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On the Number of Generators of an Invertible Ideal

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Let R be a commutative ring containing a regular element and let T be the total quotient ring of R. If F is an invertible ideal of R, then F has a finite basis; this result was originally proved by Krull in [12] for the case when R is an integral domain with identity, and Krull's proof generalizes to the case of a ring containing a regular element. In the classical case when R is a Dedekind domain, F has a basis of two elements, one of which can be chosen arbitrarily from the set of nonzero elements of F [10]. However, S. Chase has given examples which show that for any positive integer $n \geqslant 3$, there exists a Noetherian integral domain J_n with identity containing an invertible ideal with n, but no fewer, generators.

In Section 1 we show that either of the following conditions is sufficient in order that an invertible ideal A of a commutative ring R with identity have a basis of two elements: (1) A is principal over A^2. (2) $A \supset (\bigcup_\lambda AM_\lambda)$, where $\{M_\lambda\}$ is the set of maximal ideals of R containing A. Also, there is a brief consideration in Section 1 when one of a set of two generators for A can be chosen arbitrarily from the set of nonzero elements of A. In Section 2, we consider invertible ideals of a Prüfer domain D. Several known results concerning Prüfer domains indicate plausibility of the conjecture that each finitely generated ideal of D has a basis of two elements. We do not resolve this conjecture in Section 2, but we do show that conditions (1) and (2) above are equivalent, even in the local case, in D, and we give additional sufficient conditions, in terms of valuation ideals, in order that a fixed finitely generated ideal of D have a basis of two elements. It is clear, of course, that Chase's domains J_n are not Prüfer. In Section 3 we indicate a general construction of

Prüfer domains, and by means of this construction we give an example which (a) shows that none of the sufficient conditions given in Sections 1 and 2 in order that an invertible ideal have a basis of two elements are necessary, and (b) answers in the negative a question raised by Matlis in [15, p. 151].

All rings considered in this paper are assumed to be commutative.

1. Invertible Ideals of a Commutative Ring

We begin with a general result concerning generating sets of an ideal.

Lemma 1. *Suppose that A and B are finitely generated ideals of a ring R such that $A = A^2 + B$. If B has a basis of n elements, then A has a basis of $n + 1$ elements.*

Proof. In R/B, A/B is a finitely generated ideal such that $(A/B)^2 = (A^2 + B/B) = (A/B)$. Hence, A/B is principal and is generated by an idempotent element $e + B$ [16, pp. 174–5]. It follows that if $\{b_i\}_1^n$ is a basis of B, then $\{b_1, \dots, b_n, e\}$ is a basis of A.

Theorem 1[1]. *Let A be an invertible proper ideal of a ring R with identity and let $\{M_\lambda\}_{\lambda \in \Lambda}$ be the set of maximal ideals of R which contain A. If $A \supset \bigcup_{\lambda \in \Lambda} AM_\lambda$, then A is generated over A^2 by a single element. Therefore, A has a basis of two elements.*

Proof. We choose $x \in A$, $x \notin (\bigcup AM_\lambda)$. Then $xA^{-1} \not\subseteq M_\lambda$ for any λ in Λ. Therefore $A + xA^{-1}$ is an ideal of R contained in no maximal ideal of R. Hence $A + xA^{-1} = R$ and $A^2 + (x) = A$.

Remark 1. If A and R are as in Theorem 1 and if $\{P_\sigma\}_{\sigma \in S}$ is the set of all maximal ideals of R, it is straightforward to show that $A \supset (\bigcup_{\sigma \in S} AP_\sigma)$ if and only if A is principal. This result will not carry over to the case when A is assumed only to be finitely generated. For example, the maximal ideal M of a one-dimensional local domain need not be principal, but it is true that $M \supset M^2$.

Theorem 2. *If A is an invertible ideal of a ring R with identity and if $\{B_i\}_1^n$ is a finite collection of proper ideals of R, then $A \supset (\bigcup_{i=1}^n AB_i)$.*

Proof. Each B_i is contained in a maximal ideal M_i of R. Hence, it suffices to prove the theorem in the case when the B_i's are distinct maximal ideals of R. Then for any j, $M_j \not\supset (\bigcap_{i \neq j} M_i)$. Since A is invertible, $AM_j \not\supseteq A(\bigcap_{i \neq j} M_i)$.

[1] This result was stated without proof by the first author in [4, p. 337].

For each j between 1 and n, we choose $a_j \in A(\bigcap_{i \neq j} M_i)$, $a_j \notin AM_j$. If $a = \sum_{i=1}^n a_i$, then $a \in A$ and for any j between 1 and n, $a \equiv a_j \not\equiv 0(AM_j)$. Therefore, $A \supset (\bigcup_{i=1}^n AB_i)$.

COROLLARY 1.[2] *If A is an invertible ideal of a ring R with identity such that A is contained in only finitely many maximal ideals of R, then A has a basis of two elements.*

Proof. Apply Theorems 1 and 2.

COROLLARY 2. (Helms [9]) *If A is an invertible ideal of a semiquasilocal ring R, then A is principal.*

Proof. Use Remark 1 and Theorem 2.

In [15, p. 151], Matlis raises this question: If A is an invertible ideal of an integral domain R with identity such that A is contained in only finitely many maximal ideals of R, is any nonzero element of A one of a set of two generators of A? In Section 3 we give an example which shows that the answer to this question is "no." However, the answer to a form of Matlis' question is true, namely

THEOREM 3. *Suppose $a \in A$, an invertible ideal of the ring R with identity, and suppose that a belongs to only finitely many maximal ideals of R. Then there exists an element b in A for which $A = (a, b)$.*

Proof. If $a = 0$, then R is semiquasilocal and the result follows from Corollary 2. If $(a) = A$, the conclusion is obvious. In the remaining case, there is a proper ideal B of R such that $(a) = AB$. Since $(a) \subseteq B$, there are only finitely many maximal ideals $M_1, M_2, \ldots, M_n$ of R which contain B. By Theorem 2 there is an element b of A such that $b \notin (\bigcup_{i=1}^n AM_i)$. We have $(b) = AC$ for some ideal C not contained in any M_i. Thus $(a, b) = (a) + (b) = A(B + C) = A$, for $B + C$ is contained in no maximal ideal of R and hence is equal to R.

In Section 3 we shall show that neither of the following conditions is necessary in order that an invertible ideal A of a ring R with identity have a basis of two elements: (1) There is an element x in A such that $A = A^2 + (x)$; (2) $A \supset (\bigcup_{\lambda \in \Lambda} AM_\lambda)$, where $\{M_\lambda\}_{\lambda \in \Lambda}$ is the set of maximal ideals of R which contain A. Results of this section show that either (1) or (2) is sufficient to imply that A has a basis of two elements.

Before proceeding farther, we outline a construction, for $n \geqslant 3$, of a domain J_n with identity such that J_n contains an invertible ideal with a basis of n, but

[2] This result is stated as Corollary 2.5 of [15]. We may have been aware of its validity before Matlis was (cf. footnote 1).

no fewer, generators. Our construction is due to Chase, although Chase never published the result; the example has been cited at least twice in the literature [1, p. 541], [25, p. 270].

Let K be the field of real numbers, let $D_n = K[X_1, ..., X_n]$, and let $R_n = D_n/A$, where A is the principal ideal of D_n generated by $f(X_1, ..., X_n) = X_1^2 + \cdots + X_n^2 - 1$. f is prime in D_n so that R_n is an integral domain with identity. (In fact, R_n is a UFD for any field K in which -1 is not a square, see [23, p. 165], [25, p. 273].) If $E_n = K(\{X_i X_j\}_{1 \leqslant i,j \leqslant n})$ and if $J_n = (E_n + A)/A \simeq E_n/fE_n$, J_n is a Noetherian domain with identity and is the desired example. If B is the ideal of J_n generated by $\{\overline{X_1 X_i}\}_{1 \leqslant i \leqslant n}$, B is invertible since $B^2 = (\overline{X_1^2})$. In [25, pp. 270–271], Swan shows that the J_n-submodule of R_n generated by $\{\overline{X}_1, ..., \overline{X}_n\}$ has no basis of fewer than n elements, and it then follows immediately that B has no J_n-module basis (i.e., no ideal basis) of fewer than n elements.

2. Invertible Ideals of a Prüfer Domain

A Prüfer domain is defined to be an integral domain with identity in which each nonzero finitely generated ideal is invertible. Among integral domains J with identity, Prüfer domains are characterized by the property that J_P is a valuation ring for each prime P of J [14]. There are several results which might indicate that a finitely generated ideal of a Prüfer domain has a basis of two elements. We cite the following, where J denotes an integral domain with identity:

If each nonzero ideal of J with a basis of two elements is invertible, then J is a Prüfer domain [21, p. 6].

If J is Prüfer and if the prime ideal P of J is the radical of a finitely generated ideal, then P is the radical of an ideal with a basis of two elements.

The global case of the second statement is a consequence of Theorem 4 in [7, p. 288]; a proof of the local case is the following:

Suppose that P is the radical of the finitely generated ideal A and let $\{M_\lambda\}$ be the set of maximal ideals of J which don't contain P. If $J' = J_P \cap [\bigcap_\lambda J_{M_\lambda}]$, we observe that PJ' is a maximal ideal of J'. To prove this statement it suffices to show, since each prime ideal of J' is the extension of a prime ideal of J [4, p. 333], that $QJ' = J'$ for any prime ideal Q of J properly containing P. Thus if $x \in Q - P$ and if $B = A + (x)$, the B-transform is contained in J' [7, p. 283] so that $BJ' = J'$ and $QJ' = J'$ also. It follows that PJ' is maximal in J' and is the radical of the finitely generated ideal AJ'. By Corollary 1, AJ' has a basis $\{a, b\}$ of two elements. But $PJ' = P$ by [7; p. 285] so that $a, b \in P$ and $P = \sqrt{\{a, b\}J}$.

The following generalization of the result just proved can be established in similar manner:

If A is a finitely generated ideal of a Prüfer domain J, having only finitely many minimal prime ideals, then the radical of A is the radical of an ideal with a basis of two elements. Throughout the remainder of this section we use the letter D to denote a Prüfer domain.

Remark 2. It is shown in [7, p. 223] that if A and B are finitely generated nonzero ideals of D with bases of n and m elements, respectively, then $A \cap B$ and $A : B$ are finitely generated and have bases of $n + m$ and $m(n + m)$ elements, respectively.[3] The proof of this result in [7] rests on a theorem due to Jensen [10, p. 93], which shows that among integral domains J with identity, Prüfer domains are characterized by the fact that $(X + Y)(X \cap Y) = XY$ for any two ideals X, Y of J. From this equality, it follows easily that if $\{F_i\}_{i=1}^n$ is a finite collection of nonzero finitely generated fractional ideals of D, then $F_1 + F_2 + \cdots + F_n = (\bigcap_{i=1}^n F_i^{-1})^{-1}$. In particular, if $y_1, \ldots, y_n$ are nonzero elements of K, then $(y_1, \ldots, y_n) = [\bigcap_{i=1}^n (x_i)]^{-1}$, where $x_i = y_i^{-1}$ for each i. Thus, to show that each invertible fractional ideal of D has a basis of two elements, one must show, equivalently, that for any nonzero elements $x_1, \ldots, x_n$ of K, $\bigcap_{i=1}^n (x_i)$ is an intersection of two principal fractional ideals of D.

If ξ is a nonzero element of K, the quotient field of D, then we denote by A_ξ the ideal of D consisting of all elements x of D such that $x\xi \in D$. If $\xi = a/b$ where $a, b \in D$, then $A_\xi = (a) : (b) = (\xi^{-1}) \cap D$. Thus by Lemma 2, A_ξ has a basis of two elements. One wonders if each nonzero ideal of D with a basis of two elements is of the form A_ξ. We show later that this is not the case, but first we consider conditions under which a finitely generated ideal is of the form A_ξ.

LEMMA 3. *If A is a proper ideal of a ring R with identity and if $\{M_\lambda\}$ is the set of maximal ideals of R which contain A, then any ideal of R contained in $\bigcup_\lambda M_\lambda$ is contained in some M_λ. Therefore the set of maximal ideals of R_S, where $S = R - (\bigcup_\lambda M_\lambda)$, is the set of extensions of the M_λ's to R_S. If R is an integral domain, then $R_S = \bigcap_\lambda R_{M_\lambda}$.*

Proof. If B is an ideal of R which is contained in no M_λ, then $A + B = R$ so that $a + b = 1$ for some $a \in A$, $b \in B$. Then $b \in B - (\bigcup_\lambda M_\lambda)$. The statement concerning the maximal ideals of R_S is then immediate. The last

[3] A better bound on the number of generators of $A : B$ is $mn + 1$; this follows from the fact that $A : B = AB^{-1} \cap D$. Quentel in [22, p. 659] has recently observed that if J is an integral domain with identity and if the intersection of any two finitely generated ideals of J is finitely generated, then the quotient of any two finitely generated ideals of J is also finitely generated.

assertion of the lemma follows from the facts that $R_S = \bigcap_\lambda (R_S)_{M_\lambda R_S}$ in case R is an integral domain, and that $(R_S)_{M_\lambda R_S} = R_{M_\lambda}$.

THEOREM 4. *Let A be a proper finitely generated ideal of D, let $\{M_\lambda\}$ be the set of maximal ideals of D which contain A, and let $S = D - (\bigcup_\lambda M_\lambda)$. These conditions are equivalent:*

(1) $A \supset (\bigcup_\lambda AM_\lambda)$.

(2) *A is of the form A_ξ.*

(3) *AD_S is principal.*

Proof. (1) → (2): If $a \in A - (\bigcup_\lambda AM_\lambda)$, we can write $(a) = AB$ for some ideal B of D which is contained in no M_λ. Hence, $A + B = D$ and $u + v = 1$ for some $u \in A$, $v \in B$. We observe that $A = (a) : (v) = A_{a/v}$. That $A \subseteq (a) : (v)$ is clear, and if $xv \in (a)$, then $x = xu + xv \in A$ so that equality holds, $A = (a) : (v)$.

(2) → (3): If $A = A_\xi = (\xi^{-1}) \cap D$, then $AD_S = (\xi^{-1}) D_S \cap D_S$. Also for any λ, $\xi^{-1}D_{M_\lambda} \cap D_{M_\lambda} = AD_{M_\lambda}$, and since D_{M_λ} is a valuation ring, it follows that $\xi^{-1} \in D_{M_\lambda}$ for each λ. Therefore, $\xi^{-1} \in \bigcap_\lambda D_{M_\lambda}$, and Lemma 2 shows that $\bigcap_\lambda D_{M_\lambda} = D_S$. Hence $AD_S = \xi^{-1}D_S \cap D_S = \xi^{-1}D_S$, and (3) holds.

(3) → (1): Let $AD_S = aD_S$, where $a \in A$. For any λ, $M_\lambda D_S$ is a proper ideal of D_S so that $a \notin (aD_S)(M_\lambda D_S) = (AD_S)(M_\lambda D_S) = AM_\lambda D_S$. Therefore, $a \in A - (\bigcup_\lambda AM_\lambda)$ and our proof is complete.

COROLLARY 3. If A is a proper finitely generated ideal of D which is contained in only finitely many maximal ideals, then A is of the form A_ξ.

COROLLARY 4. ([2, p. 12]). If D is a Dedekind domain, then each nonzero ideal of D is of the form A_ξ.

If J is a domain with identity having quotient field L, an ideal A of J is said to be a *valuation ideal* provided there is a valuation ring V between J and L and an ideal B of V such that $A = B \cap J$; this is equivalent to the assertion that $AV \cap J = A$. We use the fact that if D is Prüfer and if $\{P_\alpha\}$ is the set of prime ideals of D distinct from D, then $\{D_{P_\alpha}\}$ is the collection of valuation rings between D and K [14, p. 554]. Hence, each valuation v on K which is nonnegative on D is uniquely determined by its center P_α on D. We say, in this case, that v is associated with the prime ideal P_α.

LEMMA 3. *Suppose that D is a semiquasilocal domain with maximal ideals $M_1, M_2, \ldots, M_n$. If d is a nonzero element of $M_1 \cap M_2 \cap \cdots \cap M_n$, then $dD_{M_1} \cap D \not\supseteq \bigcap_{i=2}^n (dD_{M_i} \cap D)$.*

Proof. We consider two cases.

Case 1. M_1 is a minimal prime ideal of (d). Then $dD_{M_1} \cap D$ is M_1-primary. Further $dD_{M_i} \cap D$ is a valuation ideal and hence $\sqrt{(dD_{M_i} \cap D)} = P_i$ is prime in D for $2 \leqslant i \leqslant n$ [27, p. 342]. No M_i contains M_1, and hence $M_1 \neq P_i$ for $2 \leqslant i \leqslant n$. Hence, there are no containment relations between M_1 and P_i. This implies that $M_1 + P_i = D$ for $2 \leqslant i \leqslant n$, so that $M_1 + (P_2 \cap \cdots \cap P_n) = D$. Because

$$\sqrt{(dD_{M_1} \cap D)} = M_1 \qquad \text{and} \qquad \sqrt{(dD_{M_i} \cap D)} = P_i ,$$

it follows that $(dD_{M_1} \cap D) + [\bigcap_{i=2}^{n} (dD_{M_i} \cap D)] = D$. In particular, $dD_{M_1} \cap D \not\supseteq \bigcap_{i=2}^{n} (dD_{M_i} \cap D)$.

Case 2. M_1 is not a minimal prime ideal of (d). Then $M_1 \supset P_1$, a minimal prime of (d). We choose in this case an element $x \in M_1$, $x \notin P_1$, $x \notin (\bigcup_{i=2}^{n} M_i)$. Then $v_i(d/x) = v_i(d) - v_i(x) = v_i(d) \geqslant 0$ for $2 \leqslant i \leqslant n$. And since $dD_{M_1} \subseteq P_1D_{M_1} \subset xD_{M_1} \subseteq M_1D_{M_1}$, $d/x \in D_{M_1}$ also. Hence,

$$d/x \in \left(\bigcap_{i=2}^{n} dD_{M_i}\right) \cap D_{M_1} = \bigcap_{i=2}^{n} (dD_{M_i} \cap D).$$

Yet $(d/x)/d = 1/x \notin D$ since $x \in M$. Therefore, $d/x \notin (d) = \bigcap_{i=1}^{n} (dD_{M_i} \cap D)$, implying that $d/x \notin dD_{M_1} \cap D$. Hence, $dD_{M_1} \cap D \not\supseteq \bigcap_{i=2}^{n} (dD_{M_i} \cap D)$.

PROPOSITION 1. *Suppose that A is an ideal of D which is a finite intersection of valuation ideals. Then A is a finite intersection of valuation ideals associated with maximal ideals of D. If A is finitely generated, A is contained in only finitely many maximal ideals of D.*

Proof. By hypothesis, there is a finite collection $\{P_i\}_1^n$ of prime ideals of D such that $A = \bigcap_1^n (AD_{P_i} \cap D)$. For each i, let M_i be a maximal ideal of D containing P_i. Then $D_{M_i} \subseteq D_{P_i}$ for each i so that

$$A = \bigcap_1^n (AD_{P_i} \cap D) \supseteq \bigcap_1^n (AD_{M_i} \cap D) \supseteq A.$$

Therefore, A is a finite intersection of valuation ideals associated with maximal ideals.

We now assume that A is finitely generated. Since $AD_{M_i} \cap D = D$ if $A \not\subseteq M_i$, we may assume that $A \subseteq M_i$ for $1 \leqslant i \leqslant n$. We show that if M_0 is a maximal ideal not in the set $\{M_1, ..., M_n\}$, then $A \not\subseteq M_0$. Thus let $V = \bigcap_{i=0}^{n} D_{M_i}$. V is a semiquasilocal Prüfer domain with $n + 1$ maximal ideals $N_0, N_1, ..., N_n$, where $N_i = M_iD_{M_i} \cap V = M_iV$ and $V_{N_i} = D_{M_i}$ for each i [17, p. 54] [18, p. 38]. $A' = \bigcap_1^n (AD_{M_i} \cap V)$ is an ideal of V lying over A so that $A' = AV$ [4, p. 333]. Thus, A' is principal. Furthermore,

$$A' = \bigcap_1^n (AD_{M_i} \cap V) = \bigcap_1^n (\mathrm{AVD}_{M_i} \cap V)$$

$$= \bigcap_1^n (A'D_{M_i} \cap V) = \bigcap_1^n (A'V_{N_i} \cap V).$$

Lemma 3 then implies that $A' = AV \not\subseteq N_0 = M_0V$. Therefore, $A \not\subseteq M_0$ and our proof of Proposition 1 is complete.

Remark 3. If the finitely generated ideal A of D can be represented as a finite intersection of valuation ideals associated with prime ideals $P_1, P_2, \ldots, P_n$, where A is contained in each P_i and there are no containment relations between distinct P_i's, then each P_i is maximal in D and such a representation is unique. To prove that each P_i is maximal, suppose it is not and assume that M_1 is a maximal ideal of D properly containing P_1. Then $M_1 \not\subseteq \bigcup_{i=1}^n P_i$ since there are no containment relations among the P_i's so we can choose $m \in M_1$, $m \notin \bigcup_{i=1}^n P_i$. The ideal $C = A + (m)$ is invertible, proper (since $C \subseteq M_1$), and C properly contains A. Hence, there is an ideal B of D properly containing A such that $A = BC$. We have $CD_{P_i} = D_{P_i}$ for any i since $m \in C$. Hence $AD_{P_i} = BD_{P_i}$ for each i and $B \subseteq \bigcap_{i=1}^n (AD_{P_i} \cap D) = A$. This contradiction shows that each P_i is maximal in D. Uniqueness of the representation then follows from the fact, established in the proof of Proposition 1, that $\{P_1, \ldots, P_n\}$ is precisely the set of maximal ideals of D which contain A.

THEOREM 5. *In D, these statements are equivalent*:

(a) *Each nonzero element of D belongs to only finitely many maximal ideals of D.*

(b) *Each ideal of D is a finite intersection of valuation ideals.*

(c) *Each finitely generated ideal of D is a finite intersection of valuation ideals.*

(d) *Each principal ideal of D is a finite intersection of valuation ideals.*

Proof. The implications (a) $\rightarrow$ (b) $\rightarrow$ (c) $\rightarrow$ (d) are clear, and Proposition 1 shows that (d) $\rightarrow$ (a).

The ring of all algebraic integers is a one-dimensional Prüfer domain J in which each nonzero finitely generated ideal is principal and is, therefore, of the form A_ξ. Yet each nonunit of J belongs to uncountably many maximal ideals of J.

3. An Example

We consider a nontrivial valuation v on a field L with valuation ring V of the form $K + M$ where K is a field and M is the maximal ideal of V. If J is a domain with identity which is a subring of K, we give some properties of the domain $J_1 = J + M$[4]. We shall not establish the truth of (a)–(e).

(a) *J_1 has quotient field L.*

(b) *The integral closure of J_1 is $J' + M$, where J' is the integral closure of J in K.*

(c) *If A is an ideal of J_1, then either $A \subseteq M$ or $M \subseteq A$.*

(d) *The ideals of J_1 which contain M are of the form $B + M$ where B is an ideal of J. If S is a subset of B which generates B as an ideal of J, then S generates $B + M$ as an ideal of J_1.*

(e) *If B is an ideal of J, $J_1/(B + M) \simeq J/B$. Hence $B + M$ is prime, maximal, or primary in J_1 if and only if B is, respectively, prime, maximal, or primary in J.*

(f) *The prime ideals of J_1 contained in M coincide with the prime ideals of V contained in M.*

Proof of (f). We must show that if P_0 is a prime ideal of J_1 properly contained in M, then P_0 is a prime ideal of V. First, P_0 is an ideal of V, for if $x \in P_0$ and $y \in V$, then yx and y^2x are in M so that $y^2x \cdot x = (yx)^2 \in P_0$. Since P_0 is prime in J_1, this implies that $yx \in P_0$. Also, P_0 is prime, for if $u, v \in V$ and $uv \in P_0$, then u or v is in M. If $u \notin M$, u is a unit of V; and therefore, $u^{-1}uv = v \in P_0$ since P_0 is an ideal of V. But if both u and v are in M, then u or v is in P_0 since P_0 is prime in J_1.

(g) *If P is a prime of J_1 properly contained in M, $(J_1)_P = V_P$ is a valuation ring. If N is a multiplicative system in J, $(J_1)_N = J_N + M$.*

Proof. Clearly $(J_1)_P \subseteq V_P$. If $\xi \in V_P$, $\xi = a/n$ for some $a \in V, n \in V - P$. We choose $m \in M - P$. Then $a/n = am/nm$ with $am \in M \subseteq J_1$ and $nm \in M - P$. Hence, $\xi = am/nm \in (J_1)_P$ and $V_P = (J_1)_P$.

For the second half of (g), we need only show that $(J_1)_N \subseteq J_N + M$. Thus if $\xi \in (J_1)_N$, $\xi = (a + m)/n$ for some $a \in J$, $m \in M$, $n \in N$. Since n is a unit of V, $n^{-1}m \in M$. Therefore, $\xi = (a/n) + mn^{-1} \in J_N + M$.

[4] We make the assumption that V contains an isomorphic copy of its residue field in order to simplify the notation; analogous results hold when J is any subring of V/M and J_1 is the inverse image of J under the natural mapping from V onto V/M. Constructions of the type $J + M$ are considered in [24], [8], [19], and [6].

(h) *J_1 is a valuation ring if and only if J is a valuation ring with quotient field K. J_1 is a Prüfer domain if and only if J is Prüfer and has quotient field K.*

Proof. If J_1 is a valuation ring (is Prüfer) then J_1/M is a valuation ring (is Prüfer), and in either case, $(J_1)_M$ must be a valuation ring. By (f), $(J_1)_M = S + M$, where S is the quotient field of J. But a valuation ring is uniquely determined by its maximal ideal. Hence, $S = K$, and J has quotient field K.

We suppose now that J is a valuation ring with quotient field K. Then J_1 is the inverse image of J under the natural homomorphism of V onto its residue field, and is therefore a valuation ring [18, p. 35].

Finally, if J is Prüfer and has quotient field K, then to show that J_1 is Prüfer, it suffices, in view of (g), to prove that $(J_1)_{P_0}$ is a valuation ring for any prime ideal P_0 of J_1 containing M. By (d) and (e), $P_0 = P + M$ for some prime ideal P of J. If $S = J - P$, we have $(J_1)_{P_0} \supseteq (J_1)_S = J_S + M$. Since J is Prüfer with quotient field K, $J_S = J_P$ is a valuation ring with quotient field K. Hence, as we have already shown, $J_S + M$ is a valuation ring with quotient field L. Since $J_S + M \subseteq (J_1)_{P_0} \subseteq L$, $(J_1)_{P_0}$ is also a valuation ring, and J_1 is Prüfer as we wished to show.

(i) *Suppose that A is a nonzero finitely generated ideal of J_1 contained in M. Then A has a basis of the form $\{a, k_1a, ..., k_na\}$ where the k_i's are nonzero elements of K. The ideal of J_1 generated by such a set $\{a, k_1a, ..., k_na\}$ is $Wa + C$ where W is the J-submodule of K generated by $\{1, k_1, ..., k_n\}$ and C is the ideal of V consisting of all elements x of L such that $x = 0$ or $v(x) > v(a)$.*

Proof. If $x \in J_1$ and if $y \in V$ is such that $v(y) > v(x)$, then $y/x \in M$ so that $y = (y/x)\, x \in Mx \subseteq J_1x$. Thus A has a basis of the form $\{a, a_1, ..., a_n\}$ where $v(a) = v(a_1) = \cdots = v(a_n)$. Hence, $v(a_i/a) = 0$ for each $i : a_i/a = k_i + m_i$ for some nonzero element k_i of K and some $m_i \in M$. Thus $\{a, a_1, ..., a_n\} = \{a, k_1a + m_1a, ..., k_na + m_na\}$. But $m_ia \in J_1a$ for each i so that A is generated by $\{a, k_1a, ..., k_na\}$. Clearly, A contains $Wa + C$, and $Wa + C$ contains the set $\{a, k_1a, ..., k_na\}$. But it is easily verified that $Wa + C$ is an ideal of J_1. Hence, $A = Wa + C$, as we wished to show.

(j) *Suppose that K is the quotient field of J. If each finitely generated ideal of J has a basis of n elements, then each finitely generated ideal of J_1 has a basis of n elements.*

Proof. (c) and (d) show that each finitely generated ideal of J_1 not contained in M has a basis of n elements. If A is a nonzero finitely generated ideal contained in M, (i) shows that $A = Wa + C$ for some element a of A, for $W = J + Jk_1 + \cdots + Jk_m$ a finitely generated J-submodule of K, and where $C = \{x \in L \mid x = 0 \text{ or } v(x) > v(a)\}$. Therefore, W is a fractional ideal of J, so that $W = B/d = \{b/d \mid b \in B\}$ for some ideal B of J and some nonzero element d of J. By assumption, $B = (b_1, ..., b_n)$, so that

$$W = J(b_1/d) + \cdots + J(b_n/d).$$

Finally, this implies that $\{b_1a/d, ..., b_na/d\}$ is a basis of the ideal A of J_1.

(k) *Suppose that K is the quotient field of J. In order that each nonzero finitely generated ideal of J be of the form A_ξ for some ξ in L, it is necessary and sufficient that J be a Bezout domain; that is, each finitely generated ideal of J is principal.*

Proof. (j) shows that if J is a Bezout domain, then J_1 is also a Bezout domain. Hence, if J is a Bezout domain, then each nonzero finitely generated ideal of J_1 is of the form A_ξ.

If J is not a Bezout domain, there are elements u, t of J such that (u, t) is not principal in J. We choose a nonzero element m of M and we show that the ideal (um, tm) of J_1 is not of the form A_ξ for any ξ in L. It is easy to see that the only candidates for elements ξ of L such that $(um, tm) = A_\xi$ are those elements with v-value equal to $-v(m)$. For any such ξ, however, we have $\xi^{-1} \in J_1$ so that $A_\xi = J_1\xi^{-1} \cap J_1 = J_1\xi^{-1}$. Since (u, t) is not principal in J, (j) shows that (um, tm) is not a principal ideal of J_1. Hence, $A_\xi \neq (um, tm)$ for any ξ in L.

EXAMPLE 1. Let J be a Dedekind domain which is not a principal ideal domain. Let K be the quotient field of J and suppose that u and t are elements of J which generate a nonprincipal ideal. $V = K[[X]]$ is a valuation ring of the form $K + M$, where $M = XV$ is the maximal ideal of V. We set $J_1 = J + M$. By (d)–(g), J_1 is a two-dimensional Prüfer domain. By (j), each finitely generated ideal of J_1 has a basis of two elements. (Note that J_1 is not Noetherian, for since J is integrally closed and $J \subset K$, K is not a finite J-module. In fact, in the general case J_1 will be Noetherian if and only if V is rank one discrete, J is a field, and $[K : J] < \infty$.) But by the proof of (k), the ideal $A = (uX, tX)$ is not of the form A_ξ for any ξ in $K[(X)]$. Furthermore, A is contained in each maximal ideal M_λ of J_1, so that $A = \bigcup_\lambda AM_\lambda$ since A is not principal. And yet A has a basis of two elements. There is no element c of A such that $A = A^2 + (c)$, for any such c would clearly have to be of the form sX for some unit s of V. But for any such s,

$$A^2 + (sX) = (t^2X^2, tuX^2, u^2X^2, sX) = (sX) \subset A.$$

Finally, the ideal (t, u) of J_1 provides us with an example showing that the answer to Matlis' question mentioned in the paragraph following Corollary 2 is "no", for if m is any element of M, there is no element a in J_1 such that $(m, a) = (t, u)$. Note that for this particular example, the only m's for which such an x does exist are those given to us by Theorem 3—i.e., those m's in (t, u) which belong to only finitely many maximal ideals of J_1.

Remark 4.[5] It is easy to show, from (a)–(k), that for any Prüfer domain J, J and J_1 have the same class group. In fact, if $\mathscr{F}(J)$ denotes the set of nonzero finitely generated fractional ideals of J, if $\mathscr{C}(J_1)$ denotes the class group of J_1, and if we denote by F_1' the element of $\mathscr{C}(J_1)$ determined by a nonzero finitely generated fractional ideal F_1 of J_1, then the mapping $\phi : F \to (FJ_1)'$ is a homomorphism of $\mathscr{F}(J)$ onto $\mathscr{C}(J_1)$ with kernel $\mathscr{P}(J)$, the set of nonzero principal fractional ideals of J. That ϕ is a homomorphism is clear, and (i) shows that ϕ is onto. The kernel of ϕ is obviously $\mathscr{P}(J)$.

Added in proof. H. S. Butts has recently communicated to us a result of one of his students, Philip Quartararo, and we have observed that this result can be obtained from Theorem 2.

If A is an invertible ideal of a commutative ring R with identity, and if $\{B_i\}_1^n$ is a finite collection of ideals of R such that $A \subseteq \cup_{i=1}^n B_i$, then $A \subseteq B_i$ for some i.

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[5] We had not observed that Remark 4 is true until L. Levy remarked to us that he could show that any finite abelian group is the class group of a non-Noetherian Prüfer domain. Remark 4 and the following results show that any abelian group is the class group of a two-dimensional (and hence non-Noetherian) Prüfer domain: Any abelian group G is the class group of a Dedekind domain J [3]. If K is the quotient field of J and if $J_1 = J + XK([X])$, then J_1 is a two-dimensional Prüfer domain with class group G.

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