Partial Differential Equations with Deviating Arguments in the Time Variable

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Some classes of partial differential equations with deviating arguments in the time variable are investigated. The equations are written as abstract ordinary functional differential equations of the form $\frac{dj}{dt} u(t) = -Au(t) + F(u_\tau), u_0 = \phi$, where $-A$ is the infinitesimal generator of a linear semigroup and $F$ is not necessarily continuous. Existence, uniqueness, and stability properties are developed by studying the equations in appropriately chosen spaces of initial functions $\phi$.

1. INTRODUCTION

Our objective is to investigate the existence, uniqueness, and stability properties of a class of partial differential equations with deviating arguments in the time variable. This work continues the investigations of [3] in which the equation considered was of the form

$$\frac{d}{dt} u(t) = -Au(t) + F(u_\tau), \quad t \geq 0, \quad u_0 = \phi,$$

where $-A$ is the infinitesimal generator of a strongly continuous semigroup of linear operators $T(t), t \geq 0$, and $F$ is an everywhere defined, nonlinear Lipschitz continuous function. In the present study we wish to weaken these conditions on $F$; which for applications will correspond to allowing deviating arguments to occur in terms which involve spatial partial derivatives.

Before proceeding to a general problem we will analyze a simple example in order to illustrate the ideas involved. Consider the delay partial differential equation

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\[ w_t(x, t) = w_{xx}(x, t - 1), \quad 0 \leq x \leq \Pi, \quad t \geq 0 \]
\[ w(0, t) - w(\Pi, t) = 0, \quad t \geq 0 \]
\[ w(x, t) = \phi(x, t), \quad 0 \leq x \leq \Pi, \quad -1 \leq t \leq 0. \] (1.2)

If we write this equation abstractly in \( X = L^2[0, \Pi] \), it becomes
\[ du(t)/dt = Bu(t - 1), \quad t \geq 0, \quad u_0 = \phi \] (1.3)
where \( u: [-1, \infty) \to X \), \( B \) is an operator from \( X \) to \( X \) defined by
\[ By = y'' \]
\[ D(B) = \{ y \in X: y, y' \text{ are absolutely continuous, } y'' \in X, y(0) = y(\Pi) = 0 \}, \]
\( \phi \) is given in \( C([-1, 0]; X) \), and \( u_0 \in C([-1, 0]; X) \) is defined by
\[ (u_0(\theta))(x) = \phi(x, \theta), \quad -1 \leq \theta \leq 0, \quad 0 \leq x \leq \Pi. \]

If we integrate Eq. (1.3) we obtain
\[ u(t) = \phi(0) + \int_0^t Bu(s - 1) \, ds, \quad t \geq 0, \quad u_0 - \phi. \] (1.4)

Using the fact that \( B \) is closed we can solve the integrated equation (1.4) by the "method of steps" provided that we have sufficient information about \( \phi \).

Suppose \( \phi \) satisfies \( \phi(\theta) \in D(B) \) for all \( \theta \in [-1, 0] \) and \( B^\theta \phi \in C([-1, 0]; X) \). Then the solution \( u \) of (1.4) for \( 0 < t < 1 \) is given by
\[ u(t) = \phi(0) + \int_0^t B\phi(s - 1) \, ds. \]
If \( \phi \) satisfies \( \phi(\theta) \in D(B^\theta) \) for all \( \theta \in [-1, 0] \) and \( B^\theta \phi \in C([-1, 0]; X) \), then the solution of (1.4) on \( 1 \leq t \leq 2 \) is given by
\[ u(t) = \phi(1) + \int_1^t B\phi(s - 1) \, ds \]
\[ = \phi(1) + (t - 1) B\phi(0) + \int_1^t \int_0^{s-1} B^\rho \phi(\rho - 1) \, d\rho \, ds. \]

Obviously, we can repeat this process to obtain a unique solution on \([-1, n]\) provided \( \phi(\theta) \in D(B^n) \) for all \( \theta \in [-1, 0] \) and \( B^n \phi \in C([-1, 0]; X) \), and if this holds for all \( n \), then we can obtain a unique global solution on \([-1, \infty)\).

Our approach to a general theory will have an essential similarity to this example in that we will treat the problem as an abstract ordinary functional
differential equation in a Banach space. To obtain existence we will choose the initial function \( \phi \) in an appropriately chosen space of functions. To obtain stability we will study the properties of the semigroup of operators associated with the solutions in this space of initial functions. In what follows \( X \) will denote a Banach space with norm \( \| \cdot \| \), \( r \) a positive real number, and if \( u \) is a function with domain \([-r, \infty)\), then for all \( t \geq 0 \), \( u_t \) will denote the function with domain \([-r, 0]\) defined by \( u_t(\theta) = u(t + \theta) \), \( -r \leq \theta \leq 0 \).

2. \( F \) is Linear and Unbounded

In this section we treat the Eq. (1.1) for a case in which \( F \) is linear and unbounded, but with a form that allows the equation to be written in a space of initial functions in which \( F \) is bounded. We make the following assumptions:

(2.1) \(-A\) is the infinitesimal generator of a strongly continuous linear semigroup \( T(t), t \geq 0 \) in \( X \) satisfying \( \| T(t) x \| \leq M e^{\omega t} \| x \| \), \( x \in X, t \geq 0 \), where \( M \geq 1 \) and \( \omega \in \mathbb{R} \) are constants.

(2.2) \( B \) is a closed linear operator on \( X \) satisfying \( BT(t) x = T(t) B x \) for all \( x \in \bigcap_{n=1}^\infty D(B^n) \).

(2.3) \( \alpha > 0 \) is a constant and \( X_1 \) is the Banach space \( \{x \in \bigcap_{n=1}^\infty D(B^n) : \sup_{n \geq 0} \| B^n x \| \alpha^{-n} < \infty\} \) with norm \( \| x \|_{X_1} = \sup_{n \geq 0} \| B^n x \| \alpha^{-n} \).

(2.4) \( C_1 = C([-r, 0], X_1) \) will denote the Banach space of \( X_1 \)-valued functions on \([-r, 0]\) with supremum norm. \( F : C_1 \to X_1 \) is given by \( F(\phi) = \int_{-r}^0 d_\eta(\theta) \phi(\theta) \) where \( \eta : [-r, 0] \to BL(X, X) \) is of bounded variation with total variation \( \beta \), and \( B_\eta(\theta) x = \eta(\theta) B x \) for all \( x \in \bigcap_{n=1}^\infty D(B^n) \) and \(-r \leq \theta \leq 0\).

Under these assumptions we can obtain a "mild solution" to an equation of type (1.1); that is, a solution to the integrated form of the equation.

**Proposition 2.1.** Suppose (2.1)-(2.4) hold. For each \( \phi \in C_1 \), there is a unique continuous function \( u : [-r, \infty] \to C_1 \) satisfying

\[
u(t) = T(t)\phi(0) + \int_0^t T(t - s) BF(u_s) \, ds, \quad t \geq 0, \quad u_0 = \phi. \quad (2.5)\]

**Proof.** We first observe that \( X_1 \) is complete by virtue of the closedness of \( B \). Also \( B \in BL(X_1, X_1) \) with norm \( \leq \alpha \) by virtue of \( \| B x \|_{X_1} \) —
$$\sup_{n \geq 0} \| B^{n+1} \phi \| \leq \alpha \| \phi \|_1$$ for all $\phi \in X_1$. Moreover, $F \in BL(C_1; X_1)$ with norm $\leq \alpha$, since for $\phi \in C_1$, $B^n\phi \in C([-r, 0]; X)$ and

$$\| B^n\phi \|_1 = \left\| \int_{-r}^{0} d\eta(\theta) B^n\phi(\theta) \right\|_1$$

$$\leq \beta \sup_{\theta \in (-r, 0)} \| B^n\phi(\theta) \|_1 \leq \beta \sup_{\theta \in (-r, 0)} \| (B^n\phi)(\theta) \|_1 \alpha^{-n} = \beta \| \phi \|_{C_1}.$$ 

Consequently, $BF \in BL(C_1; X_1)$ with norm $\leq \alpha\beta$. Let $\delta \in \mathbb{R}$ be such that

$$\omega - \delta < 0 \quad \text{and} \quad L(\delta) \overset{\text{def}}{=} \max\{e^{-\delta r}, 1\} (\delta - \omega)^{-1} < 1.$$ 

(2.6)

Define the complete linear space

$$H = \{ z : [-r, \infty) \to X_1, z \text{ is continuous and } \sup_{t \leq -r} e^{-\delta t} \| z(t) \|_{X_1} < \infty \}$$

with norm

$$\| z \|_H = \sup_{-r \leq t} e^{-\delta t} \| z(t) \|_{X_1}.$$ 

For fixed $\phi \in C_1$, define a mapping $S$ on $H$ by

$$(S_z)(t) = T(t)\phi(0) + \int_{0}^{t} T(t-s)BF(z_s)ds, \quad -r \leq t \leq 0,$$

$$(S_z)(t) = \phi(t), \quad t > 0.$$ 

The existence of this integral and the continuity of $(S_z)(t)$ in $t$ is easily shown using (2.1) and the fact that $BF \in BL(C_1; X_1)$. We will show that $S$ is a contraction from $H$ to $H$. For $z, w \in H$, $t \geq 0$, $n \geq 0$,

$$\| B^n(Sz)(t) - B^n(Sw)(t) \| \leq \alpha^{-n}$$

$$\leq M \int_{0}^{t} e^{\omega(t-s)} \left\| \int_{-r}^{0} d\eta(\theta) B^{n+1}(z_s(\theta) - w_s(\theta)) \right\|_1 ds$$

$$\leq M\alpha^2 \int_{0}^{t} e^{\omega(t-s)} \sup_{\theta \in 0} \| B^{n+1}(z_s(\theta) - w_s(\theta)) \|_1 \alpha^{-n+1} ds$$

$$\leq M\alpha^2 \int_{0}^{t} e^{\omega(t-s)} \| z_s - w_s \|_{C_1} ds$$

$$= M\alpha e^{\omega t} \int_{0}^{t} e^{\omega(t-s)} (\sup_{\theta \in 0} \| e^{\delta \theta} e^{-\delta(s+\theta)} \| \| z(s + \theta) - w(s + \theta) \|_{X_1}) ds$$

$$\leq M\alpha e^{\omega t} \max\{e^{-\delta r}, 1\} (\delta - \omega)^{-1} \| \alpha - w \|_H.$$ 

(2.7)
Therefore

\[
\sup_{-r \leq t} e^{-\delta t} \|(Sz)(t) - (Sw)(t)\|_{x_1} \leq L(\delta) \|z - w\|_H,
\]  

which implies

\[
\|Sz - Sw\|_H \leq L(\delta) \|z - w\|_H, \quad z, w \in H.
\]  

To see that \(S\) maps \(H\) into \(H\) we must show that for \(z \in H\)

\[
\sup_{-r \leq t} e^{-\delta t} \|(Sz)(t)\|_{x_1} < \infty.
\]  

For \(w \equiv 0\) we have \((Sw)(t) = T(t)\phi(0), t \geq 0\). Thus,

\[
\sup_{-r \leq t} e^{-\delta t} \|(Sz)(t)\|_{x_1} \leq M \sup_{-r \leq t} e^{(\omega - \delta) t} \|\phi\|_{c_1} < \infty,
\]  

since \(\omega - \delta < 0\). Then (2.8) and (2.11) imply (2.10), since \(Sz = (Sz - Sw) + Sw\) with \(w \equiv 0\). Since \(L(\delta) < 1\) we have by (2.9) that \(S\) has a unique fixed point \(u \in H\). Evidently, \(u\) is the unique solution of (2.5) and so the proof is complete.

The solutions of (2.5) form a strongly continuous linear semigroup in \(C_1\) and the properties of this semigroup give stability information about the solutions.

**Proposition 2.2.** Suppose (2.1)-(2.4) hold. The solutions \(u(t) = u(t)(\phi)\) of (2.5) define a strongly continuous semigroup of linear operators \(U(t), t \geq 0\) in \(C_1\) by \(U(t) \phi = u(t)(\phi)\) satisfying

\[
\|U(t) \phi\|_{c_1} \leq P(\delta) e^{\delta t} \|\phi\|_{c_1}, \quad \phi \in C_1, \quad t \geq 0,
\]  

where \(\delta\) is any real number such that

\[
\omega - \delta < 0 \quad \text{and} \quad M\alpha\beta \max\{e^{\delta t}, 1\} (\delta - \omega)^{-1} < 1,
\]

and

\[
P(\delta) = MM_1M_2(1 - MM_2\alpha\beta(\delta - \omega)^{-1})^{-1},
\]

\[
M_1 = \max\{e^{\delta t}, 1\}, \quad M_2 = \max\{e^{\delta t}, 1\}.
\]  

**Proof.** The claimed properties of the semigroup \(U(t), t \geq 0\) are easily established except for the estimate (2.12). To see (2.12) let \(\phi \in C_1, t \geq 0, n \geq 0\), and observe that

\[
\|Bu(\phi)(t)\|_{c_1} \leq Me^{\delta t} \|\phi\|_{c_1} + M\alpha\beta \int_0^t e^{\omega(t-s)} \|u_s(\phi)\|_{c_1} ds.
\]  

\[
(2.14)
\]
as in (2.7), which implies
\[
\| u(t) \|_{X_1} \leq Me^{\omega t} \| \phi \|_{C_1} + M\alpha^2 \int_0^t e^{\omega(t-s)} \| u_s(\phi) \|_{C_1} \, ds.
\] (2.15)

Let $\delta$, $M_1$, $M_2$ be as in (2.13). Fix $t_0 > 0$ and let $w = \sup_{0 \leq s \leq t_0} e^{-\delta s} \| u_s(\phi) \|_{C_1}$.
If $0 < \rho < t_0$, then (2.15) implies
\[
e^{-\delta \rho} \| u(\phi)(\rho) \|_{X_1} \leq Me^{\omega(\delta - \omega)\rho} \| \phi \|_{C_1} + M\alpha^2 \int_0^\rho e^{\omega(\delta - \omega)s} e^{-\delta s} \| u_s(\phi) \|_{C_1} \, ds
\]
\[
\leq M \| \phi \|_{C_1} + M\alpha^2(\delta - \omega)^{-1} w.
\] (2.16)

If $r \leq \rho \leq 0$, then
\[
e^{-\delta \rho} \| u(\phi)(\rho) \|_{X_1} \leq M_1 \| \phi \|_{C_1}.
\] (2.17)

Then, (2.16) and (2.17) imply
\[
\sup_{-r \leq \rho \leq t_0} e^{-\delta \rho} \| u(\phi)(\rho) \|_{X_1} \leq MM_1 \| \phi \|_{C_1} + M\alpha^2(\delta - \omega)^{-1} w.
\] (2.18)

For $0 \leq t \leq t_0$ we have
\[
e^{-\delta t} \| u(t) \|_{C_1} = \sup_{-r \leq \theta \leq 0} e^{\delta \theta} e^{\omega(\delta - \omega)\theta} \| u(\phi)(t + \theta) \|_{X_1}
\]
\[
\leq M_2 \sup_{-r \leq \theta \leq 0} e^{\delta(\theta + \delta)} \| u(\phi)(t + \theta) \|_{X_1}
\]
\[
\leq M_2 \sup_{-r \leq \theta \leq t_0} e^{-\delta \theta} \| u(\phi)(\rho) \|_{X_1}.
\] (2.19)

Then, (2.18) and (2.19) yield that for $0 \leq t \leq t_0$
\[
e^{-\delta t} \| u(t) \|_{C_1} \leq MM_1M_2 \| \phi \|_{C_1} + MM_2\alpha^2(\delta - \omega)^{-1} w
\]
which implies
\[
w = MM_1M_2 \| \phi \|_{C_1} + MM_2\alpha^2(\delta - \omega)^{-1} w
\]
which implies (2.12).

**Remark 2.1.** If $M\alpha^2 + \omega < 0$ then $\delta$ can be chosen negative in (2.13), thus insuring that $\lim_{t \to \infty} \| U(t) \phi \|_{C_1} = 0$ exponentially for all $\phi \in C_1$. 
EQUATIONS WITH DEVIATING ARGUMENTS

EXAMPLE 2.1. Proposition 2.1 and 2.2 can be applied to the following example, which includes the example in the introduction:

\[
\begin{align*}
  w_t(x, t) &= \gamma w_{xx}(x, t) + \lambda w_{xx}(x, t - r), & 0 \leq x \leq \Pi, & t \geq 0 \\
  w(0, t) &= w(\Pi, t) = 0, & t \geq 0 \\
  w(x, t) &= \phi(x, t), & 0 \leq x \leq \Pi, & -r \leq t \leq 0 \\
\end{align*}
\]

(2.20)

where \( \gamma \geq 0, \lambda \in \mathbb{R}, \) and \( r > 0. \)

Let \( X = L^2[0, \Pi]. \) If \( \gamma > 0 \) let \( A : X \to X \) by

\[
Ay = -\gamma y'',
\]

\[
D(A) = \{ y \in X : y, y' \text{ are absolutely continuous, } y'' \in X, y(0) = y(\Pi) = 0 \}.
\]

(2.21)

and if \( \gamma = 0 \) let \( A : X \to X \) by \( Ay = 0, \ D(A) = X. \) Then, \( -A \) is the infinitesimal generator of \( T(t), \ t \geq 0 \) as in (2.1) where \( M = 1 \) and \( \omega = -\gamma. \) Let \( B : X \to X \) be as in the introduction. Define \( \eta \) as in (2.4) by \( \eta(\theta) = 0 \) if \( \theta = -r, \ \eta(\theta) = \lambda I \) if \( -r < \theta \leq 0, \) so that \( F(\phi) = \lambda \phi(-r), \ \beta = |\lambda| \) as in (2.4). Then (2.1)–(2.4) are satisfied. From Remark 2.1 the solutions of (2.20) are asymptotically stable in the norm of \( C_1 \) provided that \( \alpha < |\gamma/|\lambda| \) (obviously, the size of \( C_1 \) depends on \( \alpha). \)

3. \( F \) is nonlinear and continuous with respect to a fractional power of \( A \)

In this section we treat Eq. (1.1) for a case in which \( F \) is nonlinear and continuous with respect to a fractional power of \( A. \) When applied to examples the formulation of the problem in this fashion allows for deviating arguments in terms involving lower-order spatial derivatives.

For a discussion of analytic semigroups and fractional powers of their generators one should use [1]. We shall make the following assumptions:

(3.1) \( -A \) is the infinitesimal generator of an analytic semigroup of linear operators \( T(t), \ t \geq 0 \) in \( X \) satisfying \( \| T(t) x \| \leq M e^{\omega t} \| x \| \) for \( t \geq 0, \ x \in X, \) and where \( M \geq 1 \) and \( \omega \in \mathbb{R} \) are constants.

(3.2) For some constant \( \alpha \) such that \( 0 < \alpha < 1 \) the fractional power \( A^\alpha \) satisfies \( \| A^\alpha T(t) x \| \leq M e^{\omega t - \alpha} \| x \|, \ t \geq 0, \ x \in X. \)

(3.3) \( A^{-\alpha} \in B(X; X) \) so that \( D(A^\alpha) \) is a Banach space with norm \( \| x \|_\alpha = \| A^\alpha x \| \) for \( x \in D(A^\alpha). \)
(3.4) \( F: C([-r, 0]) \rightarrow X \) satisfies
\[ \| F(\phi) - F(\psi) \| \leq \beta \| \phi - \psi \|_{C_\alpha} \quad \text{for } \phi, \psi \in C_\alpha, \] where \( \beta \) is a constant.

(3.5) \( F(0) = 0 \) (which implies \( \| F(\phi) \| \leq \beta \| \phi \|_{C_\alpha}, \phi \in C_\alpha \)).

With these assumptions we solve the integrated version of (1.1) in the space \( C_\alpha \). Our results extend those of Zamanov [5], where local existence was established for a similar equation.

**Proposition 3.1.** Suppose (3.1)–(3.5) hold. For each \( \phi \in C_\alpha \) there is a unique continuous function \( u: [-r, \infty) \rightarrow D(A^\alpha) \) satisfying

\[ u(t) = T(t)\phi(0) + \int_0^t T(t - s) F(u(s)) \, ds, \quad t \geq 0, \quad u_0 = \phi. \quad (3.6) \]

**Proof.** Let \( \delta \in \mathbb{R} \) satisfy
\[ \omega - \delta < 0 \quad \text{and} \quad L(\delta) = M\beta \max\{e^{-\delta r}, 1\} \Gamma(1 - \alpha)(\delta - \omega)^{\alpha - 1} < 1. \quad (3.7) \]

Define the complete linear space \( H = \{ z: [-r, \infty) \rightarrow X, z \text{ is continuous, and} \sup_{-r \leq t < 0} e^{-\delta t} \| z(t) \| < \infty \} \) with norm \( \| z \|_H = \sup_{-r \leq t \leq 0} e^{-\delta t} \| z(t) \| \). Fix \( \phi \in C_\alpha \) and define a mapping \( S \) on \( H \) by

\[ (Sz)(t) = T(t)A^\alpha\phi(0) + \int_0^t A^\alpha T(t - s) F(A^{-\alpha}z_\alpha(s)) \, ds, \quad t \geq 0 \]
\[ (Sz)(t) = A^\alpha\phi(t), \quad -r \leq t \leq 0. \]

The existence of the integral follows from (3.2), (3.4), and the inequality

\[ \left\| \int_0^t A^\alpha T(t - s) F(A^{-\alpha}z_\alpha(s)) \, ds \right\| \leq \int_0^t M e^{\omega(t-s)}(t - s)^{-\alpha} \beta \| A^{-\alpha}z_\alpha(s) \|_{C_\alpha} \, ds. \]

The continuity of \( (Sz)(t) \) in \( t \) follows from

\[ \| (Sz)(t_1) - (Sz)(t_2) \| \leq \| (T(t_1) - T(t_2))(A^\alpha\phi(0)) \| \]
\[ + \left\| \int_0^{t_1} A^\alpha T(s) F(A^{-\alpha}z_{t_1-s}(\cdot)) \, ds - \int_0^{t_2} A^\alpha T(s) F(A^{-\alpha}z_{t_2-s}(\cdot)) \, ds \right\| \]
\[ \leq \| (T(t_1) - T(t_2))(A^\alpha\phi(0)) \| \]
\[ + \int_0^{t_1} M e^{\omega(s-t)} \beta \| A^{-\alpha}(z_{t_1-s}(\cdot) - z_{t_2-s}(\cdot)) \|_{C_\alpha} \, ds \]
\[ + \int_0^{t_2} M e^{\omega(s-t)} \beta \| A^{-\alpha}z_{t_2-s}(\cdot) \|_{C_\alpha} \, ds. \]
We will use the gamma function formula $\Gamma(1 - \alpha) k^{\alpha-1} = \int_0^\infty e^{-ks^{\alpha}} \, ds$, where $k > 0$ (see [4, p. 265]), to prove that $S$ is a contraction from $H$ to $H$. For $z, w \in H, t \geq 0$,

$$\| (Sz) (t) - (Sw) (t) \| \leq M \beta \int_0^t e^{\omega(t-s)}(t-s)^{-\alpha} \| A^{-\alpha}(z_s(\cdot) - w_s(\cdot))\|_{C_a} \, ds$$

$$= M \beta e^{\delta t} \int_0^t e^{(\omega-\delta)(t-s)}(t-s)^{-\alpha} \left( \sup_{-\tau \leq \theta \leq 0} e^{\delta \theta} e^{\delta(t+s+\theta)} \| z(s+\theta) - w(s+\theta)\| \right) \, ds$$

$$\leq M \beta e^{\delta t} \int_0^t e^{(\omega-\delta)(t-s)}(t-s)^{-\alpha} \max\{e^{\delta r}, 1\} \| z - w \|_H \, ds$$

$$\leq M \beta e^{\delta t} \max\{e^{\delta r}, 1\} \Gamma(1 - \alpha) (\delta - \omega)^{\alpha-1} \| z - w \|_H.$$  

Therefore,

$$e^{-\delta t} \| (Sz) (t) - (Sw) (t) \| \leq L(\delta) \| z - w \|_H \tag{3.8}$$

which implies

$$\| Sz - Sw \|_H \leq L(\delta) \| z - w \|_H, \quad z, w \in H. \tag{3.9}$$

Finally, to see that $S$ maps $H$ into $H$ we must show that

$$\sup_{-r \leq t} e^{-\delta t} \| (Sz) (t) \| < \infty \quad \text{for each } z \in H. \tag{3.10}$$

Observe that for $w \equiv 0$, $(Sz) (t) = T(t) A s\phi(0)$ for $t \geq 0$, so that

$$\sup_{-r \leq t} e^{-\delta t} \| (Sz) (t) \| \leq M \sup_{-r \leq t} e^{(\omega-\delta)t} \| \phi \|_{C_a} < \infty. \tag{3.11}$$

Then (3.8) and (3.11) imply (3.10), since $Sz = (Sz - Sw) + Sw$ with $w \equiv 0$. Since $L(\delta) < 1$, (3.9) yields that $S$ is a strict contraction of $H$ into $H$. Thus, there is a unique $z \in H$ such that $Sz = z$. Define $u(t) = A^{-\alpha}z(t)$ for $-r \leq t$. Obviously, $u(t) \in D(A^\alpha)$ for $-r \leq t$, and $u$ is continuous from $[-r, \infty)$ to $D(A^\alpha)$. Furthermore, since $A^{-\alpha} \in B(X, X)$, $u(t)$ is the unique solution of (3.6), and the proof is complete.

We can say more about the solutions to (3.6). In fact, the solutions to (3.6) give rise to a strongly continuous semigroup of nonlinear Lipschitz continuous operators on $D(A^\alpha)$.

**Proposition 3.2.** Suppose (3.1)-(3.5) hold. The solutions $u(t) \overset{\text{def}}{=} u(t)$ of (3.6) define a strongly continuous semigroup of nonlinear Lipschitz continuous operators $U(t)$, $t \geq 0$ in $C_a$ by $U(t) = u(t)$. That is, the family $U(t), t \geq 0$, satisfies $U(t) \in C_C$ for all $t \geq 0$, $U(0) = I$, $U(t \mid s) = U(t) U(s)$ for all
$s, \ t \geq 0$, and $U(t) \phi$ is continuous from $[0, \infty)$ to $C_{\alpha}$ for each fixed $\phi \in C_{\alpha}$. Moreover, for $t \geq 0$ and $\phi, \psi \in C_{\alpha}$

$$\| U(t) \phi - U(t) \psi \|_{C_{\alpha}} \leq P(\delta) \varepsilon^{\delta t} \| \phi - \psi \|_{C_{\alpha}}$$  \hspace{1cm} (3.12)

where $\delta$ is any real number such that

$$\omega - \delta < 0 \quad \text{and} \quad M\beta \max\{e^{-\delta r}, 1\} \Gamma(1 - \alpha) (\delta - \omega)^{\alpha - 1} < 1,$$

and

$$P(\delta) \overset{\text{def}}{=} MM_1M_2(1 - MM_2\beta\Gamma(1 - \alpha) (\delta - \omega)^{\alpha - 1})^{-1}, \hspace{1cm} (3.13)$$

with $M_1 = \max\{e^{\delta r}, 1\}, \quad M_2 = \max\{e^{-\delta r}, 1\}$.

**Proof.** The proof of these assertions is routine except for the inequality (3.12). To see this let $\phi, \psi \in C_{\alpha}$. From (3.6) we have that for $t \geq 0$,

$$\|w(t)\|_{C_t} \leq LMe^{\omega t} + \beta M \int_0^t e^{\omega(t-s)}(t-s)^{-\alpha} \|w_s\|_{C_{\alpha}} ds,$$  \hspace{1cm} (3.14)

where $w(t) = u(\phi)(t) - u(\psi)(t)$ and $L = \|\phi - \psi\|_{C_{\alpha}}$. Let $\delta, \ M_1, \ M_2$ be as in (3.13). Fix $t_0 > 0$ and let $W = \sup_{\tau \geq t \geq t_0} e^{-\delta \tau} \|w_s\|_{C_{\alpha}}$. If $0 \leq \tau \leq t_0$, then (3.14) implies

$$e^{-\delta \tau} \|w(\tau)\|_{C_{\alpha}} \leq LMe^{\omega \tau} + \beta M \int_0^\tau e^{\omega(t-s)}(t-s)^{-\alpha} e^{-\delta s} \|w_s\|_{C_{\alpha}} ds$$

$$\leq LM + \beta MW\tau(1 - \alpha) (\delta - \omega)^{\alpha - 1}. \hspace{1cm} (3.15)$$

If $-\tau \leq \tau \leq 0$, then

$$e^{-\delta \tau} \|w(\tau)\|_{C_{\alpha}} \leq LM_1. \hspace{1cm} (3.16)$$

Therefore, (3.15) and (3.16) imply

$$\sup_{-r \leq \tau \leq 0} e^{-\delta \tau} \|w(\tau)\|_{C_{\alpha}} \leq LMM_1 + \beta MW\tau(1 - \alpha) (\delta - \omega)^{\alpha - 1}. \hspace{1cm} (3.17)$$

For $0 \leq t \leq t_0$ we have

$$e^{-\delta t} \|w_t\|_{C_{\alpha}} = \sup_{-r \leq \theta \leq 0} e^{\delta \theta} e^{-\delta (t + \theta)} \|w(t + \theta)\|_{C_{\alpha}}$$

$$\leq M_2 \sup_{-r \leq \tau \leq 0} e^{-\delta (t + \theta)} \|w(t + \theta)\|_{C_{\alpha}} \hspace{1cm} (3.18)$$

Then, (3.17) and (3.18) yield that for $0 \leq t \leq t_0$

$$e^{-\delta t} \|w_t\|_{C_{\alpha}} \leq LMM_1M_2 + \beta MM_2W\tau(1 - \alpha) (\delta - \omega)^{\alpha - 1}$$
which implies
\[ W \leq LMM_1M_2 + \beta MM_2 \Gamma(1 - \alpha) (\delta - \omega)^{\alpha - 1}, \]
which implies (3.12).

Remark 3.1. Since \( F(0) = 0 \), obviously \( U(t) 0 = 0 \). Thus, if \( \delta \) can be chosen negative in (3.12) \( \lim_{t \to \infty} ||U(t)\phi||_{C_{\omega}} = 0 \) exponentially for all \( \phi \in C_{\omega} \).

Notice that \( \delta \) can be chosen negative provided that \( \omega < 0 \) and \( M\beta \Gamma(1 - \alpha) - (\omega)^{1-\alpha} < 0 \).

**Example 3.1.** Propositions 3.1 and 3.2 can be applied to the following example:

\[ \begin{align*}
\psi_t(x, t) &= \psi_{xx}(x, t) + F(\psi(x, t - r)), & 0 < x < \Pi, \quad t > 0 \\
\psi(0, t) &= \psi(\Pi, t) = 0, & t > 0 \\
\psi(x, t) &= \phi(x, t), & 0 < x < \Pi, \quad -r < t \leq 0
\end{align*} \]

(3.19)

where \( F: \mathbb{R} \to \mathbb{R} \) satisfies \( F(0) = 0 \) and \( F \) is Lipschitz continuous with Lipschitz constant \( \beta \).

Let \( X = L^2[0, \Pi] \) and let \( A: X \to X \) be as in (2.21). Then

\[ Ay = \sum_{n=1}^{\infty} n^2(y, z_n) z_n, \quad y \in D(A) \]

where \( \{z_n\}_{n=1}^{\infty} \) is a complete set of orthonormal eigenvectors of \( A \) with \( z_n(s) = (2/\Pi)^{1/2} \sin ns \), \( Az_n = n^2 z_n \). Also, \( A \) satisfies (3.1) with \( M \geq 1, \omega \geq -1 \), and

\[ T(t) y = \sum_{n=1}^{\infty} e^{-n^2t}(y, z_n) z_n, \quad y \in X. \]

If we choose \( \alpha = \frac{1}{2} \), then \( A \) satisfies (3.2) with \(-1 < \omega < 0, M - M_\omega = 1/(2e(1 + \omega))^{1/2} \), since

\[ A^{1/2} T(t) y = \sum_{n=1}^{\infty} ne^{-n^2t}(y, z_n) z_n, \quad y \in X, \]

\[ \|A^{1/2} T(t) y\|^2 \leq \sup_{n \geq 1} \{ n^2 e^{-2n^2t} \} \|y\|^2, \quad y \in X, \]

and \( tn^2 e^{-2n^2(t - \omega)} \leq 1/2e(1 + \omega) \) for \( n = 1, 2, \ldots \). In addition, (3.3) is satisfied for \( \alpha = \frac{1}{2} \), since

\[ A^{-1/2} y = \sum_{n=1}^{\infty} (1/n)(y, z_n) z_n, \quad y \in X. \]
Let $F: C_{1/2} \to X$ by

$$(F(\phi))(s) := F(\phi(-r)'(s)), \quad \phi \in C_{1/2}, \quad s \in [0, II].$$

To see that $F$ is well-defined observe that $y \in D(A^{1/2})$ implies

$$\sum_{n=1}^{\infty} (y, z_n) z_n' = \text{def } x \text{ converges in } X \text{ and for } 0 \leq s \leq \pi$$

$$\left| \int_{0}^{s} x(u) \, du - \sum_{n=1}^{N} (y, z_n) z_n(s) \right|$$

$$= \left| \int_{0}^{s} x(u) \, du - \sum_{n=1}^{N} (y, z_n) z_n'(u) \, du \right|$$

$$\leq \pi^{1/2} \left( \int_{0}^{\pi} \left| x(u) - \sum_{n=1}^{N} (y, z_n) z_n'(u) \right|^2 \, du \right)^{1/2}.$$

Thus, $v(s) = \int_{0}^{s} x(u) \, du$, which means $y$ is absolutely continuous, $y' = x \in X$, and $y(0) = y(\pi) = 0$. It follows that $F$ satisfies (3.4), since for $\phi, \psi \in C_{1/2}$

$$\| F(\phi) - F(\psi) \|_{2}^2$$

$$= \int_{0}^{II} | F(\phi(-r)'(s)) - F(\psi(-r)'(s)) |^2 \, ds$$

$$\leq \beta^2 \int_{0}^{II} | \phi(-r)'(s) - \psi(-r)'(s) |^2 \, ds$$

$$= \beta^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\phi(-r) - \psi(-r), z_n)(\phi(-r) - \psi(-r), z_m)(z_n', z_m')$$

$$= \beta^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\phi(-r) - \psi(-r), z_n)(\phi(-r) - \psi(-r), z_m)(-z_n', z_m')$$

$$= \beta^2 \sum_{n=1}^{\infty} (\phi(-r) - \psi(-r), z_m)^2 n^2$$

$$- \beta^2 \| A^{1/2}(\phi(-r) - \psi(-r)) \|_{2}^2 \leq \beta^2 \| \phi - \psi \|_{C_{1/2}}^2.$$

Therefore, (3.4) is satisfied and $F(0) = 0$ implies (3.5) is satisfied. From Remark 3.1 we see that the solutions of (3.19) are asymptotically stable in the norm of $C_{1/2}$ provided that $-1 < \omega < 0$, $M = \{ \max 1, 1/2e(1 + \omega)^{1/2} \}$, and $M \beta(1/2) < (-\omega)^{1/2}$. 
REFERENCES