A compact finite difference scheme for 2D reaction–diffusion singularly perturbed problems

J.L. Gracia*, C. Clavero

Department of Applied Mathematics, University of Zaragoza, Zaragoza, Spain

Received 15 September 2004; received in revised form 4 February 2005

Abstract

In this work we define a compact finite difference scheme of positive type to solve a class of 2D reaction–diffusion elliptic singularly perturbed problems. We prove that if the new scheme is constructed on a piecewise uniform mesh of Shishkin type, it provides better approximations than the classical central finite difference scheme. Moreover, the uniform parameter bound of the error shows that the scheme is third order convergent in the maximum norm when the singular perturbation parameter is sufficiently small. Some numerical experiments illustrate in practice the result of convergence proved theoretically.

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PACS: 65N12; 65N30; 65N06

Keywords: Singular perturbation; Reaction–diffusion; Uniform convergence; Shishkin mesh; HOC scheme

1. Introduction

Fluid dynamics, quantum mechanics, elasticity, chemical reactions are some phenomena which lead to mathematical models, where one or several coefficients in the differential equation or in the boundary conditions can be very small with respect to the other coefficients. This kind of problems are known in the literature as singular perturbation problems. To approximate their solution it is well known (see [5]) that classical numerical methods cannot be used on uniform meshes; the reason is that the error is not bounded for arbitrary values of the singular perturbation parameter(s). To obtain robust numerical

* Corresponding author.
E-mail address: jlgracia@unizar.es (J.L. Gracia).

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methods it is necessary to fix the coefficients (fitted operator methods) or the mesh (fitted mesh methods) to the behaviour of the exact solution.

Most recently papers use finite differences or finite elements methods on piecewise uniform meshes (see [5,12] and references given therein). This type of meshes provides good approximations to the solution of a wide class of singularly perturbed problems. In [12] is proved that the upwind scheme is a first order method for 1D convection–diffusion problems and also that the central finite difference scheme has second order when the problem of reaction–diffusion type is considered.

Methods of high order of convergence reduce the computational cost to find good approximations of singularly perturbed problems; they have been developed in [2,7,14,15] for 1D stationary problems, in [1,6] for evolutionary 1D and 2D problems respectively and in [3] for a 2D elliptic problem with Neumann boundary conditions on the characteristic boundary. In our knowledge, numerical methods of order greater than two do not exist in the literature to solve 2D singularly perturbed elliptic problems with Dirichlet boundary conditions.

In this paper we are interested in to define a new numerical method having order greater than two to solve the following reaction–diffusion boundary value problem:

\[ Lu \equiv -\varepsilon\Delta u + bu = f \quad \text{if} \quad (x, y) \in \Omega = (0, 1)^2, \quad u = g \quad \text{on} \quad \partial\Omega, \]  

where the diffusion parameter, \( 0 < \varepsilon \leq 1 \), can be very small, the reaction term satisfies \( b(x, y) > 2\beta > 0 \), we suppose that \( b, f \in C^{6,2}(\Omega) \) and \( g \in C^{6,2}(\partial\Omega) \). It is well known that the solution of this problem has a regular layer in the boundary of \( \Omega \) with a width \( O(\sqrt{\varepsilon}) \). Moreover, we assume that the compatibility conditions up to third level are satisfied at each one of the four corners \( c_i, i = 1, 2, 3, 4 \) of the unit square; so, using the same notation that in [8], we assume that it holds

\[ \tilde{A}_k^l(f - bu, g) = 0, \quad 0 \leq k \leq 3, \quad 1 \leq l \leq 4, \]

where \( k \) refers to the level of the compatibility condition. In order to formulate \( \tilde{A}_k^l \) as local functionals, and therefore to have conditions which guarantee \( u \in C^{6,2}(\Omega) \), in the sequel we will suppose that \( (\partial^2 b / \partial x \partial y)(c_i) = 0, \quad i = 1, 2, 3, 4 \) (see [8]).

To simplify the notation, we denote

\[ u^{(l,k)}(x, y) = \frac{\partial^{l+k}u}{\partial x^l \partial y^k}(x, y), \quad [u]_k = \max_{0 \leq l \leq k} \| u^{(k-l,l)} \|, \]

where \( \| \cdot \| \) is the maximum norm. Henceforth, \( C \) is any positive constant independent of both the diffusion and the discretization parameters.

2. Asymptotic behaviour of the exact solution

To analyze the uniform convergence of the numerical method, in the following sections we need to know the asymptotic behaviour, with respect to the diffusion parameter \( \varepsilon \), of the solution of (1) and its derivatives. Following [10] and [16], it is straightforward to prove that

\[ \| u^{(k,j)} \| \leq C \varepsilon^{-k/2-j/2}, \quad 0 \leq k + j \leq 6. \]
Nevertheless, these bounds are not sufficient to prove our convergence result; so, we use the decomposition (see [4])

\[ u = v + \sum_{i=1}^{4} w_i + \sum_{i=1}^{4} z_i, \]  

(3)

where \( v \) is the regular component, \( w_i, i = 1, \ldots, 4 \), are the edge layer functions associated to the sides \( y = 0, x = 0, y = 1 \) and \( x = 1 \) respectively, and \( z_i, i = 1, \ldots, 4 \), are the corner layer functions associated to \((0,0), (0,1), (1,1)\) and \((1,0)\) respectively. Furthermore, in [4] it is proved that \( v \) satisfies \( Lv = f \), the boundary layer functions satisfy

\[ Lw_i = 0, \quad Lz_i = 0, \quad i = 1, \ldots, 4, \]  

(4)

and taking appropriate values of the boundary conditions, they also give bounds for the derivatives of \( v, w_i, z_i, i = 1, 2, 3, 4 \) up to fourth order. Nevertheless, in this paper we need to extend that bounds of its derivatives up to sixth order.

**Proposition 1.** The components of the decomposition (3) satisfy

(a) \( \|v^{(k,j)}\| \leq C(1 + e^{2-k/2-j/2}), \quad 0 \leq k + j \leq 6, \)

(b) \( \max\{\|w_i^{(k,j)}\|, \|z_i^{(k,j)}\|\} \leq Ce^{-k/2-j/2}, \quad 0 \leq k + j \leq 6, \)

(c) \( \|w_i^{(0,0)}\| \leq C(1 + e^{2-k/2}), \quad i = 1, 3, \quad 0 \leq k \leq 6, \)

(d) \( \|w_i^{(0,k)}\| \leq C(1 + e^{2-k/2}), \quad i = 2, 4, \quad 0 \leq k \leq 6. \)  

(5)

**Proof.** The proof is analogous to this one given in [4]. First, let \( \Omega^* = (-a, 1+a) \times (-a, 1+a), \quad a > 0 \) be an extended domain, \( b^*, f^* \) and \( g^* \) smooth extensions of the functions \( b, f \) and \( g \) to new domain, where \( b^* \) satisfies \( b^* (1.1) (e_i^*) = 0, \quad i = 1, 2, 3, 4 \) (\( e_i^* \) are the four corners of \( \partial \Omega^* \)).

Let \( v^* = v_0^* + e_1^* v_1^* + e_2^* v_2^* \) be, where \( v_0^* \) and \( v_1^* \) are given by \( b^* v_0^* = f^* \), \( b^* v_1^* = \Delta v_0^* \) and \( v_2^* \) is the solution of the problem

\[ L^* v_2^* = \Delta v_1^*, \quad (x, y) \in \Omega^*, \quad v_2^* = 0, \quad (x, y) \in \partial \Omega^*, \]  

(6)

where the extensions of the functions are taken such that the compatibility conditions up to third order (see [8]) are satisfied. Then, we define the regular solution \( v \) as the solution of the boundary value problem

\[ Lv = f, \quad (x, y) \in \Omega, \quad v = v^*, \quad (x, y) \in \partial \Omega. \]

Applying classical results to the extended problem (6), we can obtain

\[ \|v^{(k,j)}\| \leq C(1 + e^{2-k/2-j/2}), \quad 0 \leq k + j \leq 6. \]  

(7)

The bounds given in (b) are deduced in the same way as that of (2). Finally, we study the bounds for \( w_1^* \) (similarly can be done for the others components). Then, on the new extended domain \( \Omega^{**} = (-a, 1+a) \times (0, 1), \quad a > 0 \), it is possible to prove that the solution \( w_1^* \) of a particular problem defined on \( \Omega^{**} \) (see [4]) satisfies

\[ |(w_1^*)^{(k,0)}(x, y)| \leq C(1 + e^{2-k/2})e^{-\sqrt{b/xy}}, \quad 0 \leq k \leq 6, \quad (x, y) \in \Omega^{**}. \]  

(8)
Then, defining the boundary value problem

\[ Lw_1 = 0, \quad (x, y) \in \Omega, \]  
\[ w_1 = u - v, \quad (x, y) \in \Gamma_1, \quad w_1 = 0, \quad (x, y) \in \Gamma_3, \]  
\[ w_1(0, y) = w_1^*(0, y), \quad w_1(1, y) = w_1^*(1, y), \quad 0 \leq y \leq 1, \]  

(9a)
(9b)
(9c)

where \( \Gamma_1 = \{(x, 0) \mid 0 \leq x \leq 1\}, \ \Gamma_3 = \{(x, 1) \mid 0 \leq x \leq 1\} \), the result follows. □

Proposition 2. The edge layer functions satisfy

\[ \|w^{(k,j)}_i\| \leq C (1 + \varepsilon^{2-k/2}) \varepsilon^{-j/2}, \quad i = 1, 3, \ 0 \leq k + j \leq 6, \]  
\[ \|w^{(k,j)}_i\| \leq C (1 + \varepsilon^{2-j/2}) \varepsilon^{-k/2}, \quad i = 2, 4, \ 0 \leq k + j \leq 6. \]  

(10)

Proof. Again we only show the details corresponding to \( w_1 \). For \( k = 1 \), we differentiate (9) one time w.r.t. \( x \), obtaining

\[ L(w_1)^{(1,0)} = -b^{(1,0)} w_1, \quad (x, y) \in \Omega. \]  

(11)

From (8) it follows \( \|w_1\| \leq C \) and therefore (11) is similar to the original problem (1); then, it holds

\[ \|w^{(1,j)}_1\| \leq C \varepsilon^{-j/2}, \quad 0 \leq j \leq 5. \]  

(12)

Differentiating repeatedly (11) w.r.t. \( x \), in the same way it can be proved the bounds for \( \|w^{(k,j)}_1\|, \quad k \geq 2 \). □

3. The HOC finite difference scheme

To approximate the solution of (1), we construct a finite difference scheme following the HOC (High Order Compact) technique (see [3] and [7]). We take as basic scheme the central finite difference scheme

\[ L_{CD}^N Z_{i,j} = -\varepsilon(\partial_x^2 + \partial_y^2)Z_{i,j} + b_{i,j} Z_{i,j} = f_{i,j}, \quad (x_i, y_j) \in \Omega^N, \]  
\[ Z_{i,j} = g_{i,j}, \quad (x_i, y_j) \in \partial\Omega^N, \]  

(13)

where \( b_{i,j} = b(x_i, y_j) \) (similarly for \( f_{i,j}, g_{i,j} \)), \( N \) is the discretization parameter of the piecewise uniform Shishkin mesh \( \Omega^N = \{(x_i, y_j)\}_{i,j=0}^N \) and \( \partial_x^2, \partial_y^2 \) are the central differences of second order in the \( x \) and \( y \) directions respectively.

In standard way (see [12]), the Shishkin mesh is given by the tensor product of two 1D piecewise uniform Shishkin meshes, i.e., \( \Omega^N = w^N \times w^N \), where \( w^N = [0, \sigma] \cup [\sigma, 1 - \sigma] \cup [1 - \sigma, 1] \) distributes uniformly \( N/4 + 1 \) points in \([0, \sigma]\) and \([\sigma, 1 - \sigma]\), and \( N/2 + 1 \) points in \([\sigma, 1 - \sigma]\), and the transition parameter is

\[ \sigma = \min \left\{ 1/4, \sigma_0 \sqrt{\varepsilon} \ln N \right\}, \]  

(14)

where \( \sigma_0 \) is a constant to be fixed later (posteriorly we will see that it is crucial to take correctly the value of \( \sigma_0 \) to achieve the required order of uniform convergence). Below we denote by
Analogously to the 1D case (see [3, 7]), we add to the basic scheme approximations of the term twice w.r.t. $\tau_i = \frac{h_{i+1} + h_i}{2}$, $\tau_i = \frac{(k_{i+1} + k_i)}{2}$, $1 \leq i \leq N - 1$. To simplify the notation, we introduce the following subdomains of $\Omega$: $\Omega_i = \Omega_l \times \Omega_r$, $\Omega_c = \Omega_l \times \Omega_c$, $\Omega_\varnothing = \Omega_c \times \Omega_r$, $\Omega_{\varnothing} = \Omega_l \times \Omega_\varnothing$, $\Omega_2 = \Omega_l \times \Omega_\varnothing$, $\Omega_7 = \Omega_l \times \Omega_i$, $\Omega_8 = \Omega_\varnothing \times \Omega_l$, $\Omega_9 = \Omega_r \times \Omega_l$, where $\Omega_i = (0, \sigma)$, $\Omega_\varnothing = [\sigma, 1 - \sigma]$ and $\Omega_r = (1 - \sigma, 1)$ and their respective discrete subdomains $\Omega_i^N = \Omega_i \cap \Omega^N, i = 1, \ldots, 9$.

When $\sigma = 1/4$ the mesh is uniform and then the analysis of the convergence could be made in a classical way by using that $\varepsilon^{-1/2} < 4\sigma_0 \ln N$; so, here we only are interested in the case $\sigma = \sigma_0 \sqrt{\varepsilon} \ln N$. To increase the order of uniform convergence of the basic scheme, we distinguish two cases depending on the relation between the parameters $N$ and $\varepsilon$: (i) $N^{-1} < \sqrt{\varepsilon}$ and (ii) $\sqrt{\varepsilon} \leq N^{-1}$. Note that the second case is in practice the proper singularly perturbed case.

At the moment, in case (i) we cannot find a scheme of positive type having third order of convergence in the transition points of the mesh. The reason is the big ratio between the steps sizes associated to these points in both spatial directions. On the other hand, when $\sqrt{\varepsilon} \leq N^{-1}$ the construction of HOC scheme is easier (see [3, 7]). Therefore, we will only modify the central difference scheme in the mesh points of $\Omega^N \setminus \Omega_5^N$.

We start writing the truncation error associated to the mesh points $(x_i, y_j) \in \Omega^N \setminus \Omega_5^N$

$$
\tau_{i,j}^{DC} = L_{CD}^N(u(x_i, y_j) - Z_{i,j}) = -\varepsilon(\delta_x^2 u(x_i, y_j) - u^{(2,0)}(x_i, y_j)) + (\delta_y^2 u(x_i, y_j) - u^{(0,2)}(x_i, y_j)).
$$

Analogously to the 1D case (see [3, 7]), we add to the basic scheme approximations of the term

$$
-\varepsilon(\delta_x^2 u(x_i, y_j) - u^{(2,0)}(x_i, y_j)) \simeq -\varepsilon \frac{h_i^3}{12(h_i + h_{i+1})} u^{(4,0)}(x_i, y_j),
$$

in the boundary layers located at $x = 0$ and 1, i.e., in the mesh points $(x_i, y_j) \in \{ \Omega_1^N \cup \Omega_3^N \cup \Omega_4^N \cup \Omega_6^N \cup \Omega_7^N \cup \Omega_9^N \}$ and also an approximation of the term

$$
-\varepsilon(\delta_y^2 u(x_i, y_j) - u^{(0,2)}(x_i, y_j)) \simeq -\varepsilon \frac{k_j^3}{12(k_j + k_{j+1})} u^{(0,4)}(x_i, y_j),
$$

in the boundary layers located at $y = 0$ and 1, i.e., in the mesh points $(x_i, y_j) \in \{ \Omega_1^N \cup \Omega_2^N \cup \Omega_3^N \cup \Omega_7^N \cup \Omega_8^N \cup \Omega_9^N \}$.

The question is to take adequately these approximations in order that the new scheme be compact, of positive type and, of course, its order of convergence be greater than this one of the basic scheme. Firstly, to obtain a compact scheme we use the differential equation; then, differentiating (1) twice w.r.t. $x$ and twice w.r.t. $y$, we have

$$
-\varepsilon u^{(4,0)} = f^{(2,0)} - bu^{(2,0)} - 2b^{(1,0)}u^{(1,0)} - b^{(2,0)}u + \varepsilon u^{(2,2)},
$$

$$
-\varepsilon u^{(0,4)} = f^{(0,2)} - bu^{(0,2)} - 2b^{(0,1)}u^{(0,1)} - b^{(0,2)}u + \varepsilon u^{(2,2)}.
$$

(17)
Then, we approximate with central differences the derivatives of first, second and fourth order, i.e.,
we take
\[
\begin{align*}
\frac{\partial^1 u}{\partial x^1}(x_i, y_j) &\simeq D^0_x U_{i,j}, \\
\frac{\partial^1 u}{\partial y^1}(x_i, y_j) &\simeq D^0_y U_{i,j}, \\
\frac{\partial^2 u}{\partial x^2}(x_i, y_j) &\simeq D^2_x U_{i,j}, \\
\frac{\partial^2 u}{\partial y^2}(x_i, y_j) &\simeq D^2_y U_{i,j}, \\
\frac{\partial^4 u}{\partial x^4}(x_i, y_j) &\simeq D^4_x U_{i,j}.
\end{align*}
\]

Incorporating to the basic scheme the approximations of the terms appearing in (17), our HOC scheme is
\[
L^N U_{i,j} \equiv r_{NW}^{i,j} U_{i-1,j+1} + r_{N}^{i,j} U_{i,j+1} + r_{NE}^{i,j} U_{i+1,j+1} + r_{E}^{i,j} U_{i,j+1} + r_{W}^{i,j} U_{i,j} + r_{C}^{i,j} U_{i,j} + r_{SE}^{i,j} U_{i+1,j-1} + r_{SW}^{i,j} U_{i-1,j-1} + r_{S}^{i,j} U_{i,j-1} + r_{SE}^{i,j} U_{i+1,j+1} = Q^N (f_{i,j}), \quad (x_i, y_j) \in \Omega^N,
\]
where the coefficients are defined as follows.

- If \((x_i, y_j) \in \Omega_5\), they are
\[
\begin{align*}
r_{NW}^{i,j} &= -\frac{\varepsilon}{k_{j+1}k_j}, \\
r_{N}^{i,j} &= -\frac{\varepsilon}{k_jk_j}, \\
r_{NE}^{i,j} &= -\frac{\varepsilon}{h_{i+1}h_i}, \\
r_{E}^{i,j} &= -\frac{\varepsilon}{h_ih_i}, \\
r_{W}^{i,j} &= -\frac{\varepsilon}{k_jk_j}, \\
r_{C}^{i,j} &= 0,
\end{align*}
\]
\[
Q^N (f_{i,j}) = f_{i,j} + \frac{h^2}{12} (f_{i,j}^{(2,0)} + f_{i,j}^{(0,2)})
\]

- If \((x_i, y_j) \in \Omega_1 \cup \Omega_3 \cup \Omega_7 \cup \Omega_9\), they are
\[
\begin{align*}
r_{NW}^{i,j} &= -\frac{\varepsilon}{6h^2}, \\
r_{N}^{i,j} &= -\frac{2\varepsilon}{3h^2} + \frac{b_{i,j} + h^2 b_{i,j}^{(0,1)}}{12}, \\
r_{NE}^{i,j} &= -\frac{\varepsilon}{6h^2}, \\
r_{E}^{i,j} &= -\frac{\varepsilon}{6h^2}, \\
r_{W}^{i,j} &= -\frac{2\varepsilon}{3h^2} + \frac{b_{i,j} - h^2 b_{i,j}^{(1,0)}}{12}, \\
r_{C}^{i,j} &= \frac{10\varepsilon}{3h^2} + \frac{2b_{i,j} + h^2 b_{i,j}^{(2,0)}}{12} + \frac{h^2 b_{i,j}^{(0,2)}}{12}, \\
r_{SW}^{i,j} &= -\frac{\varepsilon}{6h^2}, \\
r_{SE}^{i,j} &= -\frac{\varepsilon}{6h^2}, \\
r_{S}^{i,j} &= -\frac{\varepsilon}{6h^2}, \\
r_{SE}^{i,j} &= -\frac{\varepsilon}{6h^2}, \\
Q^N (f_{i,j}) &= f_{i,j} + \frac{h^2}{12} (f_{i,j}^{(2,0)} + f_{i,j}^{(0,2)}).
\end{align*}
\]

- If \((x_i, y_j) \in \Omega_4 \cup \Omega_6\), they are
\[
\begin{align*}
r_{NW}^{i,j} &= -\frac{\varepsilon}{12k_{j+1}k_j}, \\
r_{N}^{i,j} &= -\frac{5\varepsilon}{6k_{j+1}k_j}, \\
r_{NE}^{i,j} &= -\frac{\varepsilon}{12k_{j+1}k_j}, \\
r_{E}^{i,j} &= -\frac{\varepsilon}{12k_{j+1}k_j}, \\
r_{W}^{i,j} &= -\frac{\varepsilon}{12k_{j+1}k_j}, \\
r_{C}^{i,j} &= 0.
\end{align*}
\]
\[
\begin{align*}
    r_{i,j}^W &= -\frac{\varepsilon}{h^2} + \frac{b_{i,j} - hb_{i,j}^{(1,0)}}{12} + \frac{\varepsilon}{6k_jk_{j+1}}, & r_{i,j}^{SW} &= -\frac{\varepsilon}{12k_jk_j}, \\
    r_{i,j}^C &= \frac{2\varepsilon}{h^2} + \frac{5b_{i,j}}{6} + \frac{5\varepsilon}{3k_jk_{j+1}} + \frac{h^2b_{i,j}^{(2,0)}}{12}, & r_{i,j}^S &= -\frac{5\varepsilon}{6k_jk_j}, \\
    r_{i,j}^E &= -\frac{\varepsilon}{h^2} + \frac{b_{i,j} + hb_{i,j}^{(1,0)}}{12} + \frac{\varepsilon}{6k_jk_{j+1}}, & r_{i,j}^{SE} &= -\frac{\varepsilon}{12k_jk_j}, \\
    Q^N(f_{i,j}) &= f_{i,j} + \frac{h^2f_{i,j}^{(2,0)}}{12}. \quad (21)
\end{align*}
\]

- If \((x_i, y_j) \in \Omega_2 \cup \Omega_8\), they are

\[
\begin{align*}
    r_{i,j}^N &= -\frac{\varepsilon}{h^2} + \frac{b_{i,j} + hb_{i,j}^{(0,1)}}{12} + \frac{\varepsilon}{6h_ih_{i+1}}, & r_{i,j}^{NE} &= -\frac{\varepsilon}{12h_ih_i}, \\
    r_{i,j}^W &= -\frac{5\varepsilon}{6h_ih_i}, & r_{i,j}^{NW} &= -\frac{\varepsilon}{12hihi}, & r_{i,j}^{SW} &= -\frac{\varepsilon}{12hihi}, \\
    r_{i,j}^C &= \frac{2\varepsilon}{h^2} + \frac{5b_{i,j}}{6} + \frac{5\varepsilon}{3h_ih_{i+1}} + \frac{h^2b_{i,j}^{(0,2)}}{12}, \\
    r_{i,j}^S &= -\frac{\varepsilon}{h^2} + \frac{b_{i,j} - hb_{i,j}^{(0,1)}}{12} + \frac{\varepsilon}{6h_ih_{i+1}}, & r_{i,j}^{E} &= -\frac{5\varepsilon}{6h_ih_{i+1}hi}, \\
    r_{i,j}^{SE} &= -\frac{\varepsilon}{12h_ih_{i+1}hi}, & Q^N(f_{i,j}) &= f_{i,j} + \frac{h^2f_{i,j}^{(0,2)}}{12}. \quad (22)
\end{align*}
\]

4. The \(\varepsilon\)-uniform convergence of the HOC scheme

Firstly we study the \(\varepsilon\)-uniform stability of the numerical method (18)–(22).

Lemma 3. Let \(d = (\|b\| + h[b_1])\) and \(N \geq N_0\) be, where \(N_0\) is a positive integer such that
\[
d\sigma_0^2 \ln^2 N < 8N^2.
\]
Then, the operator defined by (18)–(22) is of positive type. Moreover, the scheme is uniformly stable in the maximum norm and it satisfies a discrete maximum principle.

Proof. If \((x_i, y_j) \in \Omega_8\) the approximations of the second derivatives are given by central differences, and therefore it is trivial that \(r_{i,j}^* \leq 0\) if \(* \in \{NW, N, NE, W, E, SW, S, SE\}\) and \(r_{i,j}^C > 0\).

If \((x_i, y_j) \in \Omega_1 \cup \Omega_3 \cup \Omega_7 \cup \Omega_9\), from the definition of the step size \(h = 4\sigma_0 \sqrt{\varepsilon \ln N/N}\), we can deduce that \(r_{i,j}^* < 0\) if \(* \in \{A\}\) and \(r_{i,j}^C > 0\).
If \((x_i, y_j) \in \Omega_4 \cup \Omega_7\), from \(h = \sigma_0 \sqrt{\ln N} \leq k_j\), \(k_{j+1}\) we have \(r_{i,j}^* < 0\) if \(* \in A\) and \(r_{i,j}^C > 0\). In a similar way, we can obtain \(r_{i,j}^* < 0\) if \(* \in A\) and \(r_{i,j}^C > 0\) for all \((x_i, y_j) \in \Omega_2 \cup \Omega_8\).

Now we study the accuracy of the method. To prove appropriate bounds of the truncation error, we use that

\[
D_0^0 u_{i,j} - u_{i,j}^{(1,0)} = \frac{h_{i+1} - h_i}{2} u_{i,j}^{(2,0)} + \frac{R_2(x_i, x_{i+1}, u) + R_2(x_i, x_{i-1}, u)}{h_i + h_{i+1}},
\]

\[
D_0^0 u_{i,j} - u_{i,j}^{(0,1)} = \frac{k_{j+1} - k_j}{2} u_{i,j}^{(0,2)} + \frac{R_2(y_j, y_{j+1}, u) + R_2(y_j, y_{j-1}, u)}{k_j + k_{j+1}},
\]

where \(R_n\) denote the Taylor remainder in integral form. Similarly, the derivatives of second order satisfy

\[
\delta^2_x u_{i,j} - u_{i,j}^{(2,0)} = \frac{h_{i+1} - h_i}{3} u_{i,j}^{(3,0)} + \frac{1}{h_i} \left( \frac{R_3(x_i, x_{i+1}, u)}{h_{i+1}} + \frac{R_3(x_i, x_{i-1}, u)}{h_i} \right) = e_1,
\]

\[
\delta^2_y u_{i,j} - u_{i,j}^{(0,2)} = \frac{k_{j+1} - k_j}{3} u_{i,j}^{(0,3)} + \frac{1}{k_j} \left( \frac{R_3(y_j, y_{j+1}, u)}{k_{j+1}} + \frac{R_3(y_j, y_{j-1}, u)}{k_j} \right) = e_2.
\]

Finally, for the partial derivatives of fourth order, we use the notation \(\delta^2_x \delta^2_y u_{i,j} - u_{i,j}^{(2,2)} = e_3\). From previous expressions we can write the truncation error as follows:

- If \((x_i, y_j) \in \Omega_5\), then \(\tau_{i,j} = \tau_{i,j}^D\).
- If \((x_i, y_j) \in \Omega_1 \cup \Omega_3 \cup \Omega_7 \cup \Omega_9\), then

\[
\tau_{i,j} = \frac{h^2}{12}(b_{i,j}(e_1 + e_2) - 2e_3) - \frac{h^2}{6}(R_5(x_i, x_{i+1}, u) + R_5(x_i, x_{i-1}, u) + R_5(y_j, y_{j+1}, u) + R_5(y_j, y_{j-1}, u)) + h (b_{i,j}^{(1,0)} R_2(x_i, x_{i+1}, u) + b_{i,j}^{(0,1)} R_2(y_j, y_{j+1}, u) + b_{i,j}^{(1,0)} R_2(y_j, y_{j-1}, u)) / 24.
\]

(23)

- If \((x_i, y_j) \in \Omega_4 \cup \Omega_6\), then

\[
\tau_{i,j} = - e e_2 + h b_{i,j}^{(1,0)} (R_2(x_i, x_{i+1}, u) + R_2(x_i, x_{i-1}, u)) / 24 + (h^2 / 12)(b_{i,j} e_1 - e_3) - (e / h^2)(R_5(x_i, x_{i+1}, u) + R_5(x_i, x_{i-1}, u)).
\]

(24)

- If \((x_i, y_j) \in \Omega_2 \cup \Omega_8\), then

\[
\tau_{i,j} = - e e_1 + h b_{i,j}^{(0,1)} (R_2(y_j, y_{j+1}, u) + R_2(y_j, y_{j-1}, u)) / 24 + (h^2 / 12)(b_{i,j} e_2 - e_3) - (e / h^2)(R_5(y_j, y_{j+1}, u) + R_5(y_j, y_{j-1}, u)).
\]

(25)

To bound appropriately the truncation error, we need to consider a decomposition of the solution of the discrete problem similar to this one of the continuous problem (1); then, we write

\[
U_{i,j} = V_{i,j} + \sum_{k=1}^{4} W_{k:i,j} + \sum_{k=1}^{4} Z_{k:i,j}, \quad 0 \leq i, j \leq N,
\]

(26)
where
\[ L^N V_{i,j} = Q^N (L[v(x_i, y_j)]) \quad \text{in} \quad \Omega^N, \quad V_{i,j} = v(x_i, y_j) \quad \text{in} \quad \partial \Omega^N, \]
\[ L^N W_{k;i,j} = 0 \quad \text{in} \quad \Omega^N, \quad W_{k;i,j} = w_k(x_i, y_j) \quad \text{in} \quad \partial \Omega^N, \quad k = 1, 2, 3, 4, \]
\[ L^N Z_{k;i,j} = 0 \quad \text{in} \quad \Omega^N, \quad Z_{k;i,j} = z_k(x_i, y_j) \quad \text{in} \quad \partial \Omega^N, \quad k = 1, 2, 3, 4. \]

(27)

**Proposition 4.** Let \( N \geq N_0 \) be; then it holds
\[ |v(x_i, y_j) - V_{i,j}| \leq C N^{-2} \left( N^{-1} + \sqrt{\sigma_0^2 \ln^2 N} \right), \quad (x_i, y_j) \in \Omega^N. \]

(28)

**Proof.** From the expressions of the truncation error and bounds (5) for the derivatives of \( v \), a standard truncation error argument proves
\[
|\tau_{i,j}^\nu| \leq \begin{cases} 
C \varepsilon H(\|v^{(3,0)}\| + \|v^{(0,3)}\|) \leq CN^{-1} \varepsilon & \text{in} \ \Omega_5, \\
C h^4 (\varepsilon v_6 + \|v^{(4,0)}\| + \|v^{(0,4)}\| + \|v^{(3,0)}\| + \|v^{(0,3)}\|) & \text{in} \ \Omega_4 \cup \Omega_7 \cup \Omega_9, \\
N^{-1} \varepsilon + N^{-4} \sigma_0^4 \ln^4 N \varepsilon^2 + N^{-3} \sqrt{\varepsilon} & \text{in} \ \Omega_4 \cup \Omega_6, \\
C (\varepsilon h^2 \|v^{(3,0)}\| + h^4 \|v^{(0,6)}\| + h^4 \|v^{(0,4)}\| + H^3 \varepsilon [v_5] + h^4 \|v^{(3,0)}\|) & \text{in} \ \Omega_2 \cup \Omega_8, \\
C (N^{-1} \varepsilon + N^{-4} \sigma_0^4 \ln^4 N \varepsilon^2 + N^{-3} \sqrt{\varepsilon}) & \text{in} \ \Omega_4 \cup \Omega_9. 
\end{cases}
\]

Then, \(|\tau_{i,j}^\nu| \leq C \sqrt{\varepsilon} (N^{-1} \sqrt{\varepsilon} + N^{-3})\). If \( \sqrt{\varepsilon} \leq N^{-1} \), defining the barrier function \( \Psi_{i,j} = CN^{-3} \), the discrete maximum principle proves
\[ |v(x_i, y_j) - V_{i,j}| \leq CN^{-3}. \]

If \( N^{-1} < \sqrt{\varepsilon} \), defining the barrier function (see [13]) \( \Psi_{i,j} = C(\sigma^2 / \sqrt{\varepsilon}) N^{-2}(\theta(x_i) + \theta(y_j)) \), where \( C \) is a positive constant sufficiently large and
\[
\theta(z) = \begin{cases} 
z / \sigma & \text{if} \ 0 \leq z \leq \sigma, \\
1 & \text{if} \ \sigma \leq z \leq 1 - \sigma, \\
(1 - z) / \sigma & \text{if} \ 1 - \sigma \leq z \leq 1, 
\end{cases}
\]

(29)

we obtain
\[
L^N \Psi_{i,j} = \frac{\sigma^2}{\sqrt{\varepsilon}} N^{-2} \times \begin{cases} 
O(N^{-1}) & \text{if} \ (x_i, y_j) \in \Omega_1 \cup \Omega_3 \cup \Omega_7 \cup \Omega_9, \\
O(\frac{\varepsilon}{2\sigma N^{-1}}) & \text{if} \ x_i = \sigma_x, 1 - \sigma_x, \text{ or } y_j = \sigma_y, 1 - \sigma_y, \\
O(1) & \text{otherwise}. 
\end{cases}
\]

Therefore, it holds \( |\tau_{i,j}^\nu| \leq L^N \Psi_{i,j} \) and using the discrete maximum principle it follows
\[ |v(x_i, y_j) - V_{i,j}| \leq \Psi_{i,j} \leq C \sqrt{\varepsilon} N^{-2} \sigma_0^2 \ln^2 N. \]
To bound the global error associated to the boundary layer functions, we use the barrier function technique (see [9]). To do that we define the discrete functions

\[
B_{w_1;i,j} = \prod_{s=1}^{j} \left( 1 + k_s \sqrt{\beta/\varepsilon} \right)^{-1}, \quad j \neq 0, \quad B_{w_2;i,j} = \prod_{s=1}^{i} \left( 1 + h_s \sqrt{\beta/\varepsilon} \right)^{-1}, \quad i \neq 0,
\]

\[
B_{w_3;i,j} = \prod_{s=j+1}^{N} \left( 1 + k_s \sqrt{\beta/\varepsilon} \right)^{-1}, \quad j \neq N, \quad B_{w_4;i,j} = \prod_{s=i+1}^{N} \left( 1 + h_s \sqrt{\beta/\varepsilon} \right)^{-1}, \quad i \neq N,
\]

and \(B_{w_1;i,0} = 1, B_{w_2;0,j} = 1, B_{w_3;i,N} = 1, B_{w_4;N,j} = 1\).

**Lemma 5.** The discrete functions \(B_{w_k;i,j}\), for \(0 \leq i, j \leq N\) satisfy

\[
e^{-\sqrt{\beta/\varepsilon}y_j} \leq B_{w_1;i,j}, \quad e^{-\sqrt{\beta/\varepsilon}x_i} \leq B_{w_2;i,j},
\]

\[
e^{-\sqrt{\beta/\varepsilon}(1-y_j)} \leq B_{w_3;i,j}, \quad e^{-\sqrt{\beta/\varepsilon}(1-x_i)} \leq B_{w_4;i,j}.
\] (30)

Moreover,

\[
B_{w_1;i,j} \leq CN^{-\sqrt{\beta}a_0} \quad \text{for} \ 0 \leq i \leq N, \ N/4 \leq j \leq N,
\]

\[
B_{w_2;i,j} \leq CN^{-\sqrt{\beta}a_0} \quad \text{for} \ N/4 \leq i \leq N, \ 0 \leq j \leq N,
\]

\[
B_{w_3;i,j} \leq CN^{-\sqrt{\beta}a_0} \quad \text{for} \ 0 \leq i \leq N, \ 0 \leq j \leq 3N/4,
\]

\[
B_{w_4;i,j} \leq CN^{-\sqrt{\beta}a_0} \quad \text{for} \ 0 \leq i \leq 3N/4, \ 0 \leq j \leq N.
\] (31)

**Proof.** See [4,15]. \(\square\)

**Lemma 6.** Let \(N \geq N_1\) be, where \(N_1\) is a positive integer such that

\[
\beta - 2\sigma_0[b]_1 \sqrt{\varepsilon} \ln N_1/N_1 \geq 0, \quad \beta - 4\sigma_0^2[b]_2 \varepsilon \ln^2 N_1/N_1^2 \geq 0.
\] (32)

Then, there exists a constant \(C(\beta)\) such that

\[
L^N[B_{w_k;i,j}] \geq C(\beta)B_{w_k;i,j}, \quad 0 < i, j < N, \ k = 1, 2, 3, 4.
\] (33)

**Proof.** We only give the details of the proof associated to the function \(B_{w_1;i,j}\); for the other ones the proof is analogous. For \(0 < i, j < N\), we have

\[
L^N[B_{w_1;i,j}] = B_{w_1;i,j} \left[ \left( 1 + k_j \sqrt{\beta/\varepsilon} \right) (r^{SE}_{ij} + r^{S}_{ij} + r^{SW}_{ij}) + (r^{E}_{ij} + r^{C}_{ij} + r^{W}_{ij}) \right. \\
\left. + \frac{1}{1 + k_{j+1} \sqrt{\beta/\varepsilon}} (r^{NE}_{ij} + r^{N}_{ij} + r^{NW}_{ij}) \right].
\]
Using (19)–(22) and (32), it holds
\[
L^N [B_{w_1;i,j}] \equiv \begin{cases} 
B_{w_1;i,j} \left( -\frac{\beta k_{i+1} \sqrt{\varepsilon}}{k_j (\sqrt{\varepsilon} + k_{j+1} \sqrt{\beta})} + b_{i,j} \right) & \text{in } \Omega_5, \\
B_{w_1;i,j} \left( -\frac{\beta k_{i+1} \sqrt{\varepsilon}}{k_j (\sqrt{\varepsilon} + k_{j+1} \sqrt{\beta})} + b_{i,j} + \frac{h^2}{12} b_{i,j}^{(2,0)} \right) & \text{in } \Omega_4 \cup \Omega_6, \\
B_{w_1;i,j} \left( -\frac{\beta \sqrt{\varepsilon}}{\sqrt{\varepsilon} + h \sqrt{\beta}} + \frac{b_{i,j}}{2} \right) & \text{in } \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_7 \cup \Omega_8 \cup \Omega_9.
\end{cases}
\]

Finally, using that \( b > 2\beta \) the result follows. \( \square \)

**Proposition 7.** Let \( N \geq \max\{N_0, N_1\} \) be, then, for all \((x_i, y_j) \in \Omega^N \) and \( k = 1, 2, 3, 4 \), it holds
\[
|w_k(x_i, y_j) - W_{k;i,j}| \leq C \left( N^{-4} \sigma_0^4 \ln^4 N + N^{-3} + \sqrt{\varepsilon} N^{-2} \sigma_0 \ln N + N^{-\beta \sigma_0} \right).
\]

**Proof.** Again, we only show the details corresponding to the edge layer function \( w_1 \). Using that \(|w_1(x, y)| \leq C e^{-\sqrt{\beta \sigma_0} |y|} \), (4) and Lemma 5, we can reproduce the proof given in [4] or [11] in order to prove
\[
|w_1(x_i, y_j) - W_{1;i,j}| \leq C B_{w_1;i,j}, \quad (x_i, y_j) \in \Omega^N.
\]

From Lemma 5, we have
\[
|w_1(x_i, y_j) - W_{1;i,j}| \leq C N^{-\beta \sigma_0}, \quad (x_i, y_j) \in \Omega^N \setminus (\Omega_7^N \cup \Omega_8^N \cup \Omega_9^N).
\]

Then, applying the bounds of Proposition 1 to the truncation errors (23) and (25) at the mesh points \((x_i, y_j) \in \Omega_7^N \cup \Omega_8^N \cup \Omega_9^N \), we obtain
\[
|L^N [W_{1;i,j} - w_1(x_i, y_j)]| \\
\leq C[N^4 (\varepsilon) \|w_1^{(0.6)}\| + \|w_1^{(0.4)}\| + \|w_1^{(0.3)}\|] \\
+ N^{-1} \varepsilon \|w_1^{(3.0)}\| + \varepsilon N^{-2} \|w_1^{(4.0)}\| + \varepsilon h^2 \|\varepsilon_3\| \\
\leq C(N^{-4} \sigma_0^4 \ln^4 N + N^{-1} \varepsilon).
\]

If \( \sqrt{\varepsilon} \leq N^{-1} \), we define the barrier function \( \Psi_{i,j} = C(N^{-4} \sigma_0^4 \ln^4 N + N^{-3} + N^{-\beta \sigma_0}) \) and if \( N^{-1} < \sqrt{\varepsilon} \), we define the barrier function \( \Psi_{i,j} = C(N^{-4} \sigma_0^4 \ln^4 N + \sigma N^{-2} \theta(x_i) + N^{-\beta \sigma_0}) \), where \( \theta \) is given by (29). In both cases, using the discrete maximum principle, we obtain
\[
|w_1(x_i, y_j) - W_{1;i,j}| \leq \Psi_{i,j},
\]
and therefore the result follows. \( \square \)

**Proposition 8.** Let \( N \geq \max\{N_0, N_1\} \) be; then, for \( k = 1, 2, 3, 4 \) it holds
\[
|z_k(x_i, y_j) - Z_{k;i,j}| \leq C(N^{-4} \sigma_0^4 \ln^4 N + N^{-\beta \sigma_0}), \quad (x_i, y_j) \in \Omega^N.
\]

**Proof.** Here we only consider the corner layer function \( z_1 \). Similarly to Proposition 7, we can prove that
\[
|z_1(x_i, y_j) - Z_{1;i,j}| \leq C N^{-\beta \sigma_0}, \quad (x_i, y_j) \in \Omega^N \setminus \Omega_7^N.
\]
On the other hand, in $\Omega_{7}^{N}$, the truncation error given in (23) satisfies
\[ |L^{N}[Z_{1,i,j} - z_{1}(x_{i},y_{j})]| \leq C h^{4}(C_{1}+2\epsilon_{1}^{4}) + \left( \|z_{1}^{(4,0)}\| + \|z_{1}^{(0,4)}\| + \|z_{1}^{(3,0)}\| + \|z_{1}^{(0,3)}\| \right) \]
\[ \leq C N^{-4} \epsilon_{0}^{4} \ln^{4} N, \]
where we have used the bounds of Proposition 1. Then, the discrete maximum principle applied on $\Omega_{7}^{N}$ proves
\[ |z_{1}(x_{i},y_{j}) - Z_{1;i,j}| \leq C N^{-4} \epsilon_{0}^{4} \ln^{4} N + N^{-\sqrt{3} \sigma_{0}} \quad \text{in} \quad \Omega_{7}^{N}. \quad \Box \]

**Theorem 9.** Let $N \geq \max\{N_{0},N_{1}\}$ be, then for all $(x_{i},y_{j}) \in \Omega^{N}$ it holds
\[ |u(x_{i},y_{j}) - U_{i,j}| \leq C \left( N^{-3} + N^{-2} \sqrt{\epsilon_{0}^{2} \ln^{2} N} + N^{-4} \epsilon_{0}^{4} \ln^{4} N + N^{-\sqrt{3} \sigma_{0}} \right). \quad (36) \]

**Proof.** From the decomposition of the solution of the continuous (3) and discrete (26) problems, and Propositions 4, 7 and 8 the result follows. \quad \Box

**Remark 10.**

- Note that if $\sqrt{\epsilon} \leq N^{-1}$ and we take $\sqrt{\beta} \sigma_{0} \geq 3$, then the approximation to the solution of problem (1) has third order of uniform convergence.
- In the case $\sigma = 1/4$, from Theorem 9 we cannot have order of convergence greater than two. Nevertheless, it is possible to improve the bound of the error; to do that it is sufficient to consider in all mesh points of $\Omega^{N}$ the same discretization than we have used in the corner layers, i.e.,
\[ r_{i,j}^{NW} = -\epsilon/(6N^{-2}) < 0, \quad r_{i,j}^{N} = -2\epsilon/(3N^{-2}) + (b_{i,j} + N^{-1}b_{i,j}^{(0,1)})/12, \]
\[ r_{i,j}^{NE} = -\epsilon/(6N^{-2}) < 0, \quad r_{i,j}^{W} = -2\epsilon/(3N^{-2}) + (b_{i,j} - N^{-1}b_{i,j}^{(1,0)})/12, \]
\[ r_{i,j}^{C} = 10\epsilon/(3N^{-2}) + 2b_{i,j}/3 + N^{-2}b_{i,j}^{(2,0)}/12 + N^{-2}b_{i,j}^{(0,2)}/12 > 0, \]
\[ r_{i,j}^{E} = -2\epsilon/(3N^{-2}) + (b_{i,j} + N^{-1}b_{i,j}^{(1,0)})/12, \]
\[ r_{i,j}^{SW} = -\epsilon/(6N^{-2}) < 0, \quad r_{i,j}^{S} = -2\epsilon/(3N^{-2}) + (b_{i,j} - N^{-1}b_{i,j}^{(0,1)})/12, \]
\[ r_{i,j}^{SE} = -\epsilon/(6N^{-2}) < 0, \quad Q^{N}(f_{i,j}) = f_{i,j} + N^{-2}(f_{i,j}^{(2,0)} + f_{i,j}^{(0,2)})/12. \quad (37) \]

Then, using the crude bounds (2) and that $\epsilon^{-1/2} < 4\sigma_{0} \ln N$, we have
\[ |z_{i,j}^{u}| \leq C (N^{-1} \sigma_{0} \ln N)^{4}, \quad (x_{i},y_{j}) \in \Omega^{N}. \]

Then, from the maximum principle on $\Omega^{N}$ it follows
\[ |u(x_{i},y_{j}) - U_{i,j}| \leq C (N^{-1} \sigma_{0} \ln N)^{4} \quad \text{if} \quad \sigma = 1/4. \]

Note that the discretization (37) is incorporated to our algorithm, when $\sigma = 1/4$, to find the numerical solution of the test problems in next section.
5. Numerical experiments

In this section we show the results obtained for two examples using the scheme defined by (18)–(22) and (37). The first test problem is

$$-\varepsilon \Delta u + (1 + x^2 y^2 \exp(xy/2))u = f \quad \text{in } (0, 1)^2,$$

(38)

where the source term $f$ and the boundary conditions are such that $u(x, y) = (x + y)(E(2, x) + E(3, y))$ with $E(\alpha, z) = \exp(-\alpha z/\sqrt{\varepsilon}) + \exp(-\alpha(1 - z)/\sqrt{\varepsilon})$. Note that this solution is $C^\infty(\Omega)$ and consequently the coefficients of the differential equation and the boundary function satisfy sufficient compatibility and regularity conditions. Fig. 1 shows the solution for $\varepsilon = 2^{-10}$. From it we see that they are four boundary layers and three corner layers since the value of the solution at the corner (0,0) is zero. The maximum errors and the $\varepsilon$-uniform errors are calculated by $E^N = \max_{0 \leq i,j \leq N} |u(x_i, y_j) - U^N_{i,j}|$, $E^N = \max_{i,j} E^N_{i,j}$ respectively; from these values we calculate the orders and the $\varepsilon$-uniform orders of convergence by $p^N = \log(E^N / E^{2N}) / \log 2$, $p^N_{uni} = \log(E^N / E^{2N}) / \log 2$ respectively. Table 1 displays the maximum errors (first row) and the orders of convergence (second row) for several values of $\varepsilon$ and $N$. The $\varepsilon$-uniform errors and the $\varepsilon$-uniform orders of convergence appear in the last line. Clearly these results are in agreement with Theorem 9.

Fig. 1. Solution of problem (38) for $\varepsilon = 2^{-10}$.
Table 1
Maximum errors and orders of convergence

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$N = 32$</th>
<th>$N = 64$</th>
<th>$N = 128$</th>
<th>$N = 256$</th>
<th>$N = 512$</th>
</tr>
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<tbody>
<tr>
<td>$2^{-6}$</td>
<td>3.704e-4</td>
<td>2.389e-5</td>
<td>1.505e-6</td>
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<td>5.897e-9</td>
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<td>3.955</td>
<td>3.989</td>
<td>3.996</td>
<td>3.999</td>
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<td>$2^{-8}$</td>
<td>4.946e-3</td>
<td>3.485e-4</td>
<td>2.246e-5</td>
<td>1.415e-6</td>
<td>8.874e-8</td>
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<tr>
<td></td>
<td>3.827</td>
<td>3.956</td>
<td>3.988</td>
<td>3.995</td>
<td></td>
</tr>
<tr>
<td>$2^{-10}$</td>
<td>4.045e-2</td>
<td>4.776e-3</td>
<td>3.362e-4</td>
<td>2.167e-5</td>
<td>1.365e-6</td>
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<tr>
<td></td>
<td>3.082</td>
<td>3.828</td>
<td>3.956</td>
<td>3.989</td>
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<td>$2^{-12}$</td>
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<td>2.091e-2</td>
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<td>1.608</td>
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<td>3.087</td>
<td>3.283</td>
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<td>3.088</td>
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<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$2^{-24}$</td>
<td>6.266e-2</td>
<td>2.054e-2</td>
<td>3.211e-3</td>
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<tr>
<td>$P^N_{uni}$</td>
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Fig. 2. Numerical solution of problem (39) for $\varepsilon = 2^{-12}$.

The second problem that we consider is

$$-\varepsilon \Delta u + (4 + x + y)u = -4x - 4y + 10xy, \quad (x, y) \in (0, 1)^2,$$

$$u(x, 0) = (1 - x)^2, \quad 0 \leq x \leq 1, \quad u(0, y) = (1 - y)^2, \quad 0 \leq y \leq 1,$$

$$u(x, 1) = x^2, \quad 0 \leq x \leq 1, \quad u(1, y) = y^2, \quad 0 \leq y \leq 1,$$

(39)

whose solution is unknown. Fig. 2 shows the contour lines and the surface of the numerical solution for $\varepsilon = 2^{-12}$. Now, the solution have four boundary layers and two corner layers.
Table 2
Maximum errors and orders of convergence for $\varepsilon = 2^{-24}$

<table>
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<tr>
<th>Scheme</th>
<th>N = 32</th>
<th>N = 64</th>
<th>N = 128</th>
<th>N = 256</th>
</tr>
</thead>
<tbody>
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<td>Central difference scheme</td>
<td>2.556e−2</td>
<td>1.044e−2</td>
<td>4.005e−3</td>
<td>1.356e−3</td>
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<td>1.291</td>
<td>1.383</td>
<td>1.562</td>
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<tr>
<td>HOC scheme</td>
<td>6.471e−3</td>
<td>1.528e−3</td>
<td>3.608e−4</td>
<td>9.597e−5</td>
</tr>
<tr>
<td></td>
<td>2.082</td>
<td>2.082</td>
<td>1.911</td>
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</tbody>
</table>

The coefficients of problem (39) only satisfy compatibility conditions of level zero and therefore $u \in C^{1,2}(\Omega)$. The numerical (18)–(22) does not have order three because we cannot have sufficient terms in the Taylor expansions. Nevertheless, this scheme improves both the errors and the order of convergence of the central difference scheme. To estimate the errors we use a variant of the double mesh principle $D_N^\varepsilon = (\max_{0 \leq i,j \leq N} |\tilde{U}_{2i,2j}^N - U_{i,j}^N|)$, where $\{\tilde{U}_{2i,2j}^N\}$ is the numerical solution on a mesh which contains the mesh points of the original mesh $(x_i, y_j) \in \Omega^N$ and also the midpoints $x_{i+1/2} = (x_i + x_{i+1})/2$, $y_{i+1/2} = (y_i + y_{i+1})/2$, $i = 0, 1, \ldots, N - 1$.

We use these values to calculate the $\varepsilon$-uniform errors, $D_N^\varepsilon = \max D_N^\varepsilon$, the orders and the $\varepsilon$-uniform orders of convergence by $p_N^\varepsilon = \log(D_N^\varepsilon / D_{N/2}^\varepsilon)$ / log 2, $p_N^{\text{uni}} = \log(D_N^\varepsilon / D_{2N}^\varepsilon)$ / log 2. Table 2 displays the results for $\varepsilon = 2^{-24}$ and several values of $N$ using the central difference scheme and the (18)–(22). These values confirm the advantages of the new scheme even if we have a lack of compatibility conditions.

Acknowledgements

This research was partially supported by the Diputación General de Aragón and the project MEC/FEDER MTM2004–01905.

References


