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Positive and elementary stable nonstandard numerical methods with applications to predator–prey models

Dobromir T. Dimitrov, Hristo V. Kojouharov*,[1]

Department of Mathematics, University of Texas at Arlington, P.O. Box 19408, Arlington, TX 76019-0408, USA

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Abstract

Positive and elementary stable nonstandard (PESN) finite-difference methods, having the same qualitative features as the corresponding continuous predator–prey models, are formulated and analyzed. The proposed numerical techniques are based on a nonlocal modeling of the growth-rate function and a nonstandard discretization of the time derivative. This approach leads to significant qualitative improvements in the behavior of the numerical solution. Applications of the PESN methods to a specific Rosenzweig–MacArthur predator–prey model are also presented.

Keywords: Finite-difference; Nonstandard; Positive; Elementary stable; Predator–prey

1. Introduction

The interspecies interaction is among the most intensively explored fields of biology. The increasing amount of realistic mathematical models in that area helps in understanding the population dynamics of analyzed biological systems. Mathematical models of predator–prey systems, characterized by decreasing growth rate of one of the interacting populations and increasing growth rate of the other, consist of systems of differential equations. In most of the interactions modeled all rates of change are assumed to be time independent, which makes the corresponding systems autonomous. The positivity of the sizes of both interacting populations requires the mathematical models to preserve the invariance of the first quadrant.

* Corresponding author. Tel.: +1 8172725763; fax: +1 8172725802.

E-mail addresses: dobri@uta.edu (D.T. Dimitrov), hristo@uta.edu (H.V. Kojouharov).

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Numerical methods that approximate predator–prey systems are expected to be consistent with the original differential system, to be zero-stable and convergent. Nonstandard finite-difference rules for designing methods that preserve the physical properties, especially the stability properties of equilibria, of the approximated system have been developed in [14,16]. Many researchers have worked on developing nonstandard schemes that deal with these issues, including [2,12,8] among others. All of them have designed elementary stable nonstandard (ESN) methods for different classes of dynamical systems. An important number of contributions have also been made to guarantee the positivity of the numerical solution of those nonstandard numerical methods [3,9]. Marcus and Mickens [13] have constructed positive nonstandard methods that suppress numerically induced chaos for a system of three ordinary differential equations that models photoconductivity of semiconductors. Piyanwong et al. [17] and Jansen and Twizell [11] have designed positive and unconditionally stable schemes for the SIR and SEIR models, respectively. Many researchers have also worked on positive and bounded nonstandard finite-difference schemes for partial differential equations [1,15]. In this paper, we develop a new class of positive and elementary stable nonstandard (PESN) finite-difference methods for a general class of Rosenzweig–MacArthur predator–prey systems with a logistic intrinsic growth of the prey population.

The paper is organized as follows. In Section 2, we provide some definitions and preliminary results as well as a mathematical analysis of the general Rosenzweig–MacArthur predator–prey systems with a logistic intrinsic growth of the prey population. In Section 3, we design the PESN numerical methods for the considered class of predator–prey systems. In the last two sections we illustrate our results by numerical examples and outline some future research directions.

2. Definitions and preliminaries

The general Rosenzweig–MacArthur predator–prey model [4, p. 182] with a logistic intrinsic growth of the prey population has the following form:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= bx(1-x) - ag(x)xy, \quad x(t_0) = x_0 \geqslant 0, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= g(x)xy - \mathrm{d}y, \quad y(t_0) = y_0 \geqslant 0, \end{aligned} \tag{1}$$

where x and y represent the prey and predator population sizes, respectively, $b > 0$ represents the intrinsic growth rate of the prey, $a > 0$ stands for the capturing rate and $d > 0$ is the predator death rate. In (1) it is reasonable to assume

$$g(x) \geqslant 0, \quad g'(x) \leqslant 0, \quad [xg(x)]' \geqslant 0 \tag{2}$$

and that $xg(x)$ is bounded as $x \to \infty$. These assumptions express the idea that as prey population increases the consumption rate of prey per predator increases but that the fraction of the total prey population consumed per predator decreases [4].

The equilibrium points of System (1) are defined as the solutions of the system:

$$\begin{aligned} &bx(1-x) - ag(x)xy = 0, \\ &g(x)xy - \mathrm{d}y = 0. \end{aligned} \tag{3}$$

Depending on the values of the parameters and the functional response $xg(x)$ System (1) has the following equilibria:

(1) $E_0 = (0, 0)$;
(2) $E_1 = (1, 0)$ and
(3) $E^* = (x^*, y^*)$, where x^* is the solution of $xg(x) = d$ and $y^* = \frac{bx^*(1-x^*)}{ad}$. The equilibrium E^* exists if and only if $g(1) > d$.

According to the stability theory for general nonlinear systems [6], the following statements about the stability of the equilibria of System (1) are true:

(1) The equilibrium E_0 is always linearly unstable;
(2) The equilibrium E_1 is linearly stable if $g(1) < d$ and linearly unstable if $g(1) > d$;
(3) The equilibrium E^* is linearly stable if $b + ay^*g'(x^*) > 0$ and linearly unstable if $b + ay^*g'(x^*) < 0$;

A general one-step numerical scheme with a step size h, that approximates the solution $z(t) = (x(t), y(t))^{\mathrm{T}}$ of the system:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = F(z); \quad z(t_0) = z_0 \geqslant 0, \tag{4}$$

at $t_k = t_0 + kh$ can be written in the form

$$\mathscr{D}_h(z_k) = \mathscr{F}_h(F; z_k), \tag{5}$$

where

$$\mathscr{D}_h(z_k) \approx \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right)^{\mathrm{T}},$$

$\mathscr{F}_h(F; z_k)$ approximates the right-hand side of System (4) and $z_k \approx z(t_k)$.

Throughout this article, we assume that System (4) has a finite number of hyperbolic equilibria, i.e., $Re(\lambda) \neq 0$, for $\lambda \in \Omega$, where $\Omega = \bigcup_{z^* \in \Gamma} \sigma(J(z^*))$ and Γ represents the set of all equilibria of System (4).

Definition 1. Let $\bar{E} = (\bar{x}^*, \bar{y}^*)$ be a fixed point of Scheme (5) and the equation of the perturbed solution $z_k = \bar{E} + \varepsilon_k$, where $\varepsilon_k = (\delta_k, \eta_k)$ is small, be linearly approximated by

$$\mathscr{D}_h \varepsilon_k = J_h \varepsilon_k. \tag{6}$$

Here the right-hand side of Eq. (6) represents the linear term in ε_k of the Taylor expansion of $\mathscr{F}_h(F; \bar{E} + \varepsilon_k)$ around $\bar{E}$. The fixed point $\bar{E}$ is called stable if $\|\varepsilon_k\| \to 0$ as $k \to \infty$, and unstable otherwise, where ε_k is the solution of Eq. (6).

We introduce the next two definitions based on definitions given in [2].

Definition 2. The finite difference method (5) is called elementary stable, if, for any value of the step size h, its only fixed points $\bar{E}$ are those of the differential system (4), the linear stability properties of each $\bar{E}$ being the same for both the differential system and the discrete method.

Definition 3. The one-step method (5) is called a nonstandard finite-difference method if at least one of the following conditions is satisfied:

- $\mathscr{D}_h(z_k) = \frac{z_{k+1}-z_k}{\varphi(h)}$, where $\varphi(h) = h + \mathcal{O}(h^2)$ is a nonnegative function;
- $\mathscr{F}_h(F; z_k) = f(z_k, z_{k+1}, h)$, where $f(z_k, z_{k+1}, h)$ is a nonlocal approximation of the right-hand side of System (4).

Elementary stable nonstandard (ESN) methods for System (4) can be designed using the following theorem [8]:

Theorem 1. *Let ϕ be a real-valued function on $\mathbb{R}$ that satisfies the property*:

$$\phi(h) = h + O(h^2) \text{ and } 0 < \phi(h) < 1 \text{ for all } h > 0. \tag{7}$$

Let $q = \max_{\Omega}(|\lambda|^2/2|Re(\lambda)|)$, where $\lambda \in \Omega$ and Ω represents the union of the spectrums of the Jacobians at the equilibria of System (4). *Then the following numerical schemes for solving System* (4) *represent ESN methods*:

(a) *explicit ESN Euler method given by*

$$\frac{z_{k+1} - z_k}{\phi(hq)/q} = F(z_k), \tag{8}$$

(b) *implicit ESN Euler method given by*

$$\frac{z_{k+1} - z_k}{\phi(hq)/q} = F(z_{k+1}), \tag{9}$$

(c) *second-order ESN Runge–Kutta method given by*

$$\frac{z_{k+1} - z_k}{\phi(hq)/q} = \frac{F(z_k) + F(z_k + (\phi(hq)/q)F(z_k))}{2}. \tag{10}$$

Remark 1. There exists a variety of functions ϕ that satisfy condition (7), e.g., $\phi(h) = 1 - \mathrm{e}^{-h}$, i.e., $\varphi(h) = \phi(hq)/q = (1 - \mathrm{e}^{-hq})/q$.

3. Main results

ESN methods for solving Rosenzweig–MacArthur predator–prey systems with logistic intrinsic growth of the prey population can be design directly, based on the results of Theorem 1. However, to guarantee positivity of the discrete solution it is necessary not only to replace the traditional denominator function h with a new function $\varphi(h)$, but also to approximate the right-hand side of System (4) nonlocally. When done in a proper way, the resulting numerical methods are positive and elementary stable nonstandard (PESN) methods.

Let us first consider the case when the interior equilibrium E^* of System (1) does not exist:

Theorem 2. *Assume the function $g(x)$ satisfies* (2) *and $g(1) < d$. Then the following scheme for solving System* (1) *represents a PESN method*:

$$\frac{x_{k+1}-x_k}{h} = bx_k - bx_k x_{k+1} - ag(x_k)x_{k+1}y_k,$$
$$\frac{y_{k+1}-y_k}{h} = g(x_k)x_k y_k - \mathrm{d}y_{k+1}. \tag{11}$$

In the case when the interior equilibrium E^* of System (1) does exist the following theorem holds:

Theorem 3. *Let ϕ be a real-valued function on $\mathbb{R}$ that satisfies property* (7). *Assume the function $g(x)$ satisfies* (2), *$g(1) < d$ and $q > \frac{bd|1-2x^*|}{|b+ay^*g'(x^*)|x^*}$, where (x^*, y^*) is the interior equilibrium of System* (1). *Then the following scheme for solving System* (1) *represents a PESN method*:

$$\frac{x_{k+1}-x_k}{\phi(hq)/q} = bx_k - bx_k x_{k+1} - ag(x_k)x_{k+1}y_k,$$
$$\frac{y_{k+1}-y_k}{\phi(hq)/q} = g(x_k)x_k y_k - \mathrm{d}y_{k+1}. \tag{12}$$

Remark 2. To guarantee positivity in the PESN methods (11) and (12) we keep the positive terms of the right-hand side of System (1) at the old-time level and discretize the negative terms by a non-local expression, linear at the new-time level. Similar idea has been applied in the discretization of production–destruction systems in [5].

4. Proofs of the main results

Let us first consider the following general two-dimensional system of difference equations:

$$x_{k+1} = F(x_k, y_k),$$
$$y_{k+1} = G(x_k, y_k). \tag{13}$$

If $\bar{E}$ is a fixed point of System (13) then the equation for the perturbed solution ε_k, around $\bar{E}$, has the form

$$\varepsilon_{k+1} = J(\bar{E})\varepsilon_k,$$

where $J(\bar{E})$ denotes the Jacobian

$$\begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix} \text{ at } \bar{E}.$$

The solution $\varepsilon_k \to 0$ when $k \to \infty$ if and only if all eigenvalues of $J(\bar{E})$ are less than one in absolute values.

Proof (*Theorem 2*). Scheme (11) can be written in the following explicit form:

$$x_{k+1} = \frac{(1+hb)x_k}{1+hbx_k+hag(x_k)y_k},$$
$$y_{k+1} = \frac{(1+hg(x_k)x_k)y_k}{1+hd}. \tag{14}$$

Since the constants a, b, d and the function g are all positive then System (14) is unconditionally positive and its fixed points are exactly the equilibria E_0 and E_1 of System (1). Therefore, Eq. (6) for the perturbed solution of Scheme (14) around an equilibrium $\bar{E} = (\bar{x}, \bar{y})$ has the form

$$\varepsilon_{k+1} = J(\bar{E})\varepsilon_k,$$

where

$$J(\bar{E}) = \begin{pmatrix} \dfrac{(1+hb)(1+ha\bar{y}(g(\bar{x})-g'(\bar{x})\bar{x})}{(1+hb\bar{x}+ha\bar{y}g(\bar{x}))^2} & -\dfrac{(1+hb)ha\bar{x}g(\bar{x})}{(1+hb\bar{x}+ha\bar{y}g(\bar{x}))^2} \\ \dfrac{h\bar{y}(g(\bar{x})+\bar{x}g'(\bar{x}))}{1+hd} & \dfrac{1+h\bar{x}g(\bar{x})}{1+hd} \end{pmatrix}.$$

The Jacobian

$$J(E_0) = \begin{pmatrix} 1+hb & 0 \\ 0 & \dfrac{1}{1+hd} \end{pmatrix}$$

has eigenvalues $\lambda_1 = 1 + hb$ and $\lambda_2 = 1/(1 + hd)$. Since $|\lambda_1| > 1$ for $h > 0$ then the unstable equilibrium E_0 is also an unstable fixed point of Scheme 11. The Jacobian

$$J(E_1) = \begin{pmatrix} \dfrac{1}{1+hb} & -\dfrac{hag(1)}{1+hb} \\ 0 & \dfrac{1+hg(1)}{1+hd} \end{pmatrix}$$

has eigenvalues $\lambda_1 = 1/(1+hb)$ and $\lambda_2 = 1+hg(1)/(1+hd)$. If the equilibrium E_1 is stable then $g(1) < d$ and therefore $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Thus E_1 is a stable fixed point of Scheme (11). If the equilibrium E_1 is unstable then $g(1) > d$ and $|\lambda_2| > 1$. Therefore E_1 is an unstable fixed point of Scheme (11). Since E_0 and E_1 are the only equilibria of System (1), then the nonstandard Scheme (11) is elementary stable and represents a PESN method. □

Proof (*Theorem 3*). Let us denote $h_1 = \varphi(h) = \phi(hq)/q$. Since $0 < h_1 < \frac{1}{q}$ then

$$h_1 < \frac{|b + ay^* g'(x^*)|x^*}{bd|1 - 2x^*|}. \tag{15}$$

The explicit expression of the nonstandard Scheme (12) has the form

$$x_{k+1} = \frac{(1+h_1 b)x_k}{1+h_1 b x_k + h_1 a g(x_k) y_k},$$
$$y_{k+1} = \frac{(1+h_1 g(x_k) x_k) y_k}{1+h_1 d}. \tag{16}$$

Since the constants a, b, d and the function g are all positive then System (16) is unconditionally positive and its fixed points are exactly the equilibria E_0, E_1 and E^* of System (1). Therefore, Eq. (6) for the perturbed solution of Scheme (14) around an equilibrium $\bar{E} = (\bar{x}, \bar{y})$ has the form

$$\varepsilon_{k+1} = J(\bar{E})\varepsilon_k,$$

where

$$J(\bar{E}) = \begin{pmatrix} \dfrac{(1+h_1 b)(1+h_1 a\bar{y}(g(\bar{x}) - g'(\bar{x})\bar{x})}{(1+h_1 b\bar{x} + h_1 a\bar{y}g(\bar{x}))^2} & -\dfrac{(1+h_1 b)h_1 a\bar{x}g(\bar{x})}{(1+h_1 b\bar{x} + h_1 a\bar{y}g(\bar{x}))^2} \\ \dfrac{h_1\bar{y}(g(\bar{x}) + \bar{x}g'(\bar{x}))}{1+h_1 d} & \dfrac{1+h_1\bar{x}g(\bar{x})}{1+h_1 d} \end{pmatrix}.$$

The fact that Scheme (12) preserves the stability of E_0 and E_1 can be established similarly to the proof of Theorem 2.

The eigenvalues λ_1 and λ_2 of the Jacobian

$$J(E^*) = \begin{pmatrix} 1 - \dfrac{h_1 x^*(b + ay^* g'(x^*)}{1+h_1 b} & -\dfrac{h_1 ad}{1+h_1 b} \\ \dfrac{hy^*(g(x^*) + x^* g'(x^*))}{1+hd} & 1 \end{pmatrix}$$

are roots of the quadratic equation

$$\lambda^2 - (C+1)\lambda + C + AB = 0, \tag{17}$$

where

$$A = \frac{h_1 ad}{1+h_1 b}, \quad B = \frac{h_1 y^*(g(x^*) + x^* g'(x^*))}{1+h_1 d} \quad \text{and} \quad C = 1 - \frac{h_1 x^*(b + ay^* g'(x^*))}{1+h_1 b}.$$

Therefore, the stability of E^* as a fixed point of Scheme (12) depends of the absolute values of λ_1 and λ_2. The following fact is true for the roots of a general quadratic equation:

Fact. *For the quadratic equation* $\lambda^2 + \alpha\lambda + \beta = 0$ *both roots satisfy* $|\lambda_i| < 1$, $i = 1, 2$ *if and only if the following conditions are satisfied* [4, p. 82]:

- $1 + \alpha + \beta > 0$;
- $1 - \alpha + \beta > 0$; *and*
- $\beta < 1$.

Applying the above result on Eq. (17) we obtain that E^* is a stable fixed point of Scheme (12) if and only if the following conditions are true:

(a) $AB > 0$;
(b) $2 + 2C + AB > 0$; and
(c) $AB < 1 - C$.

E^* is an unstable fixed point if at least one of the above conditions fails. Since

$$A = \frac{h_1 a d}{1 + h_1 b} > 0 \quad \text{and} \quad B = \frac{h y^*(g(x^*) + [xg(x)]'|_{x=x^*})}{1 + hd} > 0$$

then the condition (a) is always true. Calculations yield that

$$C = \frac{1 + h_1 b(1 - x^*) - h_1 a y^* g'(x^*) x^*}{1 + h_1 b},$$

which is positive, because $0 < x^* < 1$ and $g'(x^*) < 0$. Therefore the second condition (b) is always true, as well. The third condition, $AB < 1 - C$, is equivalent to the following inequality:

$$h_1 b d(1 - 2x^*) < x^*(b + a y^* g'(x^*)). \tag{18}$$

Assume that $E^* = (x^*, y^*)$ is a stable equilibrium of System (1). Therefore $b + ay^* g'(x^*) > 0$. If $x^* \geqslant \frac{1}{2}$, Inequality (18) is satisfied because the left-hand side is nonpositive, while the right-hand side is positive. If $x^* < \frac{1}{2}$, Inequality (18) is satisfied because of Inequality (15). Therefore E^* is a stable fixed point of Scheme (12). If $E^* = (x^*, y^*)$ is an unstable equilibrium of System (1) then $b + ay^* g'(x^*) < 0$. In the case when $x^* \leqslant \frac{1}{2}$, Inequality (18) is not satisfied because the left-hand side is nonnegative, while the right-hand side is negative. If $x^* > \frac{1}{2}$, Inequality (18) is not satisfied because of Inequality (15). Therefore E^* is an unstable fixed point of Scheme (12). $\square$

5. Numerical examples

To illustrate the advantages of the designed PESN finite-difference methods, we consider the Rosenzweig–MacArthur predator–prey system (1) with a Holling-type II predator functional response of the form $xg(x) = x/(c + x)$ [10], which satisfies (2). System (1) becomes

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= bx(1 - x) - \frac{axy}{c + x}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{xy}{c + x} - \mathrm{d}y. \end{aligned} \tag{19}$$

We first examine System (19) in the case when the constants are $a = 2.0$, $b = 1.0$, $c = 0.5$ and $d = 6.0$, i.e., $g(1) = \frac{2}{3} < d$. Mathematical analysis of the system shows that there exist two equilibria $E_0 = (0, 0)$ and $E_1 = (1, 0)$, with the equilibrium $(1, 0)$ being globally stable in the interior of the first quadrant. The eigenvalues of $J(0, 0)$ are given by $\lambda_1 = 1$ and $\lambda_2 = -6.0$ and the eigenvalues of $J(1, 0)$ are given by $\lambda_3 = -1$ and $\lambda_4 = -\frac{16}{3}$. Comparison of numerical approximations of the solution of System (19) with the PESN method (11), the explicit ESN Euler method (8) using $\varphi(h) = \phi(hq)/q = (1 - \mathrm{e}^{-hq})/q$ with

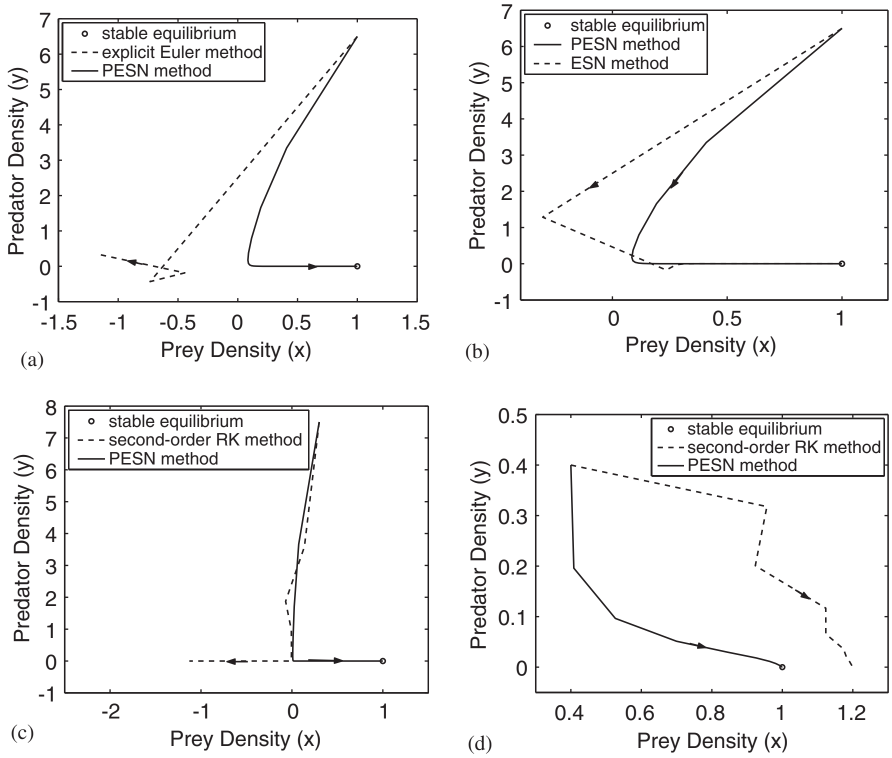

Fig. 1. Numerical approximations of the solutions of System (19) supporting the results of Theorem 2: (a) $h = 0.2$, $x_0 = 1$, $y_0 = 6.5$; (b) $h = 0.2$, $x_0 = 1$, $y_0 = 6.5$; (c) $h = 0.2$, $x_0 = 0.3$, $y_0 = 7.5$; (d) $h = 2.5$, $x_0 = 0.4$, $y_0 = 0.4$.

$q = 3.1$ and the explicit Euler method supports the results of Theorem 2. The nonstandard (ESN and PESN) methods preserve the stability of the equilibrium $(1, 0)$, while the approximation obtained by the standard method diverges (Fig. 1(a)). However, a drawback of the ESN method is that it is not positive (Fig. 1(b)). Similar behavior is observed when the standard second-order Runge–Kutta method is used to numerically solve System (19) (see Fig. 1(c)). In some cases, for relatively large step-size $h = 2.5$, the Runge–Kutta numerical solution approaches an artificially created nonexisting equilibrium (Fig. 1(d)).

Next, we examine System (19) in the case when the constants are $a = 2.0$, $b = 1.0$, $c = 1.0$ and $d = 0.2$, i.e., $g(1) = \frac{1}{2} > d$. Mathematical analysis of the system shows that there exist three equilibria $E_0 = (0, 0)$, $E_1 = (1, 0)$ and $E^* = (\frac{1}{4}, \frac{15}{32})$, with the interior equilibrium E^* being globally stable in the interior of the first quadrant. Comparison of numerical approximations of the solution of System (19) with the PESN method (11) using $\varphi(h) = \phi(hq)/q = (1 - \mathrm{e}^{-hq})/q$ with $q = 1.2$, the Patankar Euler scheme [5, p. 17], the modified Patankar Euler scheme [5, p. 18], and the second-order Runge–Kutta method supports the results of Theorem 3. The nonstandard (PESN) method preserves the stability of the equilibrium E^* (Fig. 2(b),(d),(f)), while the approximations obtained by the other three numerical methods diverge (Fig. 2(a),(c),(e)). Moreover, the modified Patankar Euler scheme (Fig. 2(c)) and the second-order Runge–Kutta method (Fig. 2(e)) produce nonpositive approximations.

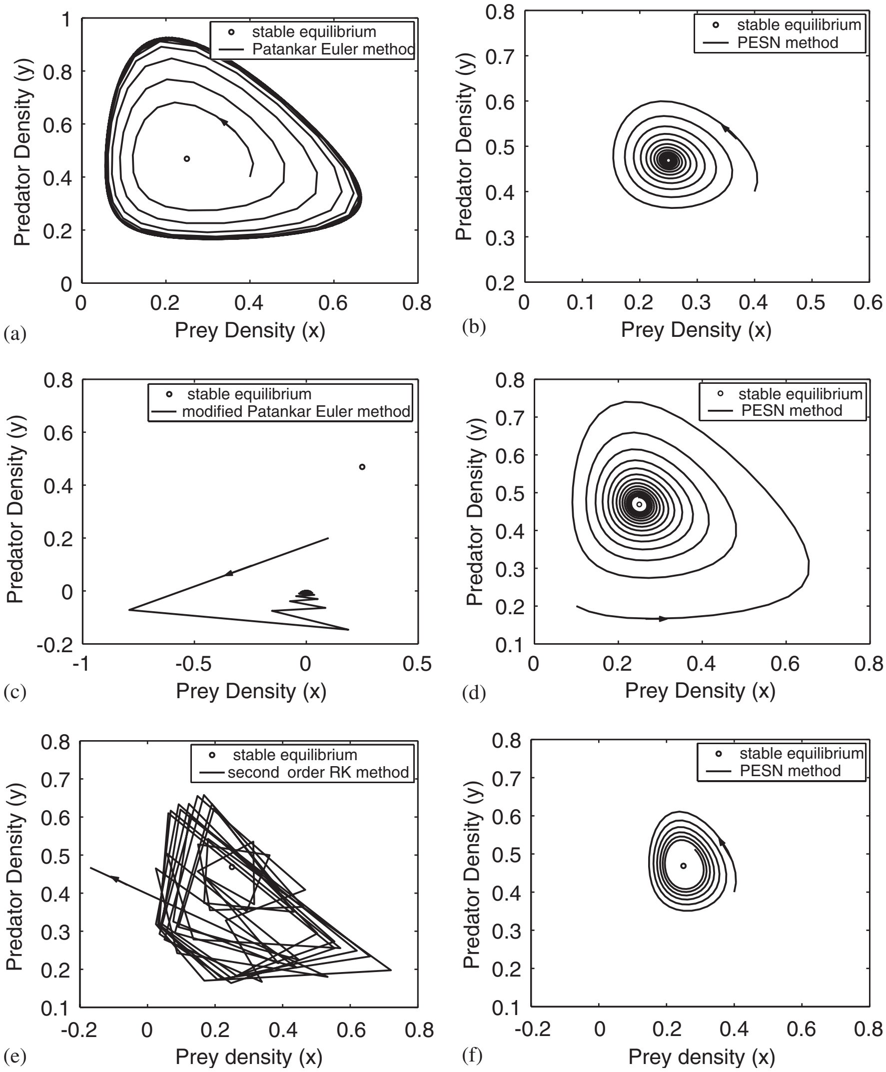

Fig. 2. Numerical approximations of the solutions of System (19) supporting the results of Theorem 3: (a) $h = 1.3$, $x_0 = 0.4$, $y_0 = 0.4$; (b) $h = 1.3$, $x_0 = 0.4$, $y_0 = 0.4$; (c) $h = 2.1$, $x_0 = 0.1$, $y_0 = 0.2$; (d) $h = 2.1$, $x_0 = 0.1$, $y_0 = 0.2$; (e) $h = 4.6$, $x_0 = 0.4$, $y_0 = 0.4$; (f) $h = 4.6$, $x_0 = 0.4$, $y_0 = 0.4$.

6. Discussion and conclusions

In this article, we applied the theory of nonstandard numerical methods to a general class of Rosenzweig–MacArthur predator–prey systems with logistic intrinsic growth of the prey population, which has a finite number of hyperbolic equilibria. Positive and elementary stable nonstandard (PESN) schemes for solving the above models were designed and analyzed. They preserve essential physical properties of exact solutions of the approximated differential systems. The proposed new methods were compared to standard numerical methods, e.g., the explicit Euler method and the second-order Runge–Kutta method, to Patankar-type methods, as presented in [5], and to the explicit ESN Euler method (8). Numerical results confirm the advantages of the new PESN method. The PESN method preserves the positivity of solutions and the stability of the equilibria for arbitrary step sizes, while the approximations obtained by the other numerical methods experience difficulties with either preserving the stability, or preserving the positivity of the solutions, or both (see Figs. 1 and 2).

The proposed new PESN numerical schemes can also be applied to other two-dimensional autonomous dynamical systems [7]. Future research directions include the construction of similar nonstandard schemes for the general case of biological systems with nonhyperbolic equilibria.

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