Statistical $\sigma$-approximation to max-product operators

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A B S T R A C T

In this paper, using the concept of statistical $\sigma$-convergence which is stronger than statistical convergence, we obtain a statistical $\sigma$-approximation theorem for a general sequence of max-product operators, including Shepard type operators, although its classical limit fails. We also compute the corresponding statistical $\sigma$-rates of the approximation.

1. Introduction

Approximation theory, which has a close relationship with other branches of mathematics, has been used in the theory of polynomial approximation and various domains of functional analysis [1], and in numerical studies of differential and integral operators [2]. In the classical approximation theory, many well-known approximating operators obey the linearity condition. In recent years, Bede et al. [3] have shown that it is possible to find some approximating operators that are not linear, such as the max-product and max–min Shepard type approximating operators. Actually, these operators are pseudo-linear which is a quite effective structure in solving the problems in many branches of applied mathematics, such as image processing [4], differential equations [5,6], idempotent analysis [7] and approximation theory [3,8]. However, so far almost all results regarding approximations by pseudo linear operators are based on the validity of the classical limit of the operators. Using the notion of statistical convergence Duman [9] obtained various statistical approximation theorems for a general sequence of max-product approximating operators, including Shepard type operators, although its classical limit fails. Recently a kind of statistical convergence (statistical $\sigma$-convergence) which is stronger than statistical convergence has been introduced by Mursaleen and Edely [10]. In this paper, using statistical $\sigma$-convergence, we improve the Duman’s results in [9].

Let $K$ be the subset of positive integers. Then the natural density of $K$ is given by $\delta(K) = \lim_{n \to \infty} \frac{1}{n} |K_n|$ if it exists, where $K_n := \{k \in K : k \leq n\}$ and the vertical bars denote the cardinality of the set $K_n$. A sequence $x = \{x_k\}$ is said to be statistically convergent to the number $L$ if for each $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

i.e. if the set $K = K(\varepsilon) := \{k \leq n : |x_k - L| \geq \varepsilon\}$ has natural density zero [11–13]. In this case we write $\text{st} \lim x_n = L$.

Let $\sigma$ be a mapping of the set of $\mathbb{N}$ into itself. A continuous linear functional $\varphi$ defined on the space $l_\infty$ of all bounded sequences is called an invariant mean (or $\sigma$-mean) [14] if and only if

(i) $\varphi(x) \geq 0$ when the sequence $x = \{x_k\}$ has $x_k \geq 0$ for all $k$.

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Lemma 3. Statistical convergence implies statistical $\sigma$-convergence.

However, one can construct an example which guarantees that the converse of Lemma 1 is not always true. Such an example was given in [10] as follows:

Example 2. Consider the case $\sigma(n) = n + 1$ and the sequence $u = \{u_m\}$ defined as

$$u_m = \begin{cases} 1, & \text{if } m \text{ is odd,} \\ -1, & \text{if } m \text{ is even,} \end{cases}$$

is statistically $\sigma$-convergent ($\delta(\sigma) - \lim u_m = 0$) but it is neither convergent nor statistically convergent.

2. Statistical $\sigma$-approximation of max-product operators

Let $(X, d)$ be an arbitrary compact metric space. By $C(X, [0, \infty))$ we denote the space of all non-negative continuous functions on $X$. Then we consider the following max-product operators:

$$L_n(f; x) = \sum_{k=0}^{n} K_n(x, x_k) \cdot f(x_k), \quad x \in X \text{ and } f \in C(X, [0, \infty)),$$

where $x_k \in X$, $k = 0, 1, \ldots, n$, are the knots; and $K_n(x, x_k)$ are non-negative continuous functions on $X$ having relatively simple expression (algebraic or trigonometric polynomials, rational functions, wavelets, etc.) such that, for any $x \in X$,

$$\delta \left( \left\{ p \in \mathbb{N} : \sum_{k=0}^{p} K_{\sigma(p)}(x, x_k) = 1 \right\} \right) = 1 \quad \text{for every } m \in \mathbb{N}.$$ 

holds. Observe that the operators mapping $C(X, [0, \infty))$ into $C(X, [0, \infty))$ are pseudo-linear, i.e., for every $f, g \in C(X, [0, \infty))$ and for any non-negative numbers $\alpha, \beta$,

$$L_n \left( \alpha \cdot f \sqrt{\beta} \cdot g; x \right) = \alpha \cdot L_n(f; x) \sqrt{\beta} \cdot L_n(g; x)$$

is satisfied [see 3].

We first recall the following lemma.

Lemma 3 (3). For any $a_k, b_k \in [0, \infty), \ k = 0, 1, \ldots, n$, we have

$$\left| \sum_{k=0}^{n} a_k - \sum_{k=0}^{n} b_k \right| \leq \sum_{k=0}^{n} |a_k - b_k|.$$ 

Now we get the following result for the max-product operators.
Theorem 4. Let \((X, d)\) be an arbitrary compact metric space. If, for the operators \(L := \{L_n\}\) given by (2) and (3),
\[
\delta(\sigma) - \lim\{|L_n(\varphi_\sigma; x) : x \in X|\} = 0 \quad \text{with } \varphi_\sigma(y) = d^2(y, x)
\]
then, for all \(f \in C(X, [0, \infty))\), we have
\[
\delta(\sigma) - \lim\{|L_n(f; x) - f(x) : x \in X|\} = 0.
\]

Proof. Let \(f \in C(X, [0, \infty))\) and \(x \in X\) be fixed. Then, using the continuity of \(f\) and also considering the compactness of \(X\), we immediately see that, for a given \(\varepsilon > 0\), there exists a positive number \(\delta\) such that
\[
|f(y) - f(x)| \leq \varepsilon + \frac{2M_f}{\delta^2} \varphi_\sigma(y)
\]
holds for all \(y \in X\), where \(M_f \coloneqq \bigvee \{|f(y)| : y \in X\}\). Now put
\[
K := \left\{ p \in \mathbb{N} : \bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x_k, x_k) = 1 \right\}
\]
for every \(m \in \mathbb{N}\). Then, by (3) we may write that
\[
\lim_{n} \frac{|\{p \leq n : p \in K\}|}{n} = 1, \quad \lim_{n} \frac{|\{p \leq n : p \in \mathbb{N}/K\}|}{n} = 0
\]
for every \(m \in \mathbb{N}\). So, by (3) and (5) and Lemma 3, we get for all \(p \in K\), that
\[
|t_{pm}(L(f; x)) - f(x)|
\]
\[
= \left| \frac{L_m(f; x) + L_{\sigma(m)}(f; x) + \cdots + L_{\sigma(m)}(f; x) - f(x)}{p + 1} \right|
\]
\[
= \left| \frac{(L_m(f; x) - f(x)) + (L_{\sigma(m)}(f; x) - f(x)) + \cdots + (L_{\sigma(m)}(f; x) - f(x))}{p + 1} \right|
\]
\[
= \left( \frac{\bigvee_{k=0}^{m} K_m(x_k, x_k) \cdot f(x_k) - \bigvee_{k=0}^{m} K_{\sigma(m)}(x_k, x_k) \cdot f(x_k)}{p + 1} \right)
\]
\[
+ \left( \frac{\bigvee_{k=0}^{m} K_{\sigma(m)}(x_k, x_k) \cdot f(x_k) - \bigvee_{k=0}^{m} K_{\sigma(m)}(x_k, x_k) \cdot f(x_k)}{p + 1} \right)
\]
\[
\leq \frac{\bigvee_{k=0}^{m} K_m(x_k, x_k) \cdot |f(x_k) - f(x)| + \bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x_k, x_k) \cdot |f(x_k) - f(x)|}{p + 1} + \cdots + \frac{\bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x_k, x_k) \cdot |f(x_k) - f(x)|}{p + 1}
\]
\[
\leq \frac{\bigvee_{k=0}^{m} K_m(x_k, x_k) \cdot \left( \varepsilon + \frac{2M_f}{\delta^2} \varphi_\sigma(x_k) \right) + \bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x_k, x_k) \cdot \left( \varepsilon + \frac{2M_f}{\delta^2} \varphi_\sigma(x_k) \right)}{p + 1}
\]
\[
+ \cdots + \frac{\bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x_k, x_k) \cdot \left( \varepsilon + \frac{2M_f}{\delta^2} \varphi_\sigma(x_k) \right)}{p + 1}
\]
\[
\leq \left( \varepsilon + \frac{2M_f}{\delta^2} \bigvee_{k=0}^{m} K_m(x_k, x_k) \cdot \varphi_\sigma(x_k) \right) + \left( \varepsilon + \frac{2M_f}{\delta^2} \bigvee_{k=0}^{\sigma(m)} K_{\sigma(m)}(x_k, x_k) \cdot \varphi_\sigma(x_k) \right)
\]
\[
+ \cdots
\]
Theorem 6

If the sequence \( \lim_{n \to \infty} \frac{K_m(x, x_k) \cdot \varphi_x(x_k)}{p + 1} \) then, using

\[
\frac{\varepsilon + \frac{2M_f}{\delta^2}}{p + 1} \left[ \sum_{k=0}^{m} K_m(x, x_k) \cdot \varphi_x(x_k) + \frac{\sigma(m)}{p + 1} \sum_{k=0}^{m} K_m(x, x_k) \cdot \varphi_x(x_k) + \cdots + \frac{\sigma(m)}{p + 1} \sum_{k=0}^{m} K_m(x, x_k) \cdot \varphi_x(x_k) \right]
\]

\[
= \varepsilon + \frac{2M_f}{\delta^2} \left[ L_m(\varphi_x; x) + L_{\sigma(m)}(\varphi_x; x) + \cdots + L_{\sigma(m)}(\varphi_x; x) \right]
\]

\[
= \varepsilon + \frac{2M_f}{\delta^2} \cdot t_{pm}(L(\varphi_x; x)).
\]

Now, taking the maximum over \( x \in X \), the last inequality gives, for all \( p \in K \), that

\[
\sqrt{\{ |t_{pm}(L(f; x)) - f(x)| : x \in X \}} \leq \varepsilon + \frac{2M_f}{\delta^2} \sqrt{\{ |t_{pm}(L(\varphi_x; x))| : x \in X \}}.
\]

For a given \( r > 0 \), choose an \( \varepsilon > 0 \) such that \( \varepsilon < r \). Then, it follows from (8) that

\[
\left[ \{ p \leq n : (\sqrt{\{ |t_{pm}(L(f; x)) - f(x)| : x \in X \}) \geq r \} \right]
\]

\[
= \left[ \{ p \leq n : p \in K \text{ and } (\sqrt{\{ |t_{pm}(L(f; x)) - f(x)| : x \in X \}) \geq r \} \right]
\]

\[
+ \left[ \{ p \leq n : p \in \mathbb{N} / K \text{ and } (\sqrt{\{ |t_{pm}(L(f; x)) - f(x)| : x \in X \}) \geq r \} \right]
\]

\[
\leq \left[ \{ p \leq n : p \in K \text{ and } (\sqrt{\{ |t_{pm}(L(\varphi_x; x))| : x \in X \}) \geq \sqrt{\frac{(r-\varepsilon)^2}{2M_f}} \} \right]
\]

\[
+ \left[ \{ p \leq n : p \in \mathbb{N} / K \text{ and } (\sqrt{\{ |t_{pm}(L(\varphi_x; x))| : x \in X \}) \geq r \} \right]
\]

\[
\leq \left[ \{ p \leq n : (\sqrt{\{ |t_{pm}(L(\varphi_x; x))| : x \in X \}) \geq \sqrt{\frac{(r-\varepsilon)^2}{2M_f}} \} \right] + \left[ \{ p \leq n : p \in \mathbb{N} / K \} \right].
\]

Then, using (7) and the hypothesis (4), we have

\[
\lim_{n \to \infty} \frac{\left[ \{ p \leq n : (\sqrt{\{ |t_{pm}(L(f; x)) - f(x)| : x \in X \}) \geq r \} \right]}{n}, \quad \text{uniformly in } m
\]

for every \( r > 0 \). The proof is complete. \( \square \)

The following theorems give the classical and the statistical approximation to a function \( f \in C(X, [0, \infty)) \) by means of the max-product operators \( L_m \), respectively.

**Theorem 5** ([9]). Let \( (X, d) \) be an arbitrary compact metric space. Assume that the operators \( L_m \) given by (2) satisfy the condition

\[
\sqrt{K_m(x, x_k)} = 1 \quad (\text{for } n \in \mathbb{N} \text{ and } x \in X).
\]

If the sequence \( \{L_n(\varphi_x; x)\}_{n \in \mathbb{N}} \) converges uniformly to zero function with respect to \( x \in X \), then, for all \( f \in C(X, [0, \infty)) \), \( \{L_n(f; x)\}_{n \in \mathbb{N}} \) is also uniformly convergent to \( f(x) \) with respect to \( x \in X \).

**Theorem 6** ([9]). Let \( (X, d) \) be an arbitrary compact metric space. If, for the operators \( L_m \) given by (2) and

\[
\delta \left( \{ n \in \mathbb{N} : \sqrt{K_m(x, x_k)} = 1 \} \right) = 1,
\]

\[
st \lim \left[ \{ |L_n(\varphi_x; x)| : x \in X \} \right] = 0 \quad \text{with } \varphi_y = d^2(y, x),
\]
then, for all \( f \in C(X, [0, \infty)) \), we have
\[
\text{st} - \lim \left\{ |\mathcal{L}_n(f; x) - f(x)| : x \in X \right\} = 0.
\]

**Remark 7.** We now show that our result Theorem 4 is stronger than its classical version (Theorem 5) and statistical version (Theorem 6).

Let \((X, d)\) be an arbitrary compact metric space. Consider the Shepard-type max-product operators (see [8]) as follows:
\[
S_m^\delta(f; x) = \sum_{k=0}^{m} \frac{1}{\delta(x)} \left( \sum_{j=0}^{\delta(x)} \frac{1}{\delta(x)} \right)^m f(x_j) = \frac{m}{\delta(x)} \mathcal{L}_m(f; x),
\]
(9)
where \( x \in X, \lambda, m \in \mathbb{N} \) and \( f \in C(X, [0, \infty)) \). We know from [8] that, for all \( f \in C(X, [0, \infty)) \), the sequence \( \{S_m^\delta(f; x)\} \) in (9) is uniformly convergent to \( f \) on \( X \). Now, consider the case \( \sigma(n) = n + 1 \). Then, we define the max-product operators on \( C(X, [0, \infty)) \) as
\[
T_m(f; x) = (1 + u_m) S_m^\delta(f; x), \quad x \in X \text{ and } f \in C(X, [0, \infty))
\]
(10)
where the operators \( S_m \) are given by (9) and \( u = \{u_m\} \) is given by (1). Since \( \delta(\sigma) - \lim u_m = 0 \), we observe that the sequence of positive linear operators \( T_m \) defined by (10) satisfy all hypotheses of Theorem 4. Therefore, for all \( f \in C(X, [0, \infty)) \), we conclude that
\[
\delta(\sigma) - \lim \left\{ |T_m(f; x) - f(x)| : x \in X \right\} = 0.
\]
However, since \( \{u_m\} \) is not convergent and statistically convergent, we conclude that Theorems 5 and 6 do not work for the operators \( T_m \) in (10) while our Theorem 4 still works.

### 3. Rate of statistical \( \sigma \)-convergence

In this section we compute the rate of statistical \( \sigma \)-convergence of Theorem 4.

**Definition 8.** A bounded sequence \( x = \{x_n\} \) is statistically \( \sigma \)-convergent to a number \( L \) with the rate of \( \beta \in (0, 1) \) if for every \( \varepsilon > 0 \)
\[
\lim_n \left| \left\{ p \leq n : \frac{|t_{pn}(x) - L|}{n^{1-\beta}} \geq \varepsilon \right\} \right| = 0, \quad \text{uniformly in } m.
\]

Then, this is denoted by
\[
x_n - L = o(n^{-\beta})(\delta(\sigma)).
\]

Using this definition, we obtain the following auxiliary result.

**Lemma 9.** Let \( x = \{x_n\} \) and \( y = \{y_n\} \) be bounded sequences. Assume that \( x_n - L_1 = o(n^{-\beta_1})(\delta(\sigma)) \) and \( y_n - L_2 = o(n^{-\beta_2})(\delta(\sigma)) \). Then we have

(i) \((x_n - L_1) \mp (y_n - L_2) = o(n^{-\beta})(\delta(\sigma))\), where \( \beta := \min \{\beta_1, \beta_2\}\),

(ii) \( \lambda(x_n - L_1) = o(n^{-\beta_1})(\delta(\sigma)) \), for any real number \( \lambda \).

**Proof.** (i) Assume that \( x_n - L_1 = o(n^{-\beta_1})(\delta(\sigma)) \) and \( y_n - L_2 = o(n^{-\beta_2})(\delta(\sigma)) \). Then, for \( \varepsilon > 0 \), observe that
\[
\frac{\left| \left\{ p \leq n : \left| t_{pn}(x) - L_1 \right| \geq \frac{\varepsilon}{2} \right\} \right|}{n^{1-\beta_1}} \leq \frac{\left| \left\{ p \leq n : \left| t_{pn}(x) - L_1 \right| \geq \frac{\varepsilon}{2} \right\} \right|}{n^{1-\beta_1}} + \frac{\left| \left\{ p \leq n : \left| t_{pn}(y) - L_2 \right| \geq \frac{\varepsilon}{2} \right\} \right|}{n^{1-\beta_2}}.
\]

Now by taking the limit as \( n \to \infty \) in (11) and using the hypotheses, we conclude that
\[
\lim_n \frac{\left| \left\{ p \leq n : \left| t_{pn}(x) - L_1 \right| \mp (t_{pn}(y) - L_2) \geq \varepsilon \right\} \right|}{n^{1-\beta}} = 0, \quad \text{uniformly in } m,
\]
which completes the proof of (i). Since the proof of (ii) is similar, we omit it. \( \square \)
We also need the following lemma.

**Lemma 10 ([9])**. For every \( a_k, b_k \geq 0 \) \((k = 0, 1, \ldots, n)\) we have

\[
\sum_{k=0}^{n} a_k b_k = \sqrt{\sum_{k=0}^{n} a_k^2} \sqrt{\sum_{k=0}^{n} b_k^2}.
\]

Now we recall the concept of the modulus of continuity. Let \( f \in C(X, [0, \infty)) \). Then the function \( \omega(f, \cdot) : [0, \infty) \rightarrow [0, \infty) \) defined by

\[
\omega(f, \delta) = \sqrt{\int [f(x) - f(y)] : x, y \in X, d(x, y) \leq \delta}
\]

is called the modulus of continuity of \( f \). In order to obtain our result, we will make use of the elementary inequality, for all \( f \in C(X, [0, \infty)) \) and for \( \lambda, \delta \in [0, \infty) \),

\[
\omega(f, \lambda \delta) \leq (\lambda + 1) \omega(f, \delta).
\]

Then we have the following result.

**Theorem 11.** Let \((X, d)\) be an arbitrary compact metric space. If the operators \( L := \{ L_\alpha \} \) given by (2) and (3) satisfy that

\[
\omega(f, \alpha_{pm}) = o(n^{-p}) (\delta (\sigma)) \quad \text{on } X
\]

where \( \alpha_{pm} := \sqrt{\int [f(x) - f(y)] : x \in X} \), then we have for all \( f \in C(X, [0, \infty)) \),

\[
\sqrt{\int [f_\lambda (x) - f(x)] : x \in X} = o(n^{-p}) (\delta (\sigma)) \quad \text{on } X.
\]

**Proof.** Let \( f \in C(X, [0, \infty)) \) and \( x \in X \) be fixed. Consider the set \( K \) given by (6), we can write for every \( p \in K \) and for any \( \delta > 0 \), that

\[
|t_{pm} (L(f; x)) - f(x)|
\]

\[
= \left| L_m (f; x) + L_{\sigma(m)} (f; x) + \cdots + L_{\sigma^p(m)} (f; x) - f(x) \right|
\]

\[
\leq \left| \left( \sum_{k=0}^{m} K_{\alpha} (x, x_k) \cdot [f(x_k) - f(x)] \right) + \left( \sum_{k=0}^{\sigma(m)} K_{\sigma(m)} (x, x_k) \cdot [f(x_k) - f(x)] \right) \right|
\]

\[
\leq \left( \sum_{k=0}^{m} K_{\alpha} (x, x_k) \cdot \omega(f, d(x_k, x)) \right) + \left( \sum_{k=0}^{\sigma(m)} K_{\sigma(m)} (x, x_k) \cdot \omega(f, d(x_k, x)) \right)
\]

\[
\leq \left( \sum_{k=0}^{m} K_{\alpha} (x, x_k) \cdot \left( 1 + \frac{d(x_k, x)}{\delta} \right) \right) + \left( \sum_{k=0}^{\sigma(m)} K_{\sigma(m)} (x, x_k) \cdot \left( 1 + \frac{d(x_k, x)}{\delta} \right) \right)
\]

\[
= \omega(f, \delta) \left( \sum_{k=0}^{m} K_{\alpha} (x, x_k) + \sum_{k=0}^{\sigma(m)} K_{\sigma(m)} (x, x_k) \right)
\]

\[
+ \left( \sum_{k=0}^{m} K_{\alpha} (x, x_k) \cdot d(x_k, x) + \sum_{k=0}^{\sigma(m)} K_{\sigma(m)} (x, x_k) \cdot d(x_k, x) \right)
\]

\[
= \omega(f, \delta) \left( \sum_{k=0}^{m} K_{\alpha} (x, x_k) + \sum_{k=0}^{\sigma(m)} K_{\sigma(m)} (x, x_k) \right)
\]

\[
+ \left( \sum_{k=0}^{m} K_{\alpha} (x, x_k) \cdot d(x_k, x) + \sum_{k=0}^{\sigma(m)} K_{\sigma(m)} (x, x_k) \cdot d(x_k, x) \right)
\]

\[
= \omega(f, \delta) \left( \sum_{k=0}^{m} K_{\alpha} (x, x_k) + \sum_{k=0}^{\sigma(m)} K_{\sigma(m)} (x, x_k) \right)
\]

\[
+ \left( \sum_{k=0}^{m} K_{\alpha} (x, x_k) \cdot d(x_k, x) + \sum_{k=0}^{\sigma(m)} K_{\sigma(m)} (x, x_k) \cdot d(x_k, x) \right)
\]
Now, by using Lemma 10, we immediately see that
\[ |t_{pm} (L(f; x)) - f(x)| \leq \omega(f, \delta) \left\{ 1 + \frac{1}{\delta} \left( \sqrt{L \left( \frac{d^2(., x)}{x} \right)} \right) \right\} \]
holds for every \( p \in K \) and for any \( \delta > 0 \). Now taking the maximum over \( x \in X \), the last inequality gives for all \( p \in K \) and \( \delta > 0 \), that
\[ \sqrt{\{ |t_{pm} (L(f; x)) - f(x)| : x \in X \} } \leq \omega(f, \delta) \left\{ \frac{1}{\delta} \left( \sqrt{L \left( \frac{d^2(., x)}{x} \right)} \right) \right\} . \]
So, we get
\[ \sqrt{\{ |t_{pm} (L(f; x)) - f(x)| : x \in X \} } \leq 2\omega(f, \delta) \tag{13} \]
where \( \delta := \alpha_{pm} := \sqrt{\{ t_{pm} \left( \sqrt{L \left( \frac{d^2(., x)}{x} \right)} \right) : x \in X \} } \). Hence, given \( \varepsilon > 0 \), it follows from (13) that
\[ \left| \{ p \leq n : (\sqrt{\{ |t_{pm} (L(f; x)) - f(x)| : x \in X \} } ) \geq \varepsilon \} \right| \]
\[ = \left| \{ p \leq n : p \in K \text{ and } (\sqrt{\{ |t_{pm} (L(f; x)) - f(x)| : x \in X \} } ) \geq \varepsilon \} \right| \]
\[ + \left| \{ p \leq n : p \in K \text{ and } (\sqrt{\{ |t_{pm} (L(f; x)) - f(x)| : x \in X \} } ) \geq \varepsilon \} \right| \]
\[ \leq \left| \{ p \leq n : p \in K \text{ and } \omega(f, \delta) \geq \frac{\varepsilon}{2} \} \right| + \left| \{ p \leq n : p \in K \} \right| \]
\[ \leq \left| \{ p \leq n : \omega(f, \delta) \geq \frac{\varepsilon}{2} \} \right| + \left| \{ p \leq n : p \in K \} \right| . \]
Then using (7) and the hypothesis (12), we have
\[ \sqrt{\{ |L(f; x)) - f(x)| : x \in X \} } = o(n^{-\beta}) (\delta(\sigma)) \text{ on } X. \]
The proof is complete. \( \square \)

References