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Thompson's Metric and Mixed Monotone Operators

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We apply Thompson's metric to prove several fixed point theorems of mixed monotone operators by appealing to various generalizations of Banach's fixed point theorem. © 1993 Academic Press, Inc.

1. INTRODUCTION

One of the most powerful and interesting tools for investigating nonlinear operators in cones is the so-called Hilbert's metric, or its variant, Thompson's metric. While Hilbert's metric is more well known, Thompson's metric has the advantage of not being restricted to a section of the cone when completeness is indispensable for solving problems. In this paper, we apply Thompson's metric to prove several fixed point theorems of mixed monotone operators.

2. PRELIMINARIES

Let E be a real Banach space which is partially ordered by a cone P, i.e., $x \leqslant y$ iff $y - x \in P$. Throughout this paper, we assume that the norm is monotone, i.e., $0 < x \leqslant y$ implies that $\|x\| \leqslant \|y\|$. Since P is normal iff E has an equivalent norm which is monotone, all the cones in this paper are normal. The interior of P is denoted by $i(P)$. If $i(P)$ is not empty, then we say P is solid.

Let D be a set in E. An operator $A: D \times D \to E$ is said to be mixed monotone if $A(x, y)$ is nondecreasing in x and nonincreasing in y. $(x^*, y^*) \in D \times D$ is called a coupled fixed point of A if

$$x^* = A(x^*, y^*) \qquad \text{and} \qquad y^* = A(y^*, x^*).$$

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If $x^* = y^*$, i.e., $x^* = A(x^*, x^*)$, then x^* is called a fixed point of A. For some recent discussions on mixed monotone operators, see [2, 4, 5, 10], and the references therein.

Let $x, y \in P - \{0\}$. We say that x, y are comparable [9] if there exist positive numbers λ and μ such that $\lambda x \leqslant y \leqslant \mu x$. This defines an equivalent relation, and splits $P - \{0\}$ into disjoint equivalent classes which are called components of P. If $i(P)$ is not empty, then $i(P)$ is a component of P.

Let C be a component of P. For $x, y \in C$, let

$$M(x/y) = \inf\{\lambda \colon x \leqslant \lambda y\}, \qquad M(y/x) = \inf\{\mu \colon y \leqslant \mu x\},$$

and Thompson's metric is defined by

$$d(x, y) = \ln\{\max[M(x/y), M(y/x)]\}.$$

Then $d(\,,\,)$ defines a metric on C. Under our assumptions on P, we have the following propositions.

PROPOSITION 2.1 (see [11]). *Each component C of P is complete with respect to $d(\,,\,)$.*

PROPOSITION 2.2. *Let C be a component of P, and $\{x_n\}$ be a Cauchy sequence in C with respect to $d(\,,\,)$. Then $\{x_n\}$ is also a Cauchy sequence in norm. Furthermore, $\lim_{n\to\infty} d(x_n, u) = 0$ implies $\lim_{n\to\infty} \|x_n - u\| = 0$. (From the proof of Lemma 3 in [11].)*

Under certain restrictions, we can prove a converse of Proposition 2.2.

PROPOSITION 2.3. *Let $\{x_n\}$ be a sequence in $i(P)$, and $\lim_{n\to\infty} \|x_n - x\| = 0$, where $x \in i(P)$. Then $\lim_{n\to\infty} d(x_n, x) = 0$.*

Proof. This is a modification of the proof of Theorem 1.5.3 in [6].

Since $x \in i(P)$, there exists $r > 0$ such that $\{y \in : \|y - x\| \leqslant r\} \subseteq P\}$. Put $\varepsilon_n = \|x - x_n\|$. Without loss of generality, assume that $\varepsilon_n \neq 0$ for all n. Then $x \pm r(x - x_n)/\varepsilon_n \in P$. It follows that

$$-\frac{\varepsilon_n - r}{r}\, x < x_n < \frac{\varepsilon_n + r}{r}\, x.$$

For n sufficiently large such that $r - \varepsilon_n > 0$,

$$d(x_n, x) \leqslant \ln\left\{\max\left(\frac{\varepsilon_n + r}{r}, \frac{r}{r - \varepsilon_n}\right)\right\}.$$

Let $n \to \infty$; we have $d(x_n, x) \to 0$.

3. Main Results

In this section, we work in a real Banach space which is partially ordered by a cone P.

Our first theorem generalizes the main part of Theorem 1 in [4], and the assumption (3.1) below is weaker than the assumption (1) in [4].

THEOREM 3.1. *Let C be a component of P, and $A: C \times C \to C$ be a mixed monotone operator. Suppose that for each $[a, b] \subseteq (0, 1)$, there exists a positive number $\alpha(a, b) \in (0, 1)$ such that*

$$A(tx, t^{-1}x) \geqslant t^{\alpha(a,b)} A(x, x) \tag{3.1}$$

for all $x \in C$, $t \in [a, b]$. Then A has exactly one fixed point x^* in C, and for any point $x_0 \in i(P)$, we have $A^n(x_0, x_0) \to x^* = A(x^*, x^*)$ as $n \to \infty$.

Proof. Define $\bar{A}(x) = A(x, x)$. Let $x, y \in C$ such that $d(x, y) \in [-\ln b, -\ln a]$. Without loss of generality, assume $M(x/y) \geqslant M(y/x)$, i.e., $d(x, y) = \ln M(x/y)$. By assumption $(1/b) \leqslant M(x/y) \leqslant (1/a)$. Now

$$\begin{aligned}
\bar{A}(x) &= A(x, x) \\
&\geqslant A(M(y/x)^{-1} y, M(x/y)\, y) \\
&\geqslant A(M(x/y)^{-1} y, M(x/y)\, y) \\
&\geqslant M(x/y)^{-\alpha(a,b)} A(y, y) \\
&\geqslant M(x/y)^{-\alpha(a,b)} \bar{A}(y).
\end{aligned}$$

Hence,

$$M(\bar{A}(y)/\bar{A}(x)) \leqslant M(x/y)^{\alpha(a,b)}.$$

On the other hand,

$$\begin{aligned}
\bar{A}(y) &= A(y, y) \\
&\geqslant A(M(x/y)^{-1} x, M(y/x)\, x) \\
&\geqslant A(M(x/y)^{-1} x, M(x/y)\, x) \\
&\geqslant M(x/y)^{-\alpha(a,b)} A(x, x) \\
&\geqslant M(x/y)^{-\alpha(a,b)} \bar{A}(x)
\end{aligned}$$

implies that

$$M(\bar{A}(x)/\bar{A}(y)) \leqslant M(x/y)^{\alpha(a,b)}.$$

Therefore

$$d(\bar{A}(x), \bar{A}(y)) \leqslant \ln\{M(x/y)^{\alpha(a,b)}\}$$
$$= \alpha(a, b)\, d(x, y). \tag{3.2}$$

By the Generalized Contraction Mapping Principle [8, Thm. 37.2], $\bar{A}$ has an unique fixed point $x^* \in C$, since C is a complete metric space under $d(\,,\,)$.

Let $x_0 \in C$ and $x_0 \neq x^*$. Denote $r_n = d(A^n(x_0), x^*)$. By (3.2), r_n is a decreasing sequence. If $r_n \to \delta > 0$, then $\delta \leqslant r_n \leqslant d(x_0, x^*)$ for all n. Again by (3.2), there exists a positive constant $\alpha < 1$, which depends on δ and $d(x_0, x^*)$, such that

$$r_n = d(\bar{A}^n(x_0), x^*) \leqslant \alpha d(\bar{A}^{n-1}(x_0), x^*)$$

for all n. It follows that

$$\delta \leqslant r_n \leqslant \alpha^n d(x_0, x^*) \to 0 \qquad \text{as} \quad n \to \infty.$$

That is a contradiction. We conclude that $r_n \to 0$ as $n \to \infty$. Hence, $\|\bar{A}^n(x_0) - x^*\| \to 0$ as $n \to \infty$, by Proposition 2.2.

COROLLARY 3.2. *Let C be a component of P, and $A\colon C \times C \to C$ be a mixed monotone operator. Suppose that there exists a lower semicontinuous function $\phi\colon (0, 1) \to (0, 1)$ such that $\phi(r) > r$ and*

$$A(tx, t^{-1}x) \geqslant \phi(t)\, A(x, x) \tag{3.3}$$

for all $x \in C$, $t \in (0, 1)$. Then A has exactly one fixed point x^* in C, and for any point $x_0 \in i(P)$, we have $A^n(x_0, x_0) \to x^* = A(x^*, x^*)$ as $n \to \infty$.

Proof. We only need to prove that condition (3.3) implies condition (3.1). Note that

$$\phi(t) = t^{\log_t \phi(t)},$$

and that $\log_t \phi(t)$ is upper semicontinuous since $\phi(t)$ is lower semicontinuous and $t \in (0, 1)$. Let $[a, b] \subseteq (0, 1)$. $\log_t \phi(t)$ attains its maximum $\alpha(a, b)$ on $[a, b]$. $0 < \alpha(a, b) < 1$ since $0 < \log_t \phi(t) < 1$. Hence for all $x \in C$ and $t \in [a, b]$, we have

$$A(tx, t^{-1}x) \geqslant t^{\log_t \phi(t)} A(x, x)$$
$$\geqslant t^{\alpha(a,b)} A(x, x).$$

This concludes the proof.

If $x \leqslant y$ and $x \neq y$, then we denote $x < y$ or $y > x$. We are to prove a fixed point theorem of mixed monotone operator which satisfies a strongly sublinear condition.

THEOREM 3.3. *Let P be a solid cone and $A: i(P) \times i(P) \to i(P)$ be a mixed monotone operator which satisfies*

$$A(tx, t^{-1}x) > tA(x), \qquad 0 < t < 1. \tag{3.4}$$

Suppose that there exists some $x \in i(P)$ such that the sequence $\{A^n(x_0, x_0)\}$ contains a subsequence which converges to a point $x^* \in i(P)$. Then x^* is a fixed point of A and $A^n(x_0, x_0) \to x^*$ as $n \to \infty$.

Proof. Put $\bar{A}(x) = A(x, x)$. Similar to the proof of (3.2) in Theorem 3.1, we can prove

$$d(\bar{A}(x), \bar{A}(y) < d(x, y) \tag{3.5}$$

by means of (3.4).

Since $\{\bar{A}^n(x_0)\}$ contains a subsequence which converges to $x^* \in i(P)$, that subsequence also converge to x^* with respect to $d(,)$ due to Proposition 2.3. Applying Edelstein's fixed point theorem [3, Thm. 1], we conclude that x^* is a fixed point of $\bar{A}(x)$ and $d(A^n(x_0), x^*) \to 0$ as $n \to \infty$. Finally, Proposition 2.2 implies $\|(A^n(x_0) - x^*\| \to 0$ as $n \to \infty$.

All the above results include the corresponding fixed point theorems of monotone operators $T: C \to C$ as special cases. In fact, we only need to put $A(x, y) = T(x)$ for all $(x, y) \in C \times C$ in case of nondecreasing operator T, and $A(x, y) = T(y)$ for all $(x, y) \in C \times C$ in case of nonincreasing operator T.

As another application of Thompson's metric, we will prove a fixed point theorem of $A: C \to C$, which may not satisfy any monotone condition, by appealing to Caristi's fixed point theorem [1, Thm. (2.1)′].

THEOREM 3.4. *Let C be a component of P and $A: C \to C$ be a mapping. Suppose that $\phi: C \to R^+$ be lower semicontinuous and*

$$x \geqslant A(x) \geqslant \frac{\phi(A(x))}{\phi(x)} x, \qquad x \in C. \tag{3.6}$$

Then A has a fixed point $x^* \in C$.

Proof. From (3.6),

$$M(A(x)/x) \leqslant 1$$

and

$$M(x/A(x)) \leqslant \frac{\phi(x)}{\phi(A(x))}.$$

Equation (3.6) also implies that $\phi(x)/\phi(A(x)) \geqslant 1$. Hence

$$d(x, A(x)) \leqslant \ln \left\{\frac{\phi(x)}{\phi(A(x))}\right\} \leqslant \ln \phi(x) - \ln \phi(A(x)).$$

Applying Caristi's fixed point theorem, A has a fixed point $x^* \in C$.

4. Application

In this section, we give an example to illustrate the possible applications of our results.

Let $C(X)$ denote the Banach space consists of all real valued continuous functions on X with the sup norm, where X is a compact metric space.

Let $\Omega \subset R^n$ be a compact set. Consider the integral operator

$$A(x, y)(t) = \int_\Omega K(t, s) f(s, x(s), y(s))\, ds, \tag{4.1}$$

where $K \in C(\Omega \times \Omega)$ is positive, $x, y \in C(\Omega)$, and the real valued functional $f(s, x, y)$ is nondecreasing in x and nonincreasing in y. Define the cone $P = \{x \in C(\Omega): x(s) \geqslant 0, s \in \Omega\}$.

Suppose that f possesses the following properties:

(i) $f(s, x(s), y(s)) > 0$, when $x, y \in C(\Omega)$.

(ii) A mixed monotone property with respect to g, i.e., $f(s, x, y)/g(s, x, y)$ is nonincreasing in x and nondecreasing in y ([7] introduced some similar notions), where g is a positive continuous function on $\Omega \times C(\Omega) \times C(\Omega)$ and

$$1 > \alpha(\lambda) = \min \left\{\frac{g(s, \lambda x(s), \lambda^{-1} x(s))}{g(s\, x(s)\, y(s))}: s \in \Omega\right\} > \lambda, \qquad 0 < \lambda < 1. \tag{4.2}$$

It is clear that $A: i(P) \times i(P) \to i(P)$. Now for $0 < \lambda < 1$.

$$\begin{aligned}
A(\lambda x, \lambda^{-1}x) &= \int_\Omega K(t,s)\, f(s, \lambda x(s), \lambda^{-1}x(s))\, ds \\
&= \int_\Omega K(t,s)\, \frac{f(s, \lambda x(s), \lambda^{-1}x(s))}{g(s, \lambda x(s), \lambda^{-1}x(s))}\, g(s, \lambda x(s), \lambda^{-1}x(s))\, ds \\
&\geqslant \int_\Omega K(t,s)\, \frac{f(s, x(s), x(s))}{g(s, x(s), x(s))}\, g(s, \lambda x(s), \lambda^{-1}x(s))\, ds \qquad \text{(by (ii))} \\
&\geqslant \alpha(\lambda) \int_\Omega K(t,s)\, f(s, x(s), x(s))\, ds \qquad \text{(by (4.2))} \\
&= \alpha(\lambda)\, A(x, x).
\end{aligned}$$

Hence we can apply Corollary 3.2 and get an $x^* \in i(P)$ such that

$$x^*(s) = \int_\Omega K(t,s)\, f(s, x^*(s), x^*(s))\, ds.$$

Furthermore, for any $x_0 \in i(P)$, $A^n(x_0, x_0) \to x^*$ in sup norm as $n \to \infty$.

References

1. J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, *Trans. Amer. Math. Soc.* **215** (1976), 241–251.
2. Yong-Zhuo Chen, Existence theorems of coupled fixed points, *J. Math. Anal. Appl.* **154** (1991), 142–150.
3. M. Edelstein, On fixed and periodic points under contractive mappings, *J. London Math. Soc.* **37** (1962), 74–79.
4. Dajun Guo, Fixed points of mixed monotone operators with applications, *Appl. Anal.* **31** (1988), 215–224.
5. Dajun Guo and V. Lakshimikantham, Coupled fixed points of nonlinear operators with applications, *Nonlinear Anal. TMA* **11** (1987) 623–632.
6. Dajun Guo and V. Lakshimikanthan "Nonlinear Problems in Abstracts Cones," Academic Press, New York, 1988.
7. G. W. Howell and V. Lakshmikantham, A new approach to monotone iterative technique for solution of differential equations, *Appl. Anal.* **39** (1990), 113–117.
8. M. A. Krasnosel'skii and P. P. Zabreiko, "Geometrical Methods of Nonlinear Analysis," Springer-Verlag, Berlin, 1984.
9. R. D. Nussbaum, Iterated nonlinear maps and Hilbert's projective metric, *Mem. Amer. Math. Soc.* **75**, No. 391 (Sept. 1988).
10. Yong Sun, A fixed point theorem for mixed monotone operators with applications, *J. Math. Anal. Appl.* **156** (1991), 240–252.
11. A. C. Thompson, On certain contraction mappings in a partially ordered vector space, *Proc. Amer. Math. Soc.* **14** (1963), 438–443.