NORTH-HOLLAND

# (0, $\frac{1}{2}$, 1) Matrices Which Are Extreme Points of the Generalized Transitive Tournament Polytope

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ABSTRACT

Following Brualdi and Hwang, given a generalized transitive tournament (GTT) matrix T of order $n$, we consider the *-graph of $T$, that is, the undirected graph with vertices $1, 2, \dots, n$ in which there is an edge $\{i, j\}$ between vertices $i$ and $j$ if and only if $0 < t_{ij} < 1$. We characterize the *-graphs of the extreme GTT (0, $\frac{1}{2}$, 1) matrices of order $n$. Using this characterization, we obtain for $n = 6$, 7 the complete list of extreme GTT (0, $\frac{1}{2}$, 1) matrices of order $n$.

## 1. INTRODUCTION

Let $T = [t_{ij}]$ be a (0, 1) matrix (that is, each entry of $T$ is 0 or 1) of order $n$ which satisfies $t_{ij} = 0$ for $i = 1, \dots, n$ and $t_{ij} + t_{ji} = 1$ for every $i \neq j$; then $T$ is said to be a *tournament matrix*. If $T$ also satisfies $1 \leqslant t_{ij} + t_{jk} + t_{ki} \leqslant 2$ for every $i, j, k$ distinct, then $T$ is said to be a *transitive* tournament matrix, abbreviated TT matrix.

A nonnegative matrix $T = [t_{ij}]$ of order $n$ which satisfies $t_{ii} = 0$ for $i = 1, \dots, n$ and $t_{ij} + t_{ji} = 1$ for every $i \neq j$ is said to be a *generalized tournament matrix*, abbreviated GT matrix. If $T$ also satisfies $1 \leqslant t_{ij} + t_{jk} + t_{ki} \leqslant 2$ for every $i, j, k$ distinct, then $T$ is said to be a *generalized transitive*

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*tournament matrix*, abbreviated GTT matrix. The convex polytope composed of all GT matrices of order $n$ will be denoted by $\mathscr{G}_n$, and the convex polytope composed of all GTT matrices of order $n$ by $\mathscr{T}_n$.

The tournament matrices of order $n$ are the extreme points of $\mathscr{G}_n$. In the same way, the TT matrices of order $n$ are extreme points of $\mathscr{T}_n$, but when $n \geqslant 6$ the polytope $\mathscr{T}_n$ has more extreme points (for $n \leqslant 5$ the TT matrices of order $n$ are the only extreme points of $\mathscr{T}_n$—see Dridi [3]). We say that a GTT matrix is *extreme* provided it is an extreme point of $\mathscr{T}_n$.

We will characterize for any $n \in \mathbb{N}$ the extreme GTT $(0, \frac{1}{2}, 1)$ matrices of order $n$ (that is, the extreme GTT matrices of order $n$ with all entries equal to $0, \frac{1}{2}$, or 1). In particular, for $n = 6, 7$ we will obtain the complete list of the extreme GTT $(0, \frac{1}{2}, 1)$ matrices. The method we have employed follows the ideas introduced into the subject by Brualdi and Hwang in [1].

NOTE: During the Workshop on Nonnegative Matrices held in Haifa in 1993, I was informed (private communication) that Z. Nutov and M. Penn had found one extreme point of $\mathscr{T}_8$ with some of its entries different from $0, \frac{1}{2}$, and 1.

## 2. GRAPHS

We will work only with graphs having neither loops nor multiple edges. Let $\Gamma_n$ denote the set composed of the undirected graphs with vertices $1, \dots, n$. Given $\gamma \in \Gamma_n$, the edge set of $\gamma$ will be denoted by $E(\gamma)$. Two vertices $a$ and $b$ of $\gamma$ are said to be *adjacent* if $E(\gamma)$ contains the edge $\{a, b\}$. The *complement* $\overline{\gamma}$ of $\gamma$ is the graph of $\Gamma_n$ in which two vertices $a$ and $b$ are adjacent if and only if they are not adjacent in $\gamma$.

Any finite sequence of edges of $\gamma \in \Gamma_n$

$$\{a_1, a_2\}, \{a_2, a_3\}, \dots, \{a_{s-1}, a_s\}, \{a_s, a_{s+1}\}$$

is said to be a *path of* $\gamma$ *of length* $s$; we will use the notation $[a_1, \dots, a_{s+1}]$ for this path. Note that it is possible that some vertex appears more than once. If $a_{s+1} = a_1$, then it is said to be a *cycle of* $\gamma$ *of length* $s$; we will use the notation $(a_1, \dots, a_s)$ for this cycle. A *triangular chord* of a path $[a_1, \dots, a_r]$ of $\gamma$ with $a_1 \neq a_r$ is one of the edges $\{a_i, a_{i+2}\}$ with $i \in \{1, \dots, r-2\}$. A *triangular chord* of a cycle $(a_1, \dots, a_s)$ of $\gamma$ is one of the edges $\{a_i, a_{i+2}\}$ with $i \in \{1, \dots, s-2\}$, or $\{a_{s-1}, a_1\}$, or $\{a_s, a_2\}$.

EXAMPLE. Let $\gamma \in \Gamma_6$ with edge set

$$E(\gamma) = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{2,6\},\{3,6\}\}.$$

Consider the cycles $c = (4,3,2,6,5)$ and $c' = (4,3,2,1,2,6,5)$: $\{3,6\}$ is a triangular chord of $c$, but $c'$ has no triangular chord.

Given $\gamma \in \Gamma_n$, there exists a partition of $E(\gamma)$

$$E(\gamma) = E_1(\gamma) \cup E_2(\gamma) \cup \cdots \cup E_s(\gamma)$$

such that two edges $\{a, b\}$ and $\{c, d\}$ of $\gamma$ are in the same element of the partition if and only if there exists a path $[a_1 = a,\ a_2 = b, \ldots, a_{t-1} = c,\ a_t = d]$ of $\gamma$ with $\{a_j,\ a_{j+2}\} \notin E(\gamma)$ for $j = 1, \ldots, t-2$. For $i = 1, \ldots, s$, $\gamma^i$ will denote a *spanning subgraph* of $\gamma$ (that is, a graph with the same vertex set as $\gamma$ and some of its edges) with edge set $E_i(\gamma)$. We will call each $E_i(\gamma)$ a *color class* of $\gamma$ and each $\gamma^i$ a *color component* of $\gamma$.

Let $\gamma \in \Gamma_n$, and let $\{a, b\}, \{c, d\} \in E(\gamma)$ be two edges of the same color class. We will say that the orientation $a \to b$ of $\{a, b\}$ *forces* the orientation $c \to d$ (respectively, $d \to c$) of $\{c, d\}$ if and only if there exists a path $[a_1 = a,\ a_2 = b, \ldots, a_{t-1} = c,\ a_t = d]$ of $\gamma$ of odd (respectively, even) length with $\{a_j,\ a_{j+2}\} \notin E(\gamma)$ for $j = 1, \ldots, t-2$. For short we write that $a \to b$ *forces* $c \to d$ or $a \to b \Rightarrow c \to d$. Note that it is possible for $a \to b$ to force both $c \to d$ and $d \to c$.

We say that $\gamma \in \Gamma_n$ is a *comparability graph* (or that $\gamma$ is *transitively orientable*) provided it is possible to orient each edge of $\gamma$ so that the resulting digraph satisfies the transitive law

$$a \to b, \quad b \to c \quad \text{implies} \quad a \to c.$$

Such an orientation is called a *transitive orientation* of $\gamma$.

In the next theorem, condition (ii) is the usual characterization of comparability graphs due to Gilmore and Hoffman [4]. Conditions (iii) and (iv), although stated in a different way, are due to Golumbic [5].

THEOREM 1. *Given $\gamma \in \Gamma_n$, the following statements are equivalent*:

(i) *$\gamma$ is a comparability graph*;
(ii) *any cycle of $\gamma$ of odd length has a triangular chord*;
(iii) *any color component of $\gamma$ is a comparability graph*;
(iv) *there does not exist any $\{a, b\} \in E(\gamma)$ such that $a \to b$ forces $b \to a$.*

It follows from Theorem 1 that given a comparability graph $\gamma \in \Gamma_n$, if we assign an orientation $a \to b$ to $\{a, b\} \in E(\gamma)$, then $a \to b$ forces a unique orientation in each edge of $E(\gamma)$ that belongs to the color class $E_i(\gamma)$ containing $\{a, b\}$. Orienting each edge of $E_i(\gamma)$ with the orientation induced by $a \to b$, we get a transitive orientation of $\gamma^i$. Moreover, it is possible to assign an orientation to one edge of each color class of $\gamma$ in such a way that, orienting all edges of $\gamma$ with the induced orientations, we get a transitive orientation of $\gamma$.

We can construct an algorithm that divides the edge set of a graph into its color classes, and another algorithm that decides for each color component of a graph whether it is a comparability graph (see [5]).

In fact, if $\gamma \in \Gamma_n$ with $n$ not too large, then it is possible to check by hand whether $\gamma$ is a comparability graph. We give an example: Let $\gamma \in \Gamma_6$ be again the graph with edge set

$$E(\gamma) = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{2,6\},\{3,6\}\}.$$

Clearly $\gamma$ has only one color class. Assign an arbitrary orientation to one edge of $\gamma$, for example, the orientation $1 \to 2$ to the edge $\{1,2\}$. If $\gamma$ were a comparability graph, then $1 \to 2$ would force a unique orientation for each edge of $\gamma$. But

$$\begin{aligned} 1 \to 2 \quad &\Rightarrow \quad 6 \to 2 \quad \Rightarrow \quad 6 \to 5 \quad \Rightarrow \quad 4 \to 5 \quad \Rightarrow \quad 4 \to 3 \\ &\Rightarrow \quad 2 \to 3 \quad \Rightarrow \quad 2 \to 1. \end{aligned}$$

and therefore we conclude that $\gamma$ is not a comparability graph.

## 3. CHARACTERIZATION OF *-GRAPHS OF EXTREME GTT $(0, \frac{1}{2}, 1)$ MATRICES

Let $G_n$ denote the set composed of the undirected graphs with $n$ *nonnumbered* vertices. All definitions given in Section 2 for graphs $\gamma \in \Gamma_n$ are easily adapted for graphs $g \in G_n$. Two graphs $\gamma \in \Gamma_n$ and $g \in G_n$ are said to be *isomorphic* if there exists a bijection between the vertices of $\gamma$ and $g$ that preserves adjacency.

Following Brualdi and Hwang [1], given a GTT matrix $T = [t_{ij}]$ of order $n$, we consider the **-graph* of $T$, that is, the graph $\gamma_T \in \Gamma_n$ in which $\{i, j\} \in E(\gamma_T)$ if and only if $0 < t_{ij} < 1$. In considering *-graphs it suffices to consider only GTT $(0, \frac{1}{2}, 1)$ matrices, since the matrix obtained from a GTT matrix by replacing each nonintegral entry with $\frac{1}{2}$ is also a GTT matrix.

A graph $g \in G_n$ is called *GTT-realizable* provided that there exists a GTT matrix $T$ whose *-graph $\gamma_T$ is isomorphic to $g$.

THEOREM 2 (Brualdi and Hwang [1]). *A graph $g \in G_n$ is GTT-realizable if and only if its complement $\bar{g}$ is a comparability graph.*

Brualdi and Hwang [1] show that if $g \in G_n$ is a comparability graph with at least one edge, then $g$ is not isomorphic to the *-graph of any extreme GTT $(0, \frac{1}{2}, 1)$ matrix. We present the following stronger result,

THEOREM 3. *Let $g \in G_n$ be a GTT-realizable graph with at least one edge. If some color component of $g$ is a comparability graph, then $g$ is not isomorphic to the *-graph of any extreme GTT $(0, \frac{1}{2}, 1)$ matrix.*

*Proof.* Let $T = [t_{ij}]$ be a GTT $(0, \frac{1}{2}, 1)$ matrix whose *-graph $\gamma = \gamma_T$ is isomorphic to $g$. By hypothesis, there exists a color component $\gamma^r$ of $\gamma$ which is a comparability graph. Consider the graph $\gamma^r$ provided with a transitive orientation. Then for $\varepsilon \in \mathbb{R}$ define the matrix $T_\varepsilon^r = [t_{ij}^r]$ as follows:

$$t_{ij}^r = \begin{cases} t_{ij} & \text{if } \{i, j\} \textit{ is not an edge of } \gamma^r, \\ t_{ij} + \varepsilon & \text{if } \{i, j\} \textit{ is an edge of } \gamma^r \textit{ with orientation } i \to j, \\ t_{ij} - \varepsilon & \text{if } \{i, j\} \textit{ is an edge of } \gamma^r \textit{ with orientation } j \to i. \end{cases}$$

For any two distinct $i, j \in \{1, \ldots, n\}$ we have $t_{ij}^r + t_{ji}^r = t_{ij} + t_{ji} = 1$. Given three distinct $i, j, k \in \{1, \ldots, n\}$ we have the following possibilities,

(1) *$\gamma$ contains none of the edges $\{i, j\}, \{j, k\}, \{i, k\}$.* In this case,

$$t_{ij}^r = t_{ij}, \quad t_{jk}^r = t_{jk}, \quad \text{and} \quad t_{ki}^r = t_{ki}.$$

(2) *$\gamma$ contains two of the edges $\{i, j\}, \{j, k\}, \{i, k\}$.* Then both edges are in the same color component of $\gamma$. If this color component is different from $\gamma^r$, then $t_{ij}^r = t_{ij}$, $t_{jk}^r = t_{jk}$, and $t_{ki}^r = t_{ki}$. Suppose this color component is $\gamma^r$. Without loss of generality we can suppose that $\gamma^r$ contains the edges $\{i, j\}$ and $\{i, k\}$. As $\gamma^r$ is provided with a transitive orientation, then if $i \to j$ it follows that $i \to k$, and if $j \to i$ it follows that $k \to i$; in both cases

$$t_{ij}^r + t_{jk}^r + t_{ki}^r = t_{ij} + t_{jk} + t_{ki} = 1 \text{ or } 2.$$

(3) *$\gamma$ contains one or three of the edges* $\{i, j\}$, $\{j, k\}$, $\{i, k\}$. Then

$$t_{ij}^{t} + t_{jk}^{r} + t_{ki}^{r} \in \left[t_{ij} + t_{jk} + t_{ki} - 3\varepsilon, t_{ij} + t_{jk} + t_{ki} + 3\varepsilon\right]$$

$$= \left[\tfrac{3}{2} - 3\varepsilon, \tfrac{3}{2} + 3\epsilon\right].$$

Therefore, for any $\varepsilon \in [-\frac{1}{6}, \frac{1}{6}]$, $T_{\varepsilon}^{r}$ is a GTT matrix and $T = \frac{1}{2}(T_{\varepsilon}^{r} + T_{-\varepsilon}^{r})$, which implies that $T$ is not extreme ■

THEOREM 4. *Let $g \in G_n$ be a GTT-realizable graph such that no color component of $g$ is a comparability graph, and let $T$ be a GTT $(0, \frac{1}{2}, 1)$ matrix whose *-graph $\gamma_T$ is isomorphic to $g$. Then $T$ is an extreme GTT matrix.*

*Proof.* Let $T = [t_{ij}]$ be equal to $\frac{1}{2}(R + S)$, where $R = [r_{ij}]$ and $S = [s_{ij}]$ are GTT matrices. We will show that $R = S = T$, which implies that $T$ is extreme.

(1) If $\gamma_T$ does not possess the edge $\{i, j\}$, then $t_{ij} = 0$ or $1$ and $r_{ij} = s_{ij} = t_{ij}$; otherwise $r_{ij}$ or $s_{ij}$ would be less than 0 or greater than 1.

(2) On the other hand, as no color component of $\gamma_T$ is a comparability graph, Theorem 1 implies that every color component of $\gamma_T$ has one edge $\{a_1, a_2\}$ such that $a_1 \to a_2$ forces $a_2 \to a_1$. It is not difficult to extend the same result to every edge of $\gamma_T$.

Let $\{i_1, i_2\}$ be any edge of $\gamma_T$. As $i_1 \to i_2$ forces $i_2 \to i_1$, then $\{i_1, i_2\}$ is contained in some cycle of $\gamma_T$ of odd length without triangular chords. Let $c = (i_1, i_2, \dots, i_{2n+1})$ be such a cycle. As

$$t_{i_1,i_2} + t_{i_2,i_3} + t_{i_3,i_1} = 1 \text{ or } 2,$$

then it follows that

$$r_{i_1,i_2} + r_{i_2,i_3} + r_{i_3,i_1} = s_{i_1,i_2} + s_{i_2,i_3} + s_{i_3+i_1} = t_{i_1,i_2} + t_{i_2,i_3} + t_{i_3,i_1};$$

otherwise

$$r_{i_1,i_2} + r_{i_2,i_3} + r_{i_3,i_1} \quad \text{or} \quad s_{i_1,i_2} + s_{i_2,i_3} + s_{i_3,i_1}$$

would be less than 1 or greater than 2. As

$$r_{i_3,i_1} = s_{i_3,i_1} = t_{i_3,i_1} = 0 \text{ or } 1,$$

it follows that

$$r_{i_2,i_3} = 1 - r_{i_1,i_2}.$$

Repeating this argument, we conclude

$$r_{i_3,i_4} = 1 - r_{i_2,i_3} = r_{i_1,i_2} \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad r_{i_{2n+1},i_1} = r_{i_1,i_2}$$
$$\Rightarrow \quad r_{i_1,i_2} = 1 - r_{i_1,i_2},$$

which implies

$$r_{i_1,i_2} = \tfrac{1}{2} = t_{i_1,i_2}.$$

From (1) and (2) we conclude that $R = T$, and therefore $S = T$ too. ■

We recall that each GTT-realizable graph is GTT-realizable by at least one GTT $(0, \frac{1}{2}, 1)$ matrix. Therefore Theorems 2, 3, and 4 imply the following characterization of the *-graphs of extreme GTT $(0, \frac{1}{2}, 1)$ matrices of order $n$ and of the extreme GTT $(0, \frac{1}{2}, 1)$ matrices of order $n$:

THEOREM 5. *A graph $g \in G_n$ is isomorphic to the *-graph of some extreme GTT $(0, \frac{1}{2}, 1)$ matrix if and only if*

(i) *no color component of g is a comparability graph, and*
(ii) *its complement $\bar{g}$ is a comparability graph.*

*A GTT $(0, \frac{1}{2}, 1)$ matrix is extreme if and only if its *-graph satisfies* (i) *and* (ii). ■

As we pointed out in the introduction, it is known that for $n \leqslant 5$ a GTT matrix $T$ of order $n$ is extreme if and only if $T$ is a TT matrix of order $n$; therefore the *-graph of any extreme GTT matrix of order $n \leqslant 5$ is the graph $\gamma \in \Gamma_n$ with edge set $E(\gamma) = \varnothing$. For $n = 6, 7$ we will calculate using Theorem 5 the complete list of graphs of $G_n$ which are isomorphic to the *-graph of some extreme GTT $(0, \frac{1}{2}, 1)$ matrix. We need the help of a computer. The scheme that we have followed is:

(1) We construct an algorithm for obtaining a subset $\Gamma_n'$ of $\Gamma_n$ such that for each $g \in G_n$ there exists one and only one $\gamma \in \Gamma_n'$ such that $\gamma$ is isomorphic to $g$. We identify $\Gamma_n'$ with $G_n$.

(2) As we pointed out at the end of Section 2, we can construct an algorithm that divides the edge set of a graph into its color classes and another algorithm that decides for each color component of a graph whether it is a comparability graph. Using them, we can obtain the subset $\Gamma_n''$ of $\Gamma_n'$

composed of those graphs $\gamma \in \Gamma_n'$ such that: (i) no color component of $\gamma$ is a comparability graph, and (ii) each color component of its complement $\bar{\gamma}$ is a comparability graph.

Now we give the results we have obtained.

THEOREM 6.

(i) *The graphs of* $G_6$ *which are isomorphic to the* *-*graph of some extreme GTT* $(0, \frac{1}{2}, 1)$ *matrix of* $\mathcal{T}_6$ *are given in Figure* 1.

(ii) *The graphs of* $G_7$ *which are isomorphic to the* *-*graph of some extreme GTT* $(0, \frac{1}{2}, 1)$ *matrix of* $\mathcal{T}_7$ *are given in Figure* 2.

NOTE. All graphs in Figure 1 are known to be isomorphic to the *-graph of some extreme GTT $(0, \frac{1}{2}, 1)$ matrix of order 6 (see [1]).

## 4. EXTREME GTT $(0, \frac{1}{2}, 1)$ MATRICES OF $\mathcal{T}_6$ AND $\mathcal{T}_7$

Let $\mathcal{T}_n'$ denote the subset of $\mathcal{T}_n$ composed of those matrices $T = [t_{ij}] \in \mathcal{T}_n$ such that if $t_{ij} = 1$ then $i < j$, and $\mathcal{S}_n'$ $(0, \frac{1}{2}, 1)$ the subset of $\mathcal{S}_n$ composed of those $(0, \frac{1}{2}, 1)$ matrices $T = [t_{ij}] \in \mathcal{S}_n$ such that if $t_{ij} = 1$ then $i < j$. Extend the definition of *-graph from $\mathcal{T}_n$ to $\mathcal{S}_n$. We will identify the sets $\mathcal{S}_n'$ $(0, \frac{1}{2}, 1)$ and $\Gamma_n$. Namely, each matrix $T \in \mathcal{S}_n'$ $(0, \frac{1}{2}, 1)$ is identified with its *-graph $\gamma_T$. Equivalently, each graph $\gamma \in \Gamma_n$ is identified with the unique matrix $T(\gamma) \in \mathcal{S}_n'$ $(0, \frac{1}{2}, 1)$ whose *-graph is $\gamma$.

Two matrices $R$ and $S$ are said to be *cogredient* if there exists a permutation matrix $P$ such that $S = PRP^t$. Given $\gamma_1, \gamma_2 \in \Gamma_n$, we will write $\gamma_2 \cong \gamma_1$ to mean that $T(\gamma_2)$ and $T(\gamma_1)$ are cogredient (we will write $\gamma_2 \ncong \gamma_1$ otherwise), and $\gamma_2 \cong \gamma_1^t$ to mean that $T(\gamma_2)$ and $[T(\gamma_1)]^t$ are cogredient. Note that if $\gamma_1$ and $\gamma_2$ are not isomorphic graphs, then $\gamma_2 \ncong \gamma_1$ and $\gamma_2 \ncong \gamma_1^t$.

Given a GTT matrix of order $n$ $T = [t_{ij}]$, define $i \angle j$ to mean $i \neq j$ and $t_{ij} = 1$; then $\angle$ is a partial order on $\{1, \ldots, n\}$. It is well known that every

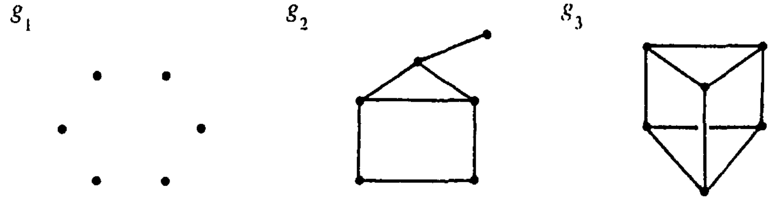


FIG. 1.

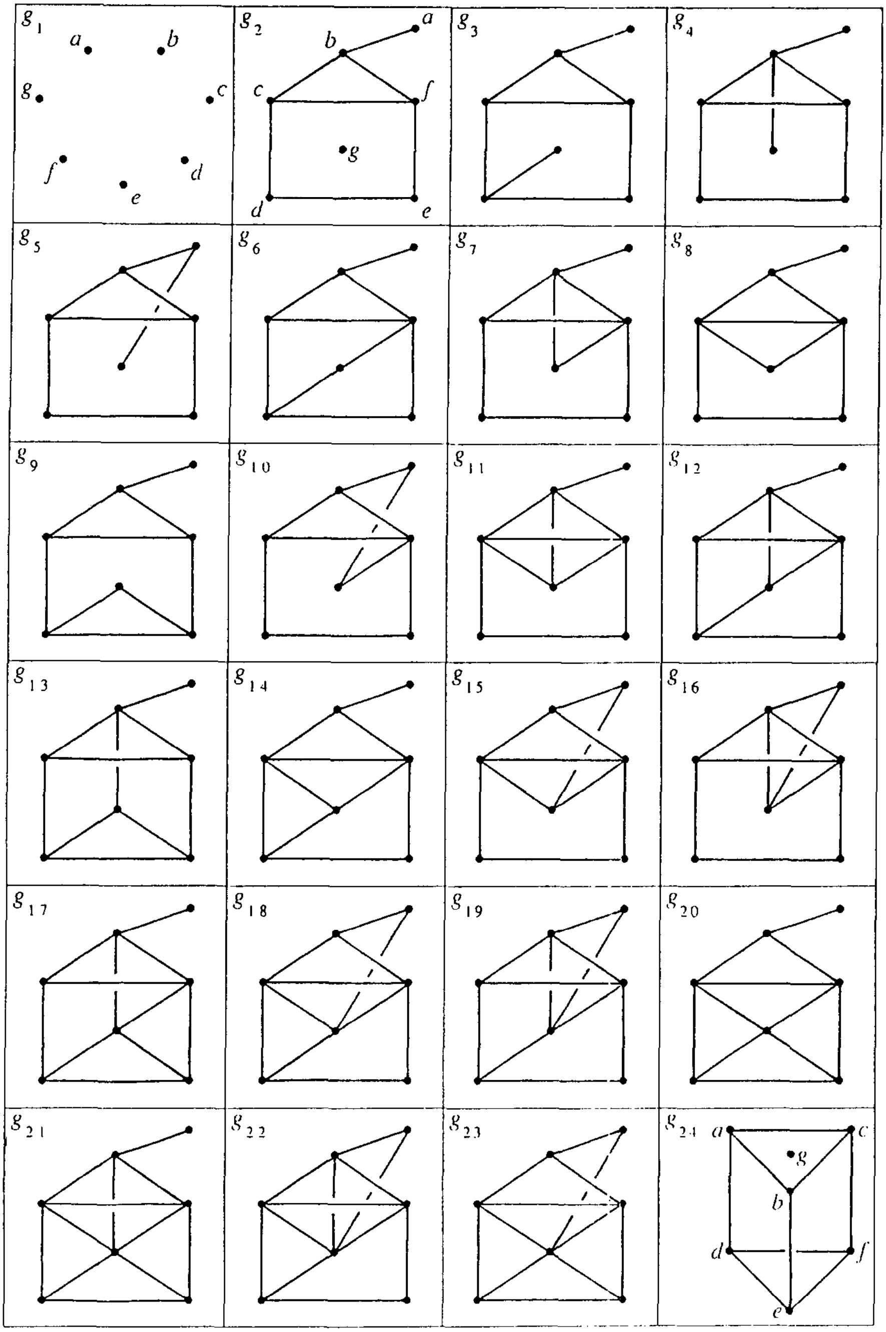
g1
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FIG. 2.

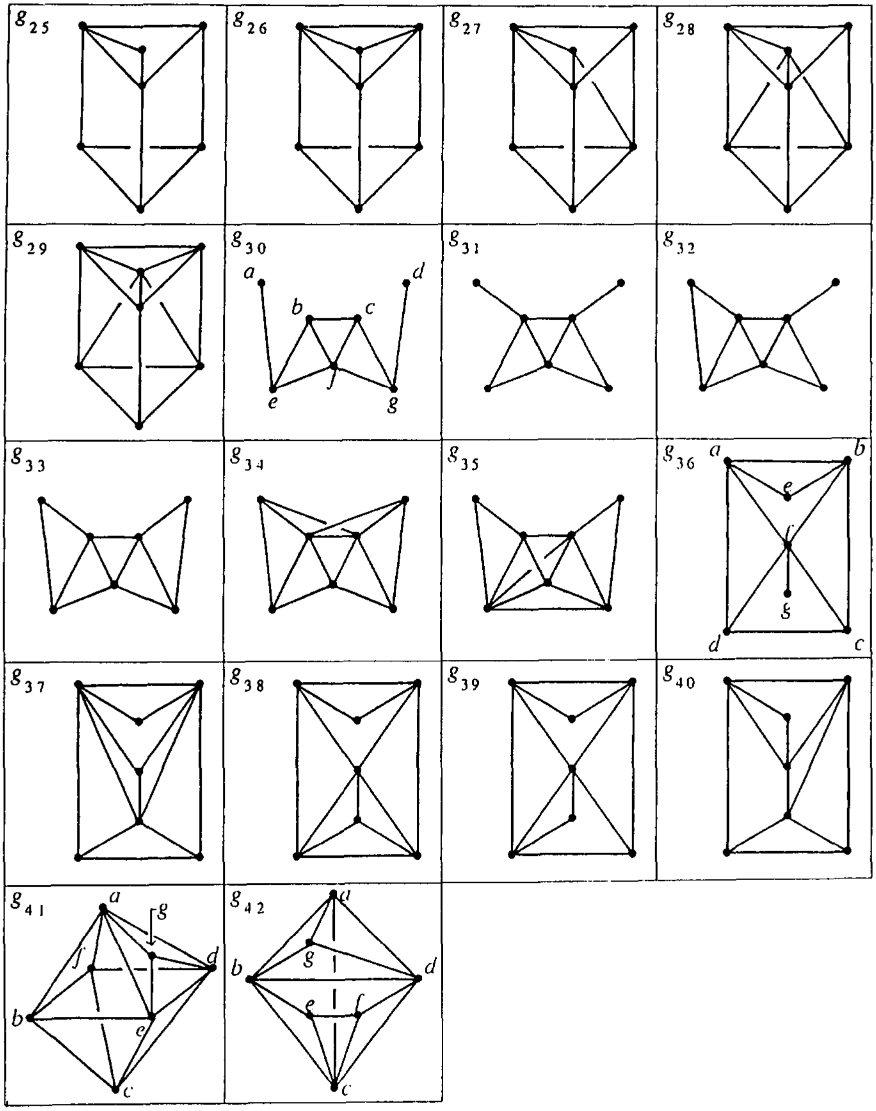


FIG. 2. (Continued)

partial order can be extended to a linear order (Szpilrajn [9]), that every linear order on $\{1, \ldots, n\}$ arises from some TT matrix, and that every TT matrix is cogredient to the matrix with 1's above the main diagonal and 0's elsewhere. It follows that

LEMMA 7. *Every matrix of $\mathcal{T}_n$ is cogredient to some matrix of $\mathcal{T}'_n$.*

The scheme we have followed (with the help of a computer) for obtaining the complete list of extreme GTT $(0, \frac{1}{2}, 1)$ matrices of $\mathcal{T}_6$ and $\mathcal{T}_7$ is:

(1) For each graph $g$ in Figure 1 for $n = 6$, and in Figure 2 for $n = 7$, we have obtained the subset $\Gamma_n(g)$ of $\Gamma_n$ composed of those $\gamma \in \Gamma_n$ such that $\gamma$ is isomorphic to $g$ and $T(\gamma)$ is a GTT matrix. Therefore

$$T(\Gamma_n(g)) := \{T(\gamma) \mid \gamma \in \Gamma_n(g)\} \subset \mathcal{G}'_n(0, \tfrac{1}{2}, 1)$$

is the set composed of all extreme GTT $(0, \frac{1}{2}, 1)$ matrices of $\mathcal{T}_n$ included in $\mathcal{G}'_n$ $(0, \frac{1}{2}, 1)$ whose *-graph is isomorphic to $g$.

(2) Lemma 7 implies that each extreme GTT $(0, \frac{1}{2}, 1)$ matrix of order $n$ whose *-graph is isomorphic to $g$ is cogredient to some matrix of $T(\Gamma_n(g))$. Therefore, it only remains to check for any two graphs $\gamma_1, \gamma_2 \in \Gamma_n(g)$ if $T(\gamma_1)$ and $T(\gamma_2)$ are cogredient (we also will check if $T(\gamma_2)$ and $[T(\gamma_1)]^t$ are cogredient).

We have obtained the following results,

THEOREM 8. *A GTT $(0, \frac{1}{2}, 1)$ matrix of order 6 is extreme if and only if it is cogredient to some $T(\gamma_{i,j}) \in \mathcal{G}'_6(0, \frac{1}{2}, 1)$ where $\gamma_{i,j} \in \Gamma_6$ is one of the graphs in Figure 3. Moreover, $\gamma_{1,1} \cong \gamma^t_{1,1}$, $\gamma_{3,1} \cong \gamma^t_{3,1}$, $\gamma_{2,2} \not\cong \gamma_{2,1}$, and $\gamma_{2,2} \cong \gamma^t_{2,1}$.*

NOTE. $T(\gamma_{2,1})$ is cogredient to the extreme GTT $(0, \frac{1}{2}, 1)$ matrix

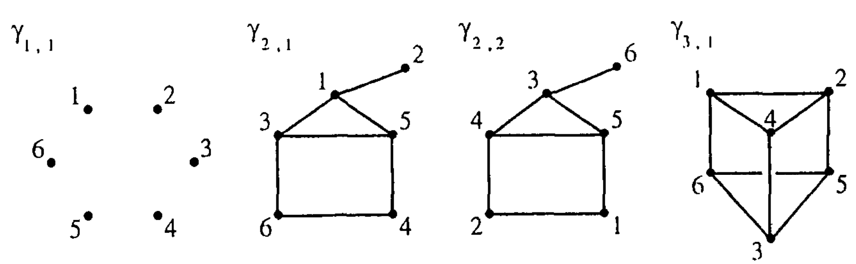


FIG. 3.

of order 6 given by Cruse [2], and $T(\gamma_{3,1})$ is cogredient to the extreme GTT $(0, \frac{1}{2}, 1)$ matrix of order 6 given by Grötschel, Jünger, and Reinelt [6].

For the result in the case $n = 7$ we will employ the following notation:

$$\gamma_{i,j} \to \{k_1, k_2, k_3, k_4, k_5, k_6, k_7\}$$

means the graph of $\Gamma_7$ obtained from the graph $g_i \in G_7$ in Figure 2 by numbering the vertices of $g_i$ as follows: $a = k_1$, $b = k_2$, $c = k_3$, $d = k_4$, $e = k_5$, $f = k_6$, and $g = k_7$ (in Figure 2, when the positions of all vertices of the graph $g_{i+1}$ coincide with the positions of all vertices of the graph $g_i$, we have omitted the lettering of the vertices of $g_{i+1}$; in that case we understand that the lettering of a vertex of $g_{i+1}$ is the same as the lettering of the vertex of $g_i$ situated in the same position).

THEOREM 9. *A GTT* $(0, \frac{1}{2}, 1)$ *matrix of order* 7 *is extreme if and only if it is cogredient to some* $T(\gamma_{i,j}) \in \mathscr{G}'_7(0, \frac{1}{2}, 1)$ *where* $\gamma_{i,j} \in \Gamma_7$ *is one of the following graphs*:

| | | |
|---|---|---|
| $\{\gamma_{1,1} \to \{1,2,3,4,5,6,7\};$ | $\gamma_{2,1} \to \{3,2,4,7,5,6,1\},$ | $\gamma_{2,2} \to \{6,3,5,1,2,4,7\},$ |
| $\gamma_{2,3} \to \{7,4,6,2,3,5,1\},$ | $\gamma_{2,4} \to \{2,1,3,6,4,5,7\};$ | $\gamma_{3,1} \to \{7,4,6,1,3,5,2\},$ |
| $\gamma_{3,2} \to \{2,1,3,6,4,5,7\};$ | $\gamma_{4,1} \to \{1,4,3,7,5,6,2\},$ | $\gamma_{4,2} \to \{6,4,5,1,3,2,7\};$ |
| $\gamma_{5,1} \to \{3,2,6,5,7,4,1\},$ | $\gamma_{5,2} \to \{5,6,2,3,1,4,7\};$ | $\gamma_{6,1} \to \{2,1,3,7,4,6,5\},$ |
| $\gamma_{6,2} \to \{7,4,5,3,1,6,2\};$ | $\gamma_{7,1} \to \{2,1,6,5,7,3,4\},$ | $\gamma_{7,2} \to \{7,4,2,3,1,5,6\};$ |
| $\gamma_{8,1} \to \{2,1,3,7,5,6,4\},$ | $\gamma_{8,2} \to \{7,4,6,1,2,5,3\};$ | $\gamma_{9,1} \to \{2,1,3,6,4,5,7\},$ |
| $\gamma_{9,2} \to \{7,4,6,1,2,5,3\};$ | $\gamma_{10,1} \to \{5,6,2,3,1,4,7\};$ | $\gamma_{11,1} \to \{2,1,3,7,5,6,4\},$ |
| $\gamma_{11,2} \to \{7,3,5,1,2,4,6\};$ | $\gamma_{12,1} \to \{1,4,3,7,5,6,2\},$ | $\gamma_{12,2} \to \{7,5,3,4,1,6,2\};$ |
| $\gamma_{13,1} \to \{2,1,3,6,4,5,7\},$ | $\gamma_{13,2} \to \{7,5,6,1,4,3,2\};$ | $\gamma_{14,1} \to \{2,1,3,7,5,6,4\},$ |
| $\gamma_{14,2} \to \{7,4,6,1,2,5,3\};$ | $\gamma_{15,1} \to \{5,7,4,1,3,2,6\},$ | $\gamma_{15,2} \to \{3,2,4,7,5,6,1\};$ |
| $\gamma_{16,1} \to \{7,4,2,3,1,6,5\},$ | $\gamma_{16,2} \to \{2,4,6,5,7,3,1\};$ | $\gamma_{17,1} \to \{2,1,5,4,7,3,6\},$ |
| $\gamma_{17,2} \to \{7,5,3,4,2,6,1\};$ | $\gamma_{18,1} \to \{5,7,4,1,3,2,6\};$ | $\gamma_{19,1} \to \{2,4,3,7,5,6,1\},$ |
| $\gamma_{19,2} \to \{7,5,2,4,1,6,3\};$ | $\gamma_{20,1} \to \{1,3,2,7,5,6,4\},$ | $\gamma_{20,2} \to \{7,4,6,2,3,5,1\};$ |
| $\gamma_{21,1} \to \{2,1,4,7,5,6,3\},$ | $\gamma_{21,2} \to \{7,4,6,1,3,2,5\};$ | $\gamma_{22,1} \to \{3,2,4,7,5,6,1\},$ |
| $\gamma_{22,2} \to \{7,5,6,1,3,2,4\};$ | $\gamma_{23,1} \to \{1,3,2,7,5,6,4\},$ | $\gamma_{23,2} \to \{5,7,4,1,3,2,6\};$ |
| $\gamma_{24,1} \to \{7,4,6,2,5,3,1\},$ | $\gamma_{24,2} \to \{6,3,5,1,4,2,7\};$ | $\gamma_{25,1} \to \{1,2,5,7,6,4,3\},$ |
| $\gamma_{25,2} \to \{6,5,3,1,2,4,7\};$ | $\gamma_{26,1} \to \{6,3,5,1,4,2,7\},$ | $\gamma_{26,2} \to \{1,5,2,7,4,6,3\};$ |
| $\gamma_{27,1} \to \{7,6,3,1,2,5,4\},$ | $\gamma_{27,2} \to \{3,5,1,6,4,7,2\};$ | $\gamma_{28,1} \to \{1,5,2,7,4,6,3\};$ |
| $\gamma_{29,1} \to \{6,3,7,2,4,1,5\},$ | $\gamma_{29,2} \to \{3,5,2,6,4,7,1\};$ | $\gamma_{30,1} \to \{7,6,4,2,5,3,1\};$ |
| $\gamma_{31,1} \to \{7,4,1,2,6,5,3\};$ | $\gamma_{32,1} \to \{7,4,1,2,5,6,3\},$ | $\gamma_{32,2} \to \{3,1,4,7,2,5,6\};$ |

$\gamma_{33,1} \to \{1,5,4,7,3,2,6\}$; $\gamma_{34,1} \to \{4,1,3,7,2,6,5\}$; $\gamma_{35,1} \to \{7,5,6,1,2,3,4\}$;
$\gamma_{36,1} \to \{5,3,4,2,7,6,1\}$, $\gamma_{36,2} \to \{1,5,4,6,2,3,7\}$; $\gamma_{37,1} \to \{6,1,7,5,2,4,3\}$,
$\gamma_{37,2} \to \{2,4,1,3,7,6,5\}$; $\gamma_{38,1} \to \{2,1,6,5,3,4,7\}$, $\gamma_{38,2} \to \{3,5,1,4,7,6,2\}$;
$\gamma_{39,1} \to \{1,5,4,6,2,3,7\}$, $\gamma_{39,2} \to \{5,3,4,1,7,6,2\}$; $\gamma_{40,1} \to \{5,3,7,4,2,1,6\}$,
$\gamma_{40,2} \to \{3,6,2,4,7,5,1\}$; $\gamma_{41,1} \to \{6,5,7,4,1,3,2\}$, $\gamma_{41,2} \to \{2,3,1,4,6,5,7\}$;
$\gamma_{42,1} \to \{5,4,6,2,1,3,7\}$, $\gamma_{42,2} \to \{1,4,3,6,7,5,2\}\}$.

*Moreover,*

(i) $\gamma_{2,2} \cong \gamma_{2,1}^t$, $\gamma_{2,4} \cong \gamma_{2,3}^t$, *and* $\gamma_{2,j} \not\cong \gamma_{2,k}$ *for* $j \neq k$;
(ii) *if* $i \in J = \{1, 10, 18, 28, 30, 31, 33, 34, 35\}$ *then* $\gamma_{i,1} \cong \gamma_{i,1}^t$;
(iii) *if* $i \in \{1, \ldots, 42\} \setminus (J \cup \{2\})$, *then* $\gamma_{i,2} \not\cong \gamma_{i,1}$ *and* $\gamma_{i,2} \cong \gamma_{i,1}^t$.

## 5. HISTORICAL REMARK

A nonnegative matrix of order $n$ such that all row and column sums are equal to 1 is said to be a *doubly stochastic matrix*. By Birkhoff's theorem, the extreme points of the polytope $\Omega_n$ of doubly stochastic matrices are the $n!$ permutation matrices of order $n$. Let $\Omega_n^0$ be the polytope composed of those doubly stochastic matrices that can be obtained as convex combinations of permutation matrices other than the identity. Mirsky [7] proposed the problem of characterizing $\Omega_n^0$ by a set of linear constrains. Cruse [2] proved that $D = [d_{ij}] \in \Omega_n^0$ if and only if

$$(T, D) := \sum_{i,j} t_{ij} d_{ij} \geqslant 1 \qquad \text{for each} \quad T \in \mathscr{T}_n.$$

Therefore, if the extreme points of $\mathscr{T}_n$ were known, then $\Omega_n^0$ would be characterized by a finite set of linear constraints.

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