $\beta \in \mathcal{O}_{\mathcal{L}X|\mathcal{L}^0 X}$. We define

$$\tau d^{-1}(\omega) = \beta.$$ 

It is clear that if $\omega = d\eta$ for an algebraic 1-form $\eta$, then $\tau d^{-1}(\omega) = \tau(\eta)$. We will call $\tau d^{-1}(\omega)$ the symplectic action functional. We will also consider the map

$$S : \Omega^2_{\mathcal{L}} \to \mathcal{O}^\times_{\mathcal{L}X|\mathcal{L}^0 X}, \quad S(\omega) = \exp(\tau d^{-1}(\omega)).$$

We call $S$ the exponentiated symplectic action map. Note that $S(\omega)$, being the exponential of the action, is the quantity entering the path integral in the Feynman interpretation of quantum field theory.

The global loop space and factorization. Let $C$ be a smooth algebraic curve, and $X$ be a smooth algebraic variety over $\mathbb{C}$. For any surjection $J \to I$ of nonempty finite sets we denote by $J_i$ the preimage of $i \in I$. To such a surjection one associates, in a standard way, the “diagonal” embedding

$$\Delta^{(J/I)} : C^I \to C^J.$$ 

Let $U^{(J/I)} \subset C^J$ be the locus of $(c_j)_{j \in J}$ such that $c_j \neq c_{j'}$ whenever the images of $j, j'$ in $I$ are different. We denote by

$$j^{(J/I)} : U^{(J/I)} \to C^J$$

the embedding. For $I = J$ it yields an open embedding

$$j^{(I/I)} : C^I, \neq U^{(I/I)} \to C^I.$$ 

For each $I$ we have constructed in [10] an ind-scheme $\mathcal{L}(X)_{C^I}$ over $C^I$ with the following properties:

(a) For $|I| = 1$, when $C^I = C$, the ind-scheme $\mathcal{L}(X)_{C}$ is given by the principal bundle construction of Gelfand and Kazhdan:

$$\mathcal{L}(X)_{C} = \mathcal{L} X \times_K \hat{C},$$

where $\hat{C}$ is the bundle of formal coordinate systems on $C$. Further, $\mathcal{L}(X)_{C}$ has a connection along $C$ induced by the action of $d/dt \in \mathfrak{g}$.
(b) The $\mathcal{L}(X)_{C^I}$ have a natural structure of a factorization monoid in the category of ind-schemes. More precisely, they are equipped with flat connections along the $C^I$. Further, for every surjection $J \to I$ there is
- an isomorphism of $C^I$-ind-schemes $\mu^{(J/I)} : \Delta^{(J/I)}*\mathcal{L}(X)_{C^I} \to \mathcal{L}(X)_{C^J}$,
- an isomorphism of $U^{(J/I)}$-ind-schemes $\kappa^{(J/I)} : j^{(J/I)}*\left(\prod_{i \in I} \mathcal{L}(X)_{C^I_i}\right) \to j^{(J/I)}*\mathcal{L}(X)_{C^J}$.

Further, these data are compatible in a natural way.

There is also a scheme $\mathcal{L}^0(X)_{C^I} \subset \mathcal{L}(X)_{C^I}$ satisfying the analogs of the properties (a) and (b) with $\mathcal{L}^0 X$ instead of $\mathcal{L} X$. The ind-scheme $\mathcal{L}(X)_{C^I}$ is a formal thickening of $\mathcal{L}^0(X)_{C^I}$, so it is given by a sheaf of topological rings $\mathcal{O}_{\mathcal{L}(X)_{C^I}}$ on $\mathcal{L}^0(X)_{C^I}$.

For each local function $f$ on $\mathcal{L}(X)_{C^I}$ we write $f|_{C^I}$ for $\mu^{(J/I)}(f)$, and $f|_{U^{(J/I)}}$ for $\kappa^{(J/I)}(f)$. Further we omit the identifications $\mu^{(J/I)}$, $\kappa^{(J/I)}$.

Let $f$ be a local section of $\mathcal{O}_{\mathcal{L}(X)}$. If $f$ is $K$-invariant, then by the principal bundle construction it gives rise to a local function $f_C \in \mathcal{O}_{\mathcal{L}(X)_C}$. For every surjection $J \to I$ define a function $f^{I,\neq}_{C^I}$ on

$$\mathcal{L}(X)_{C^I}^{I,\neq} := \mathcal{L}(X)_{C^I}|_{\Delta^{(J/I)}(C^I,\neq)} \simeq \left(\prod_{i \in I} \mathcal{L}(X)_{C^I_i}\right)|_{C^I,\neq}$$

by the formula

$$f^{I,\neq}_{C^I}(\langle a_i \rangle_{i \in I}) = \sum_{i \in I} f_C(a_i). \quad (1.5.1)$$

**1.5.2 Definition.** A local function $f$ on $\mathcal{L}(X)$ is called additive, if the following properties hold
(a) $f$ is $K$-invariant,
(b) for each $J$ there is a local function $f_{C^J}$ on $\mathcal{L}(X)_{C^J}$ restricting to $f^{I,\neq}_{C^I}$ on $\mathcal{L}(X)_{C^I}^{I,\neq}$ for every surjection $J \to I$.

We denote by $\mathcal{F} \subset p_*\mathcal{O}_{\mathcal{L}X}$ the subsheaf formed by additive functions vanishing on $\mathcal{L}^0(X)$. We now formulate the main result of this paper.

1.5.3 Theorem. The image of $\tau d^{-1} : \Omega^2_{cl} X \to p_*\mathcal{O}_{\mathcal{L}X}$ coincides with $\mathcal{F}$.

The proof will occupy the rest of the paper. Here we point out one simple property.

1.5.4 Lemma. If an additive function is $g$-invariant, then it vanishes on $\mathcal{L}^0(X)$.

**Proof.** It is enough to assume that $X = \text{Spec}(A)$ is affine. Recall that $\mathcal{L}^0(X)$ is the scheme of infinite jets of maps $A^1 \to X$. Therefore $\mathcal{L}^0(X) = \text{Spec}(A_D)$ where $A_D$ is the differential envelope of $A$, i.e., the algebra obtained by adjoining to $A$ all iterated formal derivatives $\partial^n(a)$.
of all elements of $A$ which are subject only to the Leibniz rule, see [1, (2.3.2)]. The derivation $\partial$ on $A_\mathcal{D}$ corresponds to the action of $d/dt \in g$ on $L^0(X)$. It is known that the ring of constants $\ker(\partial) \subset A_\mathcal{D}$ is equal to $\mathbb{C}$. This means that every $g$-invariant function on $L^0(X)$ is constant. The additivity condition implies that this constant is equal to 0. $\square$

(1.6) **Radon transforms are additive.** Here we prove that

$$\text{Im}(\tau d^{-1}) \subset \mathcal{F}.$$ 

The conditions (a), (b) of Definition 1.5.2 are clear. It remains to prove (c). We start by recalling from [10] the functors represented by $L(X)_CJ$. Let $S$ be a scheme and $a_J = (a_j)_{i \in J}$ be a morphism $S \to C^J$, so that $a_j : S \to C$, $j \in J$. Let $\Gamma_j = \Gamma_{a_j} \subset S \times C$ be the graph of $a_j$ and $\Gamma = \Gamma_J = \bigcup_j \Gamma_j$ be the union. This is a Cartier divisor in $S \times C$, so it is locally given by one equation. We denote by $\widehat{\mathcal{O}}_{\Gamma}$ the completion of $\mathcal{O}_{S \times C}$ along $\Gamma$, and by $K_{\Gamma}$ the localization $\widehat{\mathcal{O}}_{\Gamma}[t^{-1}]$ where $t$ is a (local) equation of $\Gamma$. Let also $K_{\Gamma, \sqrt{\Gamma}} \subset K_{\Gamma}$ be the subsheaf formed by sections whose restriction to $S_{\text{red}} \times C$ lies in $\widehat{\mathcal{O}}_{\Gamma_{\text{red}}}$, where $\Gamma_{\text{red}} = \Gamma \cap (S_{\text{red}} \times C)$. Then, see [10],

(1.6.1) **Proposition.** (a) Morphisms $h : S \to L(X)_CJ$ are in bijection with systems $(a_J, \varphi)$ where $a_J : S \to C^J$ and $\varphi$ is a morphism of locally ringed spaces

$$\left(\Gamma, K_{\Gamma, \sqrt{\Gamma}}\right) \to (X, \mathcal{O}_X).$$

(b) Similarly, morphisms $k : S \to L^0(X)_CJ$ are in bijection with systems $(a_J, \psi)$ where $a_J$ is as before and $\psi$ is a morphism of locally ringed spaces

$$\left(\Gamma, \widehat{\mathcal{O}}_{\Gamma}\right) \to (X, \mathcal{O}_X).$$

Notice that to define a function $f_{C^J} \in O_{L(X)_CJ}$ is the same as to define for each morphism $h : S \to L(X)_CJ$ with $S$ a scheme, a function $h^{*} f_{C^J} \in O_S$, in a compatible way.

Let $\omega \in \Omega^2_{X, \text{cl}}$ and $f = \tau d^{-1}(\omega) \in O_{LX}$. Let $h$ be as before, so in particular, we have morphisms $a_j : S \to C$, $j \in J$. For every surjection $J \to I$ let $S^I \subset S$ be the subscheme given by the conditions $a_j = a_{j'}$ iff the image of $j$, $j'$ in $I$ are equal. The functions $f^{I, \neq}_{C^J}$ defined in (1.5) give functions $h^{*} f^{I, \neq}_{C^J} \in O_{S^I, \neq}$, and we need to prove that they are the restrictions of a regular function $h^{*} f_{C^J} \in O_S$, which is then defined uniquely. This can be verified by working in the formal neighborhood of each point $s$ in $S^I \neq$. But for this we can replace $X$ by the union $U$ of the formal neighborhoods of the points

$$\{x_i\}_{i \in I} = \{\varphi(a_j(s))\}_{j \in J}.$$ 

Further, in $U$ the form $\omega$ can be represented as $d\eta$ for a 1-form $\eta$. So, in the rest of the proof we will assume that $\omega$ is exact, hence

$$f = \tau(\eta), \quad \eta \in \Omega^1_X.$$

(1.6.2)
Let \( \Omega = \Omega^1_{S \times C/S} \). The morphism

\[
\varphi : (\Gamma, \mathcal{K}_\Gamma^\wedge) \to (X, \mathcal{O}_X)
\]
gives rise to a section

\[
\varphi^* \eta \in H^0(\Gamma, \mathcal{K}_\Gamma^\wedge \otimes \Omega).
\]

Let \( \pi : \Gamma \to S \) be the projection. We now recall the definition and properties of the relative residue map

\[
\text{Res}_S : \pi_* (\mathcal{K}_\Gamma \otimes \Omega) \to \mathcal{O}_S.
\] (1.6.3)

Let \( \Gamma^{(m)} \subset S \times C \) be the \( m \)th infinitesimal neighborhood of \( \Gamma \), so

\[
\hat{\mathcal{O}}_\Gamma = \lim_{\leftarrow m} \mathcal{O}_{\Gamma^{(m)}} \quad \text{and} \quad \mathcal{O}_{\Gamma^{(m)}} = \mathcal{O}_{S \times C} / (t^{m+1})
\]

where \( t \) is a local equation of \( \Gamma \) in \( S \times C \). Recall the natural map

\[
\lim_{\leftarrow m} \text{Ext}^1(\mathcal{O}_{\Gamma^{(m)}}, \Omega) \to H^1_{\Gamma}(\Omega),
\] (1.6.4)

see [3, A.2.18]. Note also that the short exact sequence

\[
0 \to \hat{\mathcal{O}}_\Gamma \to \hat{\mathcal{O}}_\Gamma \to \mathcal{O}_{\Gamma^{(m)}} \to 0
\]
yields, after passing to \( \text{Ext}(\cdot, \Omega) \), that

\[
\text{Ext}^1(\mathcal{O}_{\Gamma^{(m)}}, \Omega) = \hat{\mathcal{O}}_\Gamma \otimes \Omega / t^{m+1}(\hat{\mathcal{O}}_\Gamma \otimes \Omega) \simeq t^{-m-1}(\hat{\mathcal{O}}_\Gamma \otimes \Omega) / \hat{\mathcal{O}}_\Gamma \otimes \Omega,
\]

and therefore the LHS of (1.6.4) is identified with

\[
\mathcal{K}_\Gamma \otimes \Omega / \hat{\mathcal{O}}_\Gamma \otimes \Omega.
\]

We get a morphism of sheaves

\[
\mathcal{K}_\Gamma \otimes \Omega \xrightarrow{P} H^1_{\Gamma}(\Omega).
\] (1.6.5)

We now compose \( P \) with the trace map of the Grothendieck duality theory, see [3, A.2], [8],

\[
\text{tr}_{\Gamma/S} : \pi_* H^1_{\Gamma}(\Omega) \to \mathcal{O}_S.
\]

Recall that \( \text{tr}_{\Gamma/S} \) is obtained from the canonical adjunction map

\[
\pi_* \pi^! \mathcal{O}_S \to \mathcal{O}_S
\]

via the map \( H^1_{\Gamma}(\Omega) \to \pi^! \mathcal{O}_S \). So we define the function

\[
h^* f_{CJ} = \text{tr}_{\Gamma/S}(P(\varphi^* \eta)) \in \mathcal{O}_S.
\]
It is clear that these functions are compatible for different $h: S \to \mathcal{L}(X)_{CJ}$, and thus define a function $f_{CJ} \in \mathcal{O}\mathcal{L}(X)_{CJ}$. We now verify that $f_{CJ}$ restricts to $f_{CJ}^{I,\neq}$ on $\mathcal{L}(X)^{I,\neq}$. Working with $h$ as before, it suffices to identify $h^* f_{CJ}$ on $S^{I,\neq}$.

We can replace $S$ by $S^{I,\neq}$ and assume that $a_j = a_{j'} = a_i$ for each $j, j' \in J$ whose images in $I$ are equal to $i$. Further the graphs $\Gamma_i$ of $a_i$, for $i \in I$ are disjoint. Representing locally $\varphi^* \eta$ as $\eta_0/t^d$ with $\eta_0 \in \hat{\mathcal{O}} \Gamma \otimes \Omega$, we find that, see [3, A.2.1],

$$\text{tr}_{\Gamma/S}(P(\varphi^* (\eta))) = \left[ \frac{\eta_0}{t^d} \right]. \quad (1.6.6)$$

In virtue of [3, A.1.5], the “residue symbol” on the RHS of (1.6.6) is nothing but the sum of the residues

$$\sum_{i \in I} \text{Res}_{\partial_i} (\varphi^* \eta).$$

This is precisely the definition of the function $f_{CJ}^{I,\neq}$ in (1.5.1), (1.6.2).

2. The sheafified Heisenberg module

(2.1) Reminder on vertex algebras. In this paper we will consider only $\mathbb{Z}_{\geq 0}$-graded vertex algebras as in [7]. Such an algebra is a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-vector space $V = \bigoplus_{n \geq 0} V_n$ equipped with a distinguished element $1 \in V_0$, an endomorphism $\partial: V \to V$ of degree one, and a family of bilinear operations $(x, y) \mapsto x \circ_n y$, such that $V_i \circ_n V_j \subset V_{i+j-n-1}$, subject to the axioms in [7, Definition 0.4]. One defines $Y(x, t)$ to be the operator formal series

$$Y(x, t) : y \mapsto \sum_n (x \circ_n y)t^{-n-1}.$$

Recall that a morphism of $\mathbb{Z}_{\geq 0}$-graded vertex algebras $T: V \to W$ is a homogeneous linear map such that

$$T(Y(x, t)y) = Y(Tx, t)Ty, \quad T(1) = 1, \quad \forall x, y \in V.$$

(2.1.1) Examples. (a) If $A$ is a commutative algebra with a derivation $\partial$, it is made into a vertex algebra with

$$Y(a, t)b = \exp(t \partial)(a) \cdot b.$$

Such vertex algebras will be called commutative (or holomorphic). They are characterized by the property that $Y(a, t) \in \text{End}(A)[[t]]$. Note that in this case $a \circ_0 b = a \cdot b$ is the usual product in $A$.

(b) Let $N \geq 1$ and $A_N$ be the Heisenberg algebra with generators $a_n^i, b_n^i, i = 1, \ldots, N, n \in \mathbb{Z}$, subject to

$$[a_m^j, b_n^i] = \delta_{i,j} \delta_{m,-n},$$
all other brackets being zero. Let $V_N$ be the cyclic $A_N$-module generated by a vector $1$ subject to

$$b^j_{<0} \cdot 1 = a^i_{<0} \cdot 1 = 0. \quad (2.1.2)$$

It is known that $V_N$ has a structure of a graded vertex algebra called the Heisenberg vertex algebras such that

$$Y(a^i_11, t)x = \sum_n a^i_n x t^{n-1}, \quad Y(b^j_01, t)x = \sum_n b^j_n x t^n,$$

and

$$\partial (a^i_{n+1}1) = (n+1)a^i_{n+2}1, \quad \partial (b^j_n1) = (n+1)b^j_{n+1}1, \quad n \geq 0,$$

see, e.g., [9,13].

(2.1.3) **Definition.** A filtered vertex algebra is a vertex algebra $V$ equipped with a filtration $\{F_i V\}_{i \geq 0}$ such that

$$(F_i V) \circ_n (F_j V) \subset F_{i+j-n-1} V.$$

In this case $\text{gr}^F V = \bigoplus_i F_i V / F_{i-1} V$ inherits the structure of a vertex algebra.

(2.1.4) **Example.** Let us introduce a filtration on the vertex algebra $V_N$ from 4.1.1(2). A basis of $V_N$ is formed by the monomials $a^{i_1}_{m_1} \cdots a^{i_p}_{m_p} b^{j_1}_{n_1} \cdots b^{j_q}_{n_q} \cdot 1$, $m_i > 0$, $n_j \geq 0$.

We define $F_i V_N$ to be the span of the monomials above with $p \leq i$. Note that

$$F_0 V_N = \mathbb{C}[b^j_n; j = 1, \ldots, N, n \geq 0]$$

is in fact a holomorphic vertex algebra. It corresponds to the derivation of the commutative algebra $\mathbb{C}[b^j_n]$ such that $\partial (b^j_n) = (n+1)b^j_{n+1}$. Further, $\text{gr}^F V_N$ is again holomorphic and identified with the commutative algebra

$$\mathbb{C}[a^i_m, b^j_n; m > 0, n \geq 0]$$

equipped with the derivation

$$\partial (a^i_m) = ma^i_{m+1}, \quad \partial (b^j_n) = (n+1)b^j_{n+1}.$$

(2.2) **Chiral differential operators on $A_N^\mathbb{N}$.** Let $O_{A_N}^{\text{ch}}$ be the sheaf of vertex algebras defined in [13]. Recall that $b^1, b^2, \ldots, b^N$ are the coordinates in $A_N^\mathbb{N}$. Identifying $b^i$ with $b^i_0$, localization of the $\mathbb{C}[b^j_1, \ldots, b^j_0]$-module $V_N$ yields a quasi-coherent Zariski sheaf on $A_N^\mathbb{N}$. See [13] for the localization of the vertex structure. The filtration $F$ on $V_N$ induces a filtration (also denoted $F$) on $O_{A_N}^{\text{ch}}$. We call $O_{\overline{A_N}}^{\text{ch}}$ the sheaf of chiral differential operators on $A_N^\mathbb{N}$. 
Notice further that $V_N$ is a module over the algebra
\[
\mathbb{C}[b^i_n; i = 1, \ldots, N, n \geq 0][[b^i_n; i = 1, \ldots, N, n < 0]] = H^0(\mathcal{L}^0_{A^N}, \mathcal{O}_{\mathcal{L}A^N}).
\]

This structure comes from the $A_N$-module structure, because sufficiently high degree monomials in the $b^i_{<0}$ vanish on any given vector. Therefore by localizing we get a sheaf $\mathcal{V}_N$ of discrete modules over the sheaf of topological rings $\mathcal{O}_{\mathcal{L}A^N}$ on the space $\mathcal{L}^0_{A^N}$. We have
\[
\mathcal{O}_{\mathcal{L}A^N} = p_*(\mathcal{V}_N), \quad V_N = H^0(\mathcal{V}_N). \tag{2.2.1}
\]

In particular, $\mathcal{O}_{\mathcal{L}A^N}$ is a $p_*(\mathcal{O}_{\mathcal{L}A^N})$-module.

(2.3) The sheaf of automorphisms of $\mathcal{O}_{\mathcal{L}A^N}$. We denote by $\mathcal{A}$ the Zariski sheaf on $A^N$ formed by automorphisms of $\mathcal{O}_{\mathcal{L}A^N}$ as a sheaf of filtered vertex algebras that induce the identity on $\text{gr}^F \mathcal{O}_{\mathcal{L}A^N}$. The following result is due to [6].

(2.3.1) Proposition. The sheaf $\mathcal{A}$ over $A^N$ is identified with $\Omega^{2,\text{cl}}_{A^N}$.

For future reference we recall the construction of [6]. Let
\[
\omega = \sum_{i,j} h_{ij}(b^1, \ldots, b^N) \, db^i \, db^j
\]
be a closed 2-form regular in a Zariski open set in $A^N$. Let us define for convenience the generating functions
\[
b^i(t) = \sum_{n \in \mathbb{Z}} b^i_n t^n, \quad a^i(t) = \sum_{n \in \mathbb{Z}} a^i_n t^{n-1},
\]
see [13, (1.11), (1.17)]. The formulas (4.4a), (4.4b) of [6] (taken for the case $g = \text{Id}$) yield the following expression for the automorphism $T_\omega : \mathcal{O}_{\mathcal{L}A^N} \to \mathcal{O}_{\mathcal{L}A^N}$ corresponding to $\omega$ by 2.3.1
\[
b^i(t) \mapsto b^i(t), \tag{2.3.2}
\]
\[
a^i(t) \mapsto a^i(t) + \sum_{k=1}^N b^k(t)' h_{ki}(b^1(t), \ldots, b^N(t)). \tag{2.3.3}
\]

Proof. It was proved in [6] that (2.3.2) and (2.3.3) indeed define an automorphism $T_\omega$ of sheaves of graded vertex algebras. It is clear that $T_\omega$ preserves the filtration $F$ introduced in Example 2.1.4 and induces the identity on the quotients of this filtration. This is because $F$ is defined in terms of the number of monomials in the $a^i_n$. Thus $T_\omega$ is a section of $\mathcal{A}$.

Further, the filtration induced by $F$ on the degree one part
\[
(\mathcal{O}_{\mathcal{L}A^N})_1 \subset \mathcal{O}_{\mathcal{L}A^N}
\]
coincides with the two-step filtration introduced in [6, 2.2]. It now follows from the results of [6] that each automorphism of \( \mathcal{O}_{\mathbb{A}^N} \) preserving the grading and the filtration \( F \), preserves, in particular, the two-step filtration on \( (\mathcal{O}_{\mathbb{A}^N})_1 \) and so is of the form \( T_\omega, \omega \in \Omega^{2,\text{cl}}_{\mathbb{A}^N} \). \( \square \)

The following is the key calculation of the paper.

(2.3.4) Proposition. Let \( \omega \) be a local section of \( \Omega^{2,\text{cl}}_{\mathbb{A}^N} \) and \( S(\omega) = \exp(\tau d^{-1}(\omega)) \in \mathcal{O}_{\mathbb{L}^A}^\times \) be the corresponding symplectic action functional. Then \( T_\omega : \mathcal{O}_{\mathbb{A}^N} \to \mathcal{O}_{\mathbb{A}^N} \) becomes equal, after the identification \( \mathcal{O}_{\mathbb{A}^N} = p_* \mathcal{V}_N \), to the operator of multiplication by \( S(\omega) \) on the \( \mathcal{O}_{\mathbb{L}^A} \)-module \( \mathcal{V}_N \).

Proof. Recall that the algebra \( H^0(\mathcal{O}_{\mathbb{L}(\mathbb{A}^N)}) \) acts on

\[ \mathcal{V}_N = H^0(\mathbb{A}^N, \mathcal{O}_{\mathbb{A}^N}^{\text{ch}}) = H^0(\mathcal{L}(\mathbb{A}^N), \mathcal{V}_N). \]

We can think of the element \( 1 \in \mathcal{V}_N \) as global section of the sheaf \( \mathcal{V}_N \) over \( \mathcal{L}(\mathbb{A}^N) \). Further, relations (2.1.2) means that the sheaf of \( \mathcal{O}_{\mathbb{L}(\mathbb{A}^N)} \)-modules \( \mathcal{V}_N \) is isomorphic to the sheaf of distributions on \( \mathcal{L}(\mathbb{A}^N) \), supported on \( \mathcal{L}^0(\mathbb{A}^N) \), with \( 1 \) identified with the delta function \( \delta = \delta_{\mathcal{O}_{\mathbb{L}(\mathbb{A}^N)}^\times} \). So the element \( a^i_n \in \mathcal{V}_N \) corresponds to the derivative \( \partial\delta / \partial b_i^n \). Let now \( \Phi = \Phi(b) \) be any element of \( H^0(\mathcal{O}_{\mathbb{L}(\mathbb{A}^N)}) \). Then we have

\[ \Phi \cdot \frac{\partial \delta}{\partial b_i^n} = (\Phi|_{\mathcal{L}^0(\mathbb{A}^N)}) \cdot \left( \frac{\partial \delta}{\partial b_i^n} \right) + \frac{\partial \Phi}{\partial b_i^n} \cdot \delta. \]

Let us now specialize to \( \Phi = S(\omega) \). In this case \( \Phi \) is equal to \( 1 \) on \( \mathcal{L}^0(\mathbb{A}^N) \) so we get

\[ \Phi \cdot (a^i_n 1) = a^i_n 1 + \frac{\partial \Phi}{\partial b_i^n} \cdot \delta. \]

Next, since \( S(\omega) = \exp(\tau(d^{-1}(\omega))) \), we have

\[ \frac{\partial \Phi}{\partial b_i^n} = \frac{\partial}{\partial b_i^n} \tau (d^{-1}(\omega)) \cdot \Phi = \left( \tau (\omega), \frac{\partial}{\partial b_i^n} \right) \cdot \Phi. \] (2.3.5)

Here the expression in the angle brackets is the contraction of the 1-form \( \tau(\omega) \) and the vector field \( \partial / \partial b_i^n \) on \( \mathcal{L}(\mathbb{A}^N) \).

Recall now the definition of the transgression map \( \tau \) on 2-forms. As before, assume that \( \omega = \sum h_{ij} db^i db^j \). Let \( (b^i_n) \) be a point of \( \mathbb{L}^A \) (with values in some ring) and \( (\delta b_i^n) \) be a tangent vector to \( \mathbb{L}^A \) at \( (b^i_n) \). We write the generating series

\[ b^i(t) = \sum_n b^i_n t^n, \quad \delta b^i(t) = \sum_n (\delta b^i_n) t^n. \]

Then by definition of \( \tau \) (Example 1.3.8) we have

\[ \tau(\omega)((b^i_n), (\delta b^i_n)) = \text{Res} \sum_{i,j} h_{ij} (b^1(t), \ldots, b^N(t)) \delta b^i(t) \cdot b^j(t) \cdot dt, \]
see (1.3). We apply this to the RHS of (2.3.5). Contracting with $\partial / \partial b^{i_n}$ means that we take

$$\delta b_m^j = \begin{cases} 1, & \text{if } j = i, m = -n, \\ 0, & \text{otherwise}. \end{cases}$$

On the level of generating series this entails

$$\delta b^j(t) = \begin{cases} t^n, & j = i, \\ 0, & j \neq i. \end{cases}$$

Substituting this into (2.3.5), we get that $\langle \tau(\omega), \partial / \partial b^{i_n} \rangle$ is equal to the coefficient at $t^n$ in

$$\sum_k h_{kj}(b(t)) \cdot b^k(t)'.$$ So $\Phi \cdot \tau(\omega) \cdot 1 = \tau(\omega) \cdot 1$, because $T_\omega$ is a morphism of vertex algebras satisfying (2.3.3).

Next, we verify the equality on elements of the form $b^i_1$. We get, for any $\Phi$

$$\Phi \cdot b^i_1 = b^i_1 \Phi = b^i_1 (\Phi | L_0^{0, AN}) \cdot \delta L_0^{0, AN}.$$ Let us now specialize to $\Phi = S(\omega)$. Since $\Phi$ is identically equal to 1 on $L_0^{0, AN}$, the above element is equal to $b^i_1$. So $\Phi \cdot b^i(t) = b^i(t)$.

Finally, we extend the equality from the generators $a^i_1, b^i_1$ of $V_N$ to the entire vertex algebra $V_N$. For this, we notice that both transformations we consider are in fact automorphisms of vertex algebras, and that $V_N$ is generated by $a^i_1, b^i_1$. More precisely, the transformations $T_\omega$ of [6] are proved in [6] to be automorphisms of vertex algebras. On the other hand the multiplication by $S(\omega) = \exp(\tau(d^{-1} \omega))$ is a morphism of vertex algebras, because so is multiplication by $\exp(f)$ for any factorizing function $f$, see (3.3) below, and Radon transforms are factorizing by (1.6). Alternatively, up to localizing we may assume that $\omega = d\eta$ for some $\eta \in \Omega^1_{AN}$. Set $\eta = \sum_i h_i \, dB^i$. Then $S(\omega) = \exp(\tau(\eta))$, and

$$\tau(\eta)(b^i_1) = \text{Res} \sum_i h_i (b^1(t), \ldots, b^N(t)) b^i(t)' \, dt = \text{Res} Y \left( \sum_i h_i (b^1_0, \ldots, b^N_0) b^i_1, t \right)$$

(no normal ordering because the $b^i_1$'s commute with each other). Thus multiplication by $S(\omega)$ is a morphism of vertex algebras, as an instance of the automorphism

$$\exp(\text{Res} Y(x, t))$$

valid for any vertex algebra $V$ and any element $x \in V$ such that the exponential is well defined, see [13, Section 1.8]. The proposition is proved. \qed
(2.3.7) Corollary. If $f \in \mathcal{O}^\times_{\mathbb{A}^N}$ is a local invertible function such that the multiplication by $f$ is an automorphism of $\mathcal{O}^{\text{ch}}_{\mathbb{A}^N}$ as a filtered graded vertex algebra, then $f = S(\omega)$ for some $\omega \in \Omega_{\mathbb{A}^N}^{2,\text{cl}}$.

Proof. By Proposition 2.3.4 there is an $\omega \in \Omega_{\mathbb{A}^N}^{2,\text{cl}}$ such that $f - S(\omega)$ acts by 0 on $\mathcal{O}^{\text{ch}}_{\mathbb{A}^N}$. Since $\mathcal{O}^{\text{ch}}_{\mathbb{A}^N}$ is a submodule (generated by the vacuum vector $\mathbf{1}$) and the automorphism given by $f$ preserves $\mathbf{1}$, we find that $f = 1$ on $\mathcal{O}^{\text{ch}}_{\mathbb{A}^N}$. Since both $f$ and $S(\omega)$ are equal to 1 on $\mathcal{O}^{\text{ch}}_{\mathbb{A}^N}$, $f - S(\omega)$ vanishes on $\mathcal{O}^{\text{ch}}_{\mathbb{A}^N}$. Further, $\mathcal{O}^{\text{ch}}_{\mathbb{A}^N}$ can be seen as the module of distributions on $\mathbb{A}^N$ supported on $\mathcal{O}^{\text{ch}}_{\mathbb{A}^N}$, so vanishing of the operator of multiplication by $f - S(\omega)$ implies that all the iterated normal derivatives of $f - S(\omega)$ vanish, so $f = S(\omega)$.

(2.4) Chiral differential operators in a coordinate chart. Let $X$ be a smooth algebraic variety equipped with an étale map $\phi : X \to \mathbb{A}^N$. We have then an étale morphism of schemes $\mathcal{L}^0 : \mathcal{L}^0(X) \to \mathcal{L}^0(\mathbb{A}^N)$ and a morphism of ind-schemes $\mathcal{L}\phi : \mathcal{L}(X) \to \mathcal{L}(\mathbb{A}^N)$. We have a sheaf of vertex algebras on $X$

$$\mathcal{O}^{\text{ch}}_{X,\phi} = \phi^*\mathcal{O}^{\text{ch}}_{\mathbb{A}^N} = (\phi^{-1}\mathcal{O}^{\text{ch}}_{\mathbb{A}^N}) \otimes_{\phi^{-1}\mathcal{O}_{\mathbb{A}^N}} \mathcal{O}_X,$$

(2.4.1)
called the sheaf of chiral differential operators (CDO) corresponding to the étale coordinate chart $\phi$. It inherits a grading and a filtration from $\mathcal{O}^{\text{ch}}_{\mathbb{A}^N}$. This sheaf is also a sheaf of $p_*\mathcal{O}_{\mathcal{L}(X)}$-modules so

$$\mathcal{O}^{\text{ch}}_{X,\phi} = p_*\mathcal{V}_{X,\phi}, \quad \mathcal{V}_{X,\phi} = (\mathcal{L}\phi)^*\mathcal{V}_N.$$

(2.4.2)

(2.4.3) Proposition. If $f \in \mathcal{O}^\times_{\mathcal{L}^0_X}$ is a local invertible function such that the multiplication by $f$ is an automorphism of $\mathcal{O}^{\text{ch}}_{X,\phi}$ as a filtered graded vertex algebra, then $f = S(\omega)$ for some $\omega \in \Omega_{\mathbb{A}^N}^{2,\text{cl}}$.

Proof. As in (2.3.7), this follows from the fact that the sheaf of automorphisms of $\mathcal{O}^{\text{ch}}_{X,\phi}$ as a filtered graded vertex algebra, is identified with $\Omega_{\mathbb{A}^N}^{2,\text{cl}}$, see [6]. The arguments in (2.3.1)–(2.3.4) can be repeated verbatim using the étale coordinates on $X$ given by $\phi$ instead of the standard coordinates in $\mathbb{A}^N$.

3. The factorization structure

(3.1) Reminder on factorization algebras and vertex algebras. We recall the concept of a factorization algebra, see [1] for more details.

(3.1.1) Definition. Let $\mathcal{E}$ be a quasi-coherent sheaf on $C$. A structure of a factorization algebra on $\mathcal{E}$ is a collection of quasi-coherent $\mathcal{O}_{\mathcal{C}^I}$-modules $\mathcal{E}_I$ for each non-empty finite set $I$, such that $\mathcal{E}_I$ is flat along the diagonal strata, $\mathcal{E}_{[1]} = \mathcal{E}$, and

(a) an isomorphism of $\mathcal{O}_{\mathcal{C}^I}$-modules $\nu^{(J/I)} : \Delta^{(J/I)} \mathcal{E}_J \simeq \mathcal{E}_I$ for every $J \to I$, compatible with the compositions of $J \to I$,,
(b) an isomorphism of $O_{U(J/I)}$-modules

$$\chi^{(J/I)} : j^{(J/I)*}((\mathbb{E}_J)) \sim j^{(J/I)*}\mathbb{E}_J$$

for every $J \to I$, compatible with the compositions of $J \to I$ and with $\nu$,

(c) a global section $1 \in H^0(C, \mathcal{E})$ such that for every $f \in \mathcal{E}$ one has $1 \otimes f \in \mathcal{E}_{[1,2]} \subset j_*(E \otimes f)$ and $\Delta^* (1 \otimes f) = f$.

A morphism of factorization algebras $\{E_I\}_I \to \{F_I\}_I$ is defined in the obvious way: it is a collection of morphisms of sheaves of $O_{CI}$-modules $f_I : E_I \to F_I$ which is compatible with $\nu$, $\chi$, and is such that $f = f_{[1]}$ sends $1 \otimes f$ to $1 \otimes f$.

(3.1.2) Proposition. Let $\mathcal{E}$ be a quasi-coherent sheaf on $C$ equipped with a structure of a factorization algebra. Let $0 \in C$ be a $C$-point, and $t$ be a formal coordinate on $C$ near $0$. Then $\mathcal{E}_0 = \mathcal{E} \otimes_{O_C} O_0$ has a natural structure of a vertex algebra (depending on the choice of $t$). Further, a morphism of factorization algebras yields a morphism of vertex algebras.

(3.2) The factorization algebra corresponding to $O_{\mathbb{A}^N}^{ch}$. We now assume that the curve $C$ is $\mathbb{A}^1$ with coordinate $t$ and $X = \mathbb{A}^N$. We identify $\mathcal{L}(X), \mathcal{L}^0(X)$ with the fibers of $\mathcal{L}(X)_C, \mathcal{L}^0(X)_C$ over $0 \in C$. For each $I$, let

$$q_I : \mathcal{L}^0(X)_{CI} \to X \times CI$$

be the structural morphism. The aim of this section is to describe a sheaf of $O_{\mathcal{L}(X)_{CI}}$-modules $\mathcal{V}_{N,I}$ over $\mathcal{L}^0(X)_{CI}$ such that

(a) the collection $\{(q_I)_*(\mathcal{V}_{N,I})\}_I$ of sheaves of $O_{CI}$-modules over $X$ form a sheaf of factorization algebras over $X$,

(b) the identification $\mathcal{L}(X) \simeq \mathcal{L}(X)_{C,0}$ with the fiber of $\mathcal{L}(X)_C$ at $0$ takes $\mathcal{V}_N$ to $\mathcal{V}_{N,\{1\}|\mathcal{L}(X)_{C,0}}$,

(c) $p^*(\mathcal{V}_N)$ is isomorphic to $O^{ch}_{\mathbb{A}^N}$ as a sheaf of vertex algebras.

In order to avoid technical complications in proving (c), we give a proof which relies on the results of [10]. To do so, we first recall the main steps in the construction of [10].

First, recall that $X = \mathbb{A}^N$ and $C = \mathbb{A}^1$ and that, see [10, 3.7.3]:

$$\mathcal{L}^0(\mathbb{A}^N)_{\mathbb{A}^1} = \text{Spec} \mathbb{C}[\lambda_i, b^j_{n,v}; i \in I, n \geq 0, j = 1, \ldots, N, v = 1, \ldots, |I|], \quad (3.2.1)$$

$$O_{\mathcal{L}(\mathbb{A}^N)_{\mathbb{A}^1}} = O_{\mathcal{L}^0(\mathbb{A}^N)_{\mathbb{A}^1}}[[[b^j_{n,v}; n < 0, j = 1, \ldots, N, v = 1, \ldots, |I|]]. \quad (3.2.2)$$

More precisely, if $R$ is a commutative algebra and $(\lambda_i, b^j_{n,v})$ is a system of elements of $R$, i.e., an $R$-point of $\mathcal{L}^0(\mathbb{A}^N)_{\mathbb{A}^1}$, one associates to it a morphism

$$c_I : \text{Spec}(R) \to \mathbb{A}^I = \text{Spec} \mathbb{C}[t_i; i \in I]$$
taking \( t_i \) to \( \lambda_i \) and the homomorphism

\[
\rho : \mathbb{C}[b^1, \ldots, b^N] \to H^0(\Gamma_\ell, \hat{\Omega}_\ell), \quad b^j \mapsto \sum_{n=0}^\infty \sum_{v=1}^{\vert I \vert} b^j_{n,v} t^{v-1} \prod_{i \in I} (t - \lambda_i)^n,
\]

and similarly for \( L(\mathbb{A}^N)_{\hat{A}^I} \).

Put also

\[
L^\varepsilon_\ell(\mathbb{A}^N)_{\hat{A}^I} = \text{Spec}(\mathbb{C}[\lambda_i, b^j_{n,v}; \ell \geq n]/(b^j_{n,v_1} b^j_{n,v_2} \ldots b^j_{n,v_1 + \varepsilon_n}; n < 0)), \quad (3.2.3)
\]

where \( \varepsilon = (\varepsilon_{-1}, \varepsilon_{-2}, \ldots) \) with \( \varepsilon_{n} \geq 0, \varepsilon_{n} = 0 \) for almost all \( n \), and \( v_1, \ldots, v_{1+\varepsilon_n} = 1, \ldots, \vert I \vert \).

Then the ind-scheme \( L(\mathbb{A}^N)_{\hat{A}^I} \) represents the ind-pro-object

\[
\left\{ \left[ \text{CDR}(\omega_{\mathbb{A}^N})_{\hat{A}^I} \right]_I \right\}
\]

is a sheaf of factorization algebras over \( X \), and the fiber of \( \text{CDR}(\omega_{\mathbb{A}^N})_{\hat{A}^I} \) at the point \( 0 \in \mathbb{A}^I \) is a complex of sheaves of \( \mathbb{Z}_{\geq 0} \)-graded vertex algebras which is isomorphic to the chiral de Rham complex \( \Omega_{\mathbb{A}^N}^{\text{ch}} \) of [13].

By [10, 5.5.5], the complex \( \Gamma(\mathbb{A}^N, \text{CDR}(\omega_{\mathbb{A}^N})_{\hat{A}^I}) \) is also equipped with a left action of the \( \mathbb{C}[\mathbb{A}^I] \)-algebra \( \widehat{CD}_I \) generated by symbols \( a^i_{n,v}, b^i_{n,v}, \phi^i_{n,v}, \psi^i_{n,v}, i = 1, \ldots, N, n \in \mathbb{Z}, v = 1, \ldots, \vert I \vert \), subject to

\[
[a^i_{m,\mu}, b^j_{n,v}] = \delta_{i,j} \delta_{v,\mu} \delta_{m,-n}, \quad [\phi^j_{m,\mu}, \psi^i_{n,v}] = \delta_{i,j} \delta_{v,\mu} \delta_{m,-n},
\]

all other brackets being zero. Recall that \( \mathbb{A}^I \simeq \text{Spec} \mathbb{C}[\lambda_i; i \in I] \). This module is generated by a cyclic vector \( 1_I \) subject to

\[
b^j_{<0,v} \cdot 1_I = a^j_{<0,v} \cdot 1_I = \phi^j_{<0,v} \cdot 1_I = \psi^j_{<0,v} \cdot 1_I = 0.
\]

By localizing \( \Gamma(\mathbb{A}^N, \text{CDR}(\omega_{\mathbb{A}^N})_{\hat{A}^I}) \) we get a sheaf \( \mathcal{V}_{N,I} \) of discrete \( \mathcal{O}_L(\mathbb{A}^N)_{\hat{A}^I} \)-modules over \( L^0(\mathbb{A}^N)_{\hat{A}^I} \).

Now, consider the subalgebra \( A_{N,I} \subset \widehat{CD}_I \) generated by \( \{a^i_{n,v}, b^i_{n,v}\} \). Under localization the \( \mathbb{C}[\mathbb{A}^I] \)-submodule

\[
A_{N,I} 1_I \subset \Gamma(\mathbb{A}^N, \text{CDR}(\omega_{\mathbb{A}^N})_{\hat{A}^I}),
\]

yields a subsheaf \( \mathcal{V}_{N,I} \subset \mathcal{V}_{N,I} \) of discrete \( \mathcal{O}_L(\mathbb{A}^N)_{\hat{A}^I} \)-modules over \( L^0(\mathbb{A}^N)_{\hat{A}^I} \). It may be viewed as the sheaf of distributions on \( L(\mathbb{A}^N)_{\hat{A}^I} \) supported on \( L^0(\mathbb{A}^N)_{\hat{A}^I} \). The following is immediate from [10, 5.4, 5.5].
(3.2.4) Lemma. The collection of sheaves of $\mathcal{O}_{\mathcal{C}^1}$-modules $\{(q_I)_*(\mathcal{V}_{N,I})\}_I$ over $X$ is a subsheaf of factorization algebras of $(\mathcal{C}D\mathcal{R}(\omega_\mathcal{C}^1))_{\mathcal{A}^1}$. Further, the corresponding subsheaf of vertex algebra of $H^0(\Omega^\mathcal{C}^1_{\mathcal{A}^1})$ is isomorphic to $\mathcal{V}_N = H^0(\Omega^\mathcal{C}^1_{\mathcal{A}^1})$.

(3.3) The factorization algebra corresponding to $\mathcal{O}^\mathcal{C}_{X,\phi}$. Let $\phi : X \to \mathbb{A}^N$ be an étale map as in (2.4). We have then the morphism of factorization groupoids $\{\mathcal{L}_{\mathcal{C}^1}(\phi) : \mathcal{L}_{\mathcal{C}^1}(X) \to \mathcal{L}_{\mathcal{C}^1}(\mathbb{A}^N)\}$. Using this, we obtain the factorization algebra corresponding to $\mathcal{O}^\mathcal{C}_{X,\phi}$ as a pullback. More precisely, we define sheaves $\mathcal{V}_{X,\phi,I} \subset \mathcal{W}_{X,\phi,I}$ of discrete $\mathcal{O}_{\mathcal{L}(\mathcal{C}^1)_{\mathcal{A}^1}}$-modules over $\mathcal{L}^0(X)_{\mathcal{A}^1}$ as

$$\mathcal{V}_{X,\phi,I} = (\mathcal{L}_{\mathcal{C}^1}(\phi))^*\mathcal{V}_{N,I}, \quad \mathcal{W}_{X,\phi,I} = (\mathcal{L}_{\mathcal{C}^1}(\phi))^*\mathcal{W}_{N,I}. \tag{3.3.1}$$

It follows from [10] and the definition of $\mathcal{O}^\mathcal{C}_{X,\phi}$ that the next proposition holds.

(3.3.2) Proposition. (a) The collections of sheaves $\{(q_I)_*(\mathcal{V}_{X,\phi,I})\}_I$ and $\{(q_I)_*(\mathcal{W}_{X,\phi,I})\}_I$ form sheaves of factorization algebras on $X$.

(b) The corresponding sheaves of vertex algebras are $\mathcal{O}^\mathcal{C}_{X,\phi}$ and $\mathcal{O}^\mathcal{C}_{X,\phi}$.

(3.4) Additive functions are Radon transforms. We can now finish the proof of Theorem 1.5.3. We must prove the inclusion

$$\mathcal{F} \subset \text{Im}(\tau d^{−1}).$$

As above, we take $C = \mathbb{A}^1$. Further, it is enough to work Zariski locally on $X$ and therefore we can assume that $X$ possesses an étale map $\phi : X \to \mathbb{A}^N$. We fix such $\phi$. Fix also a local additive function $f$ on $\mathcal{L}(X)$ vanishing on $\mathcal{L}^0(X)$, and local functions $f_{\mathcal{C}^1}$ over $\mathcal{L}(X)_{\mathcal{C}^1}$ as in (1.5.2). By the definition of $f_{\mathcal{C}^1}$, we have that $f_{\mathcal{C}^1}$ vanishes on $\mathcal{L}^0(X)_{\mathcal{C}^1}$. Thus the section $\exp(f_{\mathcal{C}^1})$ of $\mathcal{O}^\mathcal{C}_{\mathcal{L}(\mathcal{C}^1)_{\mathcal{C}^1}}$ is well defined, and multiplication by $\exp(f_{\mathcal{C}^1})$ is a morphism of sheaf of $\mathcal{O}_{\mathcal{C}^1}$-modules $(q_I)_*(\mathcal{V}_{X,\phi,I})$ over $X$. When $I$ varies, such morphisms are obviously compatible with the factorization structure of $(q_I)_*(\mathcal{V}_{N,I})$ in (3.2.4). Further, the unit maps to itself because $\exp(f_{\mathcal{C}^1}) = 1$ on $\mathcal{L}^0(X)_{\mathcal{C}^1}$. Thus we get a factorization algebra automorphism. By (3.1.2) multiplication by $\exp(f)$ is an automorphism of the sheaf of vertex algebras

$$\mathcal{O}^\mathcal{C}_{X,\phi} = p_*(\mathcal{V}_{X,\phi}).$$

So, by (2.4.3), we have $\exp(f) = S(\omega)$ for some closed two-form $\omega$ on $X$. Hence $f = \tau d^{-1}(\omega)$, yielding the desired inclusion. Note also that the main result of [13] says that the sheaves $\Omega^\mathcal{C}^1_{X,\phi} = \Omega^\mathcal{C}^1_X$ are in fact independent of $\phi$ up to a canonical isomorphism. Theorem 1.5.3 is proved.

We now finish the proof of Theorem 0.2.1. To see the equivalence of (i) and (ii), note that $\mathcal{O}^\mathcal{C}_{X,\phi}$ is a $p_*(\mathcal{O}_{\mathcal{L}X})$-submodule of $\mathcal{O}^\mathcal{C}_{X,\phi}$, as well as a sheaf of vertex subalgebras. So if the operator of multiplication by $\exp(f)$ is an automorphism of $\Omega^\mathcal{C}^1_{X,\phi}$ as a sheaf of vertex algebras, it is an automorphism of $\mathcal{O}^\mathcal{C}_{X,\phi}$ as a sheaf of vertex algebras and so $f$ has the form $\tau(d^{-1}(\omega))$ by (2.4.3). On the other hand, if $f$ is of the form $\tau(d^{-1}(\omega))$, then the multiplication by $\exp(f)$ is a vertex automorphism by the same argument as in (2.3.6) (proof of Proposition 2.3.4).

The equivalence of (i) and (iii) follows from Theorem 1.5.3 and the next lemma.
(3.4.1) Lemma. If \( f \) is 2-additive, then the multiplication by \( \exp(f) \) is an automorphism of \( \mathcal{O}^{ch}_{X,\phi} \) and \( \Omega^{ch}_{X,\phi} \).

Proof. We need to prove that the multiplication by \( \exp(f) \) preserves the operations \( a \circ_n b, n \in \mathbb{Z} \). But in the language of factorization algebras these operations are obtained by expanding the factorization isomorphism on \( C^2 \) near the diagonal. See [1, (3.5.14)]. So the existence of \( f_{C^2} \) implies that the \( a \circ_n b \) are preserved. \( \square \)

(3.4.2) Corollary. A 2-additive function is additive.

This is natural to expect if we view additivity as an infinitesimal analog of additivity in \( n \)-punctured disks for all \( n \geq 2 \). Any such disk can be decomposed into 2-punctured ones (pairs of pants) so additivity in pairs of pants implies additivity in \( n \)-punctured disks.

References