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An ancient Chinese mathematical algorithm and its application to nonlinear oscillators

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ABSTRACT

An ancient Chinese mathematical method is briefly introduced, and its application to nonlinear oscillators is elucidated where He's amplitude–frequency formulation is outlined. Three examples are given to show the extremely simple solution procedure and remarkably accurate solutions.

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1. Introduction

An ancient Chinese mathematics book *The Nine Chapters on the Mathematical Art* (2nd century BC) introduced several ancient algorithms [1–7]. In this paper we will introduce one of those methods, the *Ying Buzu Shu*, which is the oldest method for solving algebraic equations [1,2].

Consider an algebraic equation

$$f(x) = 0. \tag{1}$$

The basic idea of the *Ying Buzu Shu* is to guess two initial solutions, x_1 and x_2, which lead to the remainders $f(x_1)$ and $f(x_2)$, respectively, and the approximate solution is updated as [1]

$$x = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}. \tag{2}$$

This solution procedure was further developed by a Chinese mathematician, Ji-Huan He, to solve nonlinear differential equations [8].

2. He's amplitude–frequency formulation

According to the above ancient Chinese mathematical method, He's amplitude–frequency formulation first appeared in Ref. [8]. To illustrate the basic solution procedure, we consider the following general nonlinear oscillator

$$u'' + f(u)u = 0, \qquad u(0) = A, \qquad u'(0) = 0, \tag{3}$$

where $f(u) > 0$ is a known function of u.

According to the ancient Chinese mathematical method, we should choose two trial solutions. We can begin with [8]

$$u_1 = A\cos\omega_1 t \tag{4}$$

$$u_2 = A\cos\omega_2 t \tag{5}$$

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where ω_1 and ω_2 can be freely chosen, generally we choose $\omega_1 = 1$ and $\omega_2 = \omega$, where ω is the frequency of the nonlinear oscillator. Substituting u_1 and u_2 into Eq. (3) results in reminders $R_1(t)$ and $R_2(t)$, respectively. Ji-Huan He suggested the following amplitude–frequency formulation in Ref. [8]

$$\omega^2 = \frac{\omega_1^2 R_2(0) - \omega_2^2 R_1(0)}{R_2(0) - R_1(0)}. \tag{6}$$

Geng and Cai suggested a modification, which is [9]

$$\omega^2 = \frac{\omega_1^2 R_2\left(\frac{T_2}{N}\right) - \omega_2^2 R_1\left(\frac{T_1}{N}\right)}{R_2\left(\frac{T_2}{N}\right) - R_1\left(\frac{T_1}{N}\right)} \tag{7}$$

where T_1 and T_2 are the periods of the trial solutions, u_1 and u_2, respectively, N is generally chosen as $N = 12$.

In 2008, Ji-Huan He improved the formulation, which reads [10,11]

$$\omega^2 = \frac{\omega_1^2 \widetilde{R_2} - \omega_2^2 \widetilde{R_1}}{\widetilde{R_2} - \widetilde{R_1}} \tag{8}$$

where

$$\widetilde{R_1} = \frac{4}{T_1}\int_0^{T_1/4} R_1(t)\cos t\,\mathrm{d}t, \tag{9}$$

$$\widetilde{R_2} = \frac{4}{T_2}\int_0^{T_2/4} R_2(t)\cos \omega t\,\mathrm{d}t. \tag{10}$$

Due to its simplicity, many authors applied the amplitude–frequency formulation to various nonlinear oscillators [12–18] with great success.

3. Examples

Example 1. We consider a problem of some importance in plasma physics concerning an electron beam injected into a plasma tube where the magnetic field is cylindrical and increases towards the axis in inverse proportion to the radius. The governing equation reads [19,20]:

$$u'' + \frac{1}{u} = 0, \qquad u(0) = A, \qquad u'(0) = 0. \tag{11}$$

We re-write Eq. (11) in the form

$$R(t) = u''u + 1 = 0, \qquad u(0) = A, \qquad u'(0) = 0. \tag{12}$$

According to He's amplitude–frequency formulation, we choose two trial functions $u_1(t) = A\cos t$ and $u_2(t) = A\cos\omega t$, where ω is assumed to be the frequency of the nonlinear oscillator. Using Eqs. (9) and (10), we have

$$\widetilde{R_1} = \frac{4}{T_1}\int_0^{T_1/4} R_1(t)\cos t\,\mathrm{d}t = -\frac{4A^2}{3\pi} + \frac{2}{\pi}, \tag{13}$$

and

$$\widetilde{R_2} = \frac{4}{T_2}\int_0^{T_1/4} R_2(t)\cos \omega t\,\mathrm{d}t = -\frac{4\omega^2 A^2}{3\pi} + \frac{2}{\pi}. \tag{14}$$

Applying He's amplitude–frequency formulation, Eq. (8), yields the following result

$$\omega^2 = \frac{\omega_1^2 \widetilde{R_2} - \omega_2^2 \widetilde{R_1}}{\widetilde{R_2} - \widetilde{R_1}} = \frac{3}{2A^2}. \tag{15}$$

The exact frequency reads [19]

$$\omega_{\mathrm{ex}}(A) = \frac{2\pi}{T_{\mathrm{ex}}(A)} = \frac{2\pi}{2\sqrt{2\pi}A} = \frac{\sqrt{2\pi}}{2A} = \frac{1.2533141}{A}. \tag{16}$$

The 2.28% accuracy is remarkably good as illustrated in Fig. 1.

$A = 1, \omega_{\mathrm{app}} = 1.224744871 \qquad A = 10, \omega_{\mathrm{app}} = 0.1224744871.$

Example 2. We consider a nonlinear oscillator of the form [20,21]:

$$u'' + \frac{u^3}{1+u^2} = 0, \qquad u(0) = A, \qquad u'(0) = 0. \tag{17}$$

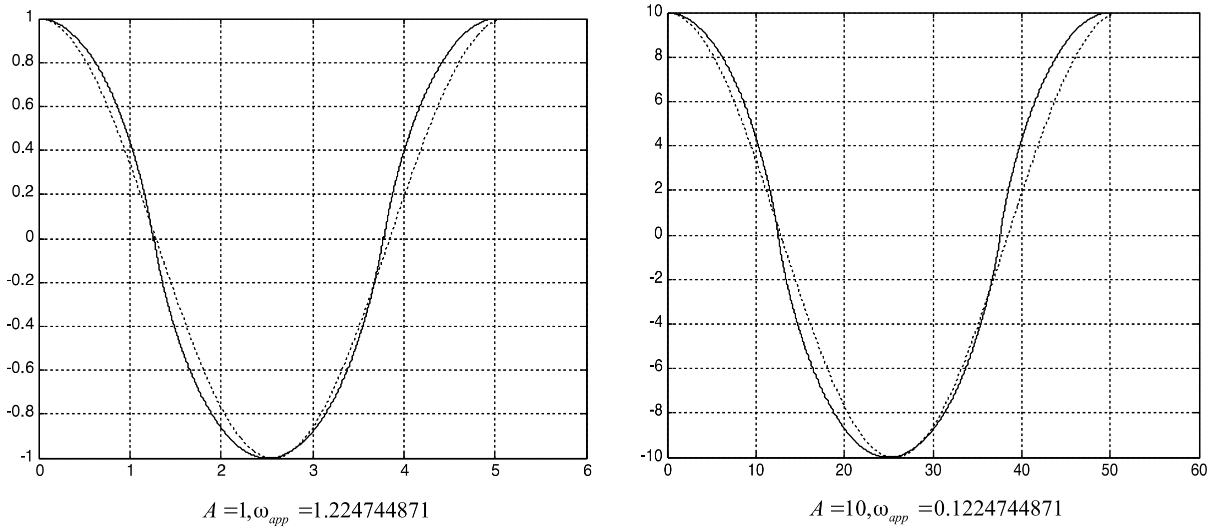

Fig. 1. Comparison of the approximate solution with the exact solution of Eq. (11) (dashed line: the approximated solution and solid line: the exact solution).

Similarly we choose $u_1(t) = A\cos t$ and $u_2(t) = A\cos\omega t$ as trial solutions, this leads to the following residuals

$$R_1(t) = -A\cos t + \frac{A^3\cos^3 t}{1 + A^2\cos^2 t}, \tag{18}$$

and

$$R_2(t) = -A\omega^2\cos\omega t + \frac{A^3\cos^3\omega t}{1 + A^2\cos^2\omega t}. \tag{19}$$

In view of Eq. (7), we have

$$\omega^2 = \frac{\omega_1^2 R_2\left(\frac{T_2}{12}\right) - \omega_2^2 R_1\left(\frac{T_1}{12}\right)}{R_2\left(\frac{T_2}{12}\right) - R_1\left(\frac{T_1}{12}\right)} = \frac{\frac{3}{4}A^2}{1 + \frac{3}{4}A^2}. \tag{20}$$

The result obtained is the same as those in Refs. [21,22]. Fig. 2 shows excellent agreement between the approximate solution and the exact one.

$$A = 10,\ \omega_{\text{app}} = 0.9934 \qquad A = 100,\ \omega_{\text{app}} = 0.9999.$$

Example 3. We consider the nonlinear oscillator with high nonlinearities [23]:

$$u'' + \alpha u + \gamma u^{2n+1} = 0, \quad \alpha \geq 0,\ \gamma > 0,\ n = 1, 2, 3, \ldots, \qquad u(0) = A, \qquad u'(0) = 0. \tag{21}$$

Proceeding in the same way as Example 2, we have

$$\omega^2 = \frac{\omega_1^2 R_2\left(\frac{T_2}{12}\right) - \omega_2^2 R_1\left(\frac{T_1}{12}\right)}{R_2\left(\frac{T_2}{12}\right) - R_1\left(\frac{T_1}{12}\right)} = \alpha + \gamma A^{2n}\left(\frac{\sqrt{3}}{2}\right)^{2n} \tag{22}$$

where

$$R_1(t) = -A\cos t + A\alpha\cos t + \gamma A^{2n+1}\cos^{2n+1} t \tag{23}$$

and

$$R_2(t) = -A\omega^2\cos\omega t + A\alpha\cos\omega t + \gamma A^{2n+1}\cos^{2n+1}\omega t. \tag{24}$$

The exact frequency reads [12]

$$\omega_{\text{ex}} = \frac{2\pi}{4\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{\alpha + \frac{\gamma}{n+1}A^{2n}\left(1 + \sin^2\theta + \sin^4\theta + \cdots + \sin^{2n}\theta\right)}}}. \tag{25}$$

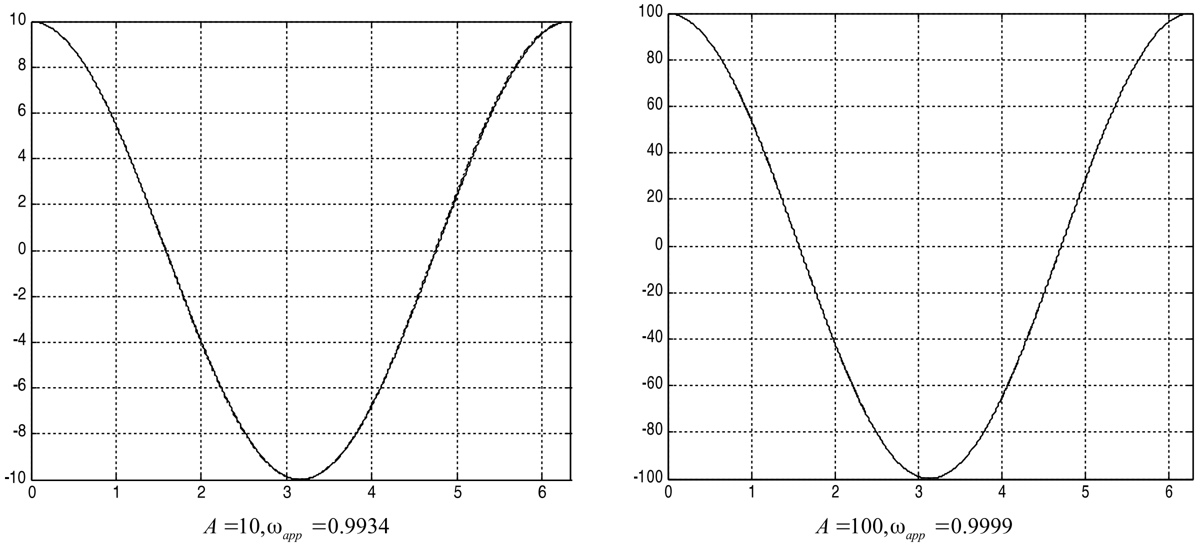

Fig. 2. Comparison of the approximate solution with the exact solution of Eq. (17) (dashed line: the approximated solution and solid line: the exact solution).

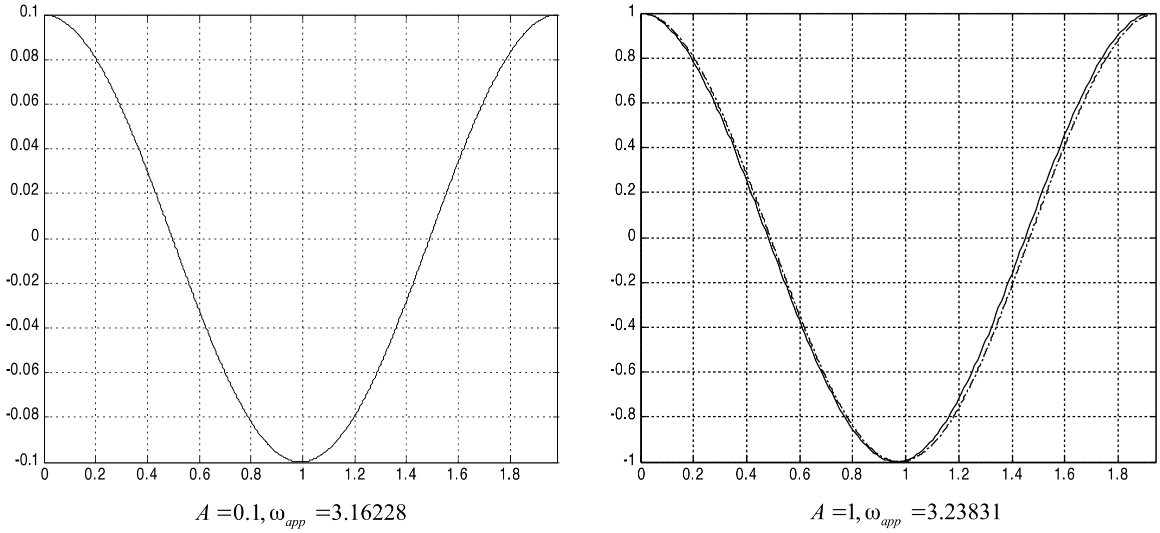

Fig. 3. Comparison of the approximate solution with the exact solution of Eq. (26) (dashed line: the approximated solution and solid line: the exact solution).

In order to verify the accuracy of the obtained frequency, we consider the following special case.

$$u'' + 10u + u^9 = 0, \qquad u(0) = A, \qquad u'(0) = 0. \tag{26}$$

The approximate frequency agrees very well with the exact one as shown in Fig. 3.

$$A = 0.1, \omega_{\text{app}} = 3.16228 \qquad A = 1, \omega_{\text{app}} = 3.23831.$$

4. Conclusion

He's amplitude–frequency formulation was derived using the basic solution procedure of an ancient Chinese mathematical method, the formulation was proved to be very effective to various nonlinear oscillators. It can be used in engineering to determine the period of a nonlinear oscillator using only pencil and hand.

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