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Labeling matched sums with a condition at distance two

Sarah Spence Adams [a], Denise Sakai Troxell [b,*]

[a] Franklin W. Olin College of Engineering, Olin Hall, Olin Way, Needham, MA 02492, USA
[b] Mathematics and Sciences Division, Babson College, Babson Park, MA 02457, USA

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ABSTRACT

An $L(2, 1)$-labeling of a graph G is a function $f: V(G) \to \{0, 1, \ldots, k\}$ such that $|f(x) - f(y)| \geq 2$ if x and y are adjacent vertices, and $|f(x) - f(y)| \geq 1$ if x and y are at distance 2. Such labelings were introduced as a way of modeling the assignment of frequencies to transmitters operating in close proximity within a communications network. The *lambda number* of G is the minimum k over all $L(2, 1)$-labelings of G. This paper considers the lambda number of the matched sum of two same-order disjoint graphs, wherein the graphs have been connected by a perfect matching between the two vertex sets. Matched sums have been studied in this context to model possible connections between two different networks with the same number of transmitters. We completely determine the lambda number of matched sums where one of the graphs is a complete graph or a complete graph minus an edge. We conclude by discussing some difficulties that are encountered when trying to generalize this problem by removing more edges from a complete graph.

1. Introduction

In a generalization of Hale's channel assignment problem [1], transmitters in close proximity within a communications network must receive frequencies that are sufficiently far apart to avoid interference. Griggs and Yeh modeled this problem in 1992 [2] by introducing $L(2, 1)$-*labelings*, a variation of the classical graph coloring of vertices. In an $L(2, 1)$-labeling, a nonnegative integer (frequency) is assigned to each vertex (transmitter) in the graph (network) so that adjacent vertices (very close transmitters) must be assigned not only different but also non-consecutive integers, while vertices at distance two (close transmitters) must be assigned different integers. An $L(2, 1)$-labeling of a graph G that uses labels in the set $\{0, 1, \ldots, k\}$ is called a k-*labeling*. The minimum k so that G has a k-labeling is called the λ-*number* of G and is denoted by $\lambda(G)$. A $\lambda(G)$-labeling will be referred to simply as a λ-labeling if there is no confusion about the underlying graph.

The $L(2, 1)$-labeling and its various generalizations have spawned a vast literature motivated by efficiency issues in different network structures arising in practical applications. Griggs and Yeh [2] conjectured that $\lambda(G) \leq \Delta^2(G)$ where $\Delta(G)$ is the maximum degree of a vertex in G. This conjecture has kept the field active for almost two decades and although it has been verified for various families of graphs, it has not been settled in general. Well over 100 works on $L(2, 1)$-labelings published before 2006 are listed in the surveys by Calamoneri [3] and Yeh [4]. As of June 2010, a Google Scholar advanced search on the exact term "$L(2, 1)$-labeling" returned 360 works with 141 of them published since 2006 and with a significantly large number of them focusing on determining the lambda number of particular classes of graphs.

In channel assignment applications, desirable network topologies attempt to balance connectivity, efficiency and reliability. When connecting two different networks with the same number of transmitters, such a balance could be attempted by connecting each transmitter in one network to exactly one transmitter in the other. In 2002, Georges and Mauro [5] were the first to study the $L(2, 1)$-labelings of such structures. If G and H are two graphs on n vertices and M is a

* Corresponding author.
E-mail addresses: sarah.adams@olin.edu (S.S. Adams), troxell@babson.edu (D.S. Troxell).

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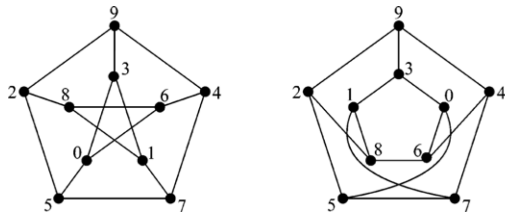

Fig. 1. A 9-labeling of the Petersen graph and its alternative drawing as $C_5 +_M C_5$.

Table 1
A k-labeling $f: V(G +_M H) \to \{0, 1, \ldots, k\}$ in its table format.

$f(v_1)$	$f(v_2)$	$\ldots$	$f(v_n)$
$f(w_1)$	$f(w_2)$	$\ldots$	$f(w_n)$

perfect matching between $V(G)$ and $V(H)$, then the *matched sum* of G and H is defined as the graph $G +_M H$ with vertex set $V(G +_M H) = V(G) \cup V(H)$ and edge set $E(G +_M H) = E(G) \cup E(H) \cup M$. The following result provides an upper bound for $\lambda(G +_M H)$ based on the individual $\lambda(G)$ and $\lambda(H)$.

Theorem 1.1 (*[5]*). *Let G and H be two graphs on n vertices and let M be a perfect matching between $V(G)$ and $V(H)$. Then $\lambda(G +_M H) \leq \lambda(G) + \lambda(H) + 2$.* $\square$

Surprisingly, the determination of exact values or tighter upper bounds for the λ-number of matched sums of two graphs proved to be very complex even when focusing on graphs as simple as cycles on n vertices C_n. The well-known Petersen graph is 3-regular, has λ-number 9 and is isomorphic to the matched sum $C_5 +_M C_5$ for the perfect matching M given in Fig. 1. Griggs and Yeh [2] showed that $\lambda(C_n) = 4$ if $n \geq 3$, so the bound of 10 in Theorem 1.1 is not tight. Furthermore, among matched sums of two cycles, only the Petersen graph has a λ-number as high as 9, as indicated by the following result:

Theorem 1.2 (*[5]*). *Let n be an integer so that $n \geq 3$. Therefore, $\lambda(C_n +_M C_n) \leq 8$ if and only if $C_n +_M C_n$ is not isomorphic to the Petersen graph.* $\square$

Interestingly, Georges and Mauro could not verify that the upper bound in Theorem 1.2 is tight and, consequently, conjectured that not only is there no matched sum of two cycles with λ-number exactly 8, but also that there is no 3-regular graph with λ-number 8. In investigating this conjecture, they showed that $\lambda(C_n +_M C_n) \leq 7$ for $n \leq 6$. They also studied one particular matched sum of two cycles, the n-prism Pr_n, which is isomorphic to the Cartesian product $P_2 \times C_n$ where P_2 is a path on two vertices, showing that either $\lambda(Pr_n) = 5$, if n is a multiple of 3, or 6 otherwise.

Other authors attempted to verify Georges and Mauro's conjecture in the case of a matched sum of two cycles. Adams et al. [6] proved $\lambda(C_n +_M C_n) \leq 7$ for $n = 7$ and 8, and showed that the previously known upper bound of 7 for $\lambda(C_6 +_M C_6)$ could be brought down to 6 by providing exact values for the λ-numbers of all graphs in this family. Subsequently, Adams et al. [7] provided $\lambda(C_n +_M C_n)$ for $n = 5$, 7, and 8, thereby closing all cases with $n \leq 8$. Recently, Huang [8] showed that $\lambda(C_n +_M C_n) \leq 7$ for $n = 9$, 10, 11, and 12.

In this article, we determine the λ-number of matched sums of two graphs where one of them is a complete graph K_n or a *near-complete* graph $K_n - e$ where e is an edge in K_n.

Throughout this paper, we will be using the following well-known result on the λ-numbers of some elementary families of graphs.

Result 1.3 (*[2]*). Let P_n, C_n, and K_n be the families of paths, cycles and complete graphs on n vertices, respectively. Therefore,

$$\text{(i)}\ \lambda(P_n) = \begin{cases} 0, & \text{if } n = 1, \\ 2, & \text{if } n = 2, \\ 3, & \text{if } n = 3 \text{ or } 4, \\ 4, & \text{if } n \geq 5; \end{cases}$$

(ii) $\lambda(C_n) = 4$, if $n \geq 3$;

(iii) $\lambda(K_n) = 2n - 2$, if $n \geq 1$. $\square$

2. λ-number of matched sums involving complete and near-complete graphs

We start this section by introducing some notation. Given two graphs G and H with vertex sets $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{w_1, w_2, \ldots, w_n\}$, respectively, and a perfect matching M with edges $E(M) = \{v_i w_i,\ i = 1, 2, \ldots, n\}$, we will often present a k-labeling $f: V(G +_M H) \to \{0, 1, \ldots, k\}$ in its table format given in Table 1. For instance, Table 2 contains one possible way to present the 9-labeling in Fig. 1. The largest label in the table has been boxed for easy reference.

Table 2
A 9-labeling of the Petersen graph in table format.

0	1	3	6	8
5	7	$\boxed{9}$	4	2

Table 3
A 5-labeling of $K_3 +_M G$.

0	2	4
3	$\boxed{5}$	1

Table 4
A $(2n-1)$-labeling of $K_n +_M G$ for $n \geq 4$ and $\text{diam}(G) \leq 2$.

0	2	…	$2n-4$	$2n-2$
3	5	…	$\boxed{2n-1}$	1

Table 5
A $(2n-1)$-labeling of $K_n +_M G$ for $n \geq 4$ and $\text{diam}(G) > 2$ where u and v are far.

0	2	…	$2n-6$	$2n-4$	$\boxed{2n-2}$
3	5	…	$2n-3$	**1**	**1**
				u	v

The *diameter* of a graph G is the maximum distance between any pair of vertices in G and will be denoted by $\text{diam}(G)$. We say that two vertices are *close* (resp., *far*) in a graph G if they are at distance at most 2 (resp., at least 3) in G. Note that two close vertices must be assigned different labels in any $L(2, 1)$-labeling but two far vertices may be assigned the same label. Consequently, if $\text{diam}(G) \leq 2$, then all the vertices in G are pairwise close and must be assigned different labels by any $L(2, 1)$-labeling of G; hence, if G has n vertices, then $\lambda(G) \geq n-1$.

We are now prepared to present our first major result. Theorem 2.1 provides the λ-number of the matched sum of a complete graph with any other graph of the same order.

Theorem 2.1. *Let G be a graph on n vertices and let M be a perfect matching between $V(K_n)$ and $V(G)$. Then,*

(i) $\lambda(K_1 +_M G) = 2$;

(ii) $\lambda(K_2 +_M G) = \begin{cases} 3, & \textit{if } G \textit{ is not a path}, \\ 4, & \textit{otherwise}; \end{cases}$

(iii) $\lambda(K_3 +_M G) = 5$;

(iv) *if* $n \geq 4$, $\lambda(K_n +_M G) = \begin{cases} 2n-1, & \textit{if } \text{diam}(G) \leq 2, \\ 2n-2, & \textit{otherwise}. \end{cases}$

Proof. If $n < 3$, it follows that $K_1 +_M G = P_2$ while $K_2 +_M G = P_4$, if G is not a path, and $K_2 +_M G = C_4$ if G is a path. Therefore, items (i) and (ii) follow from items (i) and (ii) in Result 1.3. If $n = 3$, then $\lambda(K_3 +_M G) \leq 5$ since Table 3 contains a 5-labeling of $K_3 +_M G$. In addition, $K_3 +_M G$ contains $K_3 +_M H$ where H is the graph with three isolated vertices and consequently $\lambda(K_3 +_M G) \geq \lambda(K_3 +_M H)$. Since K_3 is a subgraph of $K_3 +_M H$, by item (iii) in Result 1.3 we must have $\lambda(K_3 +_M H) \geq \lambda(K_3) = 4$. If there existed a 4-labeling of $K_3 +_M H$, then one of the vertices in K_3 would have to be labeled 2 which in turn would make it impossible to label one of its three neighbors. Therefore, $\lambda(K_3 +_M H) \geq 5$. When all the previous inequalities are combined, we have $5 \geq \lambda(K_3 +_M G) \geq \lambda(K_3 +_M H) \geq 5$ so item (iii) holds. Let us assume that $n \geq 4$ for the remainder of this proof.

If the $\text{diam}(G) \leq 2$, then $\text{diam}(K_n +_M G) \leq 2$ so $\lambda(K_n +_M G) \geq 2n-1$. On the other hand, $\lambda(K_n +_M G) \leq 2n-1$ since Table 4 contains a $(2n-1)$-labeling of $K_n +_M G$ (we shade consecutive columns following a certain pattern when it facilitates checking the conditions of the $L(2, 1)$-labeling). This fact can be verified by noticing that all the $2n$ labels are different and the n labels used in each row of the table (that is, the labels within K_n and within G, respectively) are pairwise non-consecutive. Moreover, the labels used in each column of the table (that is, labels assigned to vertices incident to an edge in M) are exactly 3 apart, except for the labels on the last column which are $2n-3 \geq 5$ apart. We conclude that $\lambda(K_n +_M G) = 2n-1$.

If the $\text{diam}(G) > 2$, then there are two far vertices u and v in G. Table 5 contains a $(2n-2)$-labeling of $K_n +_M G$ where we assume, without loss of generality, that u and v are labeled **1** in the last two columns of the second row. All the labels used are different and the labels in each row are pairwise non-consecutive, except for the labels of u and v. Moreover, the labels used on each column of the table are 3 apart, except for the labels on the last two columns which are $2n-5 \geq 3$ and $2n-3 \geq 5$ apart, respectively. But $\lambda(K_n +_M G) \geq \lambda(K_n) = 2n-2$ by item (iii) of Result 1.3, so we have $\lambda(K_n +_M G) = 2n-2$. □

The remainder of this section will be dedicated to determining the λ-number of matched sums of two graphs where one of them is a near-complete graph. We begin by finding the λ-number of near-complete graphs using Result 2.2 which relates the λ-number of a graph G with the path covering number of its complement G^c. This result has been instrumental in determining exact λ-numbers for other classes of graphs as illustrated in [9,10]. Recall that a *path covering* of a graph G is a set of vertex disjoint paths of G containing all the vertices of G, and the *path covering number* of G, denoted by $c(G)$, is the minimum number of paths in a path covering of G.

Result 2.2 ([11]). Let G be a graph on n vertices. Then, $c(G^c) \geq 2$ if and only if $\lambda(G) = n + c(G^c) - 2$. □

Lemma 2.3. *Let $n \geq 2$, and let e be an edge in K_n. Then, $\lambda(K_n - e) = 0$, if $n = 2$, and $\lambda(K_n - e) = 2n-3$, otherwise.*

Proof. If $n = 2$, then $K_2 - e$ is the graph with two isolated vertices and therefore $\lambda(K_2 - e) = 0$. Let us assume that $n \geq 3$. The graph $(K_n - e)^c$ is isomorphic to the graph with one edge and $n-2$ isolated vertices, so if $G = K_n - e$ then $c(G^c) = n-1 \geq 2$ and therefore, Result 2.2 implies $\lambda(G) = n + c(G^c) - 2 = n + (n-1) - 2 = 2n-3$. □

The next result establishes lower and upper bounds for the λ-number of matched sums of two graphs where one of them is a near-complete graph.

Lemma 2.4. *Let G be a graph on $n \geq 3$ vertices, e be an edge in K_n, and M be a perfect matching between $V(K_n)$ and $V(G)$. Then, $2n-3 \leq \lambda((K_n - e) +_M G) \leq 2n-1$.*

Proof. Note that $K_n - e$ is a subgraph of $(K_n - e) +_M G$ which in turn is a subgraph of $K_n +_M G$, so $\lambda(K_n - e) \leq \lambda((K_n - e) +_M G) \leq \lambda(K_n +_M G)$. Recall from Lemma 2.3 that $\lambda(K_n - e) = 2n-3$, and from Theorem 2.1 item (iv) that $\lambda(K_n +_M G) \leq 2n-1$. Therefore, the desired result follows. □

We proceed towards determining the exact λ-number of any matched sum involving a near-complete graph by providing complete descriptions for the graphs G so that the λ-number of a matched sum of a near-complete graph and G attains each one of the possible three values given in Lemma 2.4.

Let us initially examine the case $\lambda((K_n - e) +_M G) = 2n-3$ for $n \geq 2$. It is not possible to have $n \leq 3$ because P_2 is a subgraph of $(K_2 - e) +_M G$ and P_5 is a subgraph of $(K_3 - e) +_M G$, where $\lambda(P_2) = 2$ and $\lambda(P_5) = 4$ by item (i) of Result 1.3, and therefore $\lambda((K_n - e) +_M G) > 2n-3$.

Theorem 2.5. *Let G be a graph on $n \geq 4$ vertices, e be an edge in K_n, and M be a perfect matching between $V(K_n)$ and $V(G)$.Then, $\lambda((K_n - e) +_M G) = 2n-3$ if and only if G contains either*

(i) *two disjoint pairs of far vertices in G; or*
(ii) *three pairwise far vertices in G.*

Proof. Let x and y be the ends of e.

Suppose first that $\lambda((K_n - e) +_M G) = 2n-3$ and consider an arbitrary $(2n-3)$-labeling of $(K_n - e) +_M G$. Since $\mathrm{diam}(K_n - e) = 2$, the labels used in $K_n - e$ must be all different. Moreover, the labels of x and y must be consecutive; otherwise the labels assigned to $K_n - e$ would also be a k-labeling of K_n with $k \leq 2n-3$, contradicting $\lambda(K_n) = 2n-2$ in item (iii) of Result 1.3. If a vertex w in G is not adjacent to either x or y, then it is close to all the vertices in $K_n - e$ and consequently the label of w cannot be equal to any label used in $K_n - e$. On the other hand, if a vertex w in G is adjacent to x (resp., y), then the label of w cannot be equal to any label used in $K_n - e$ except possibly for the label of y (resp., x); but y (resp., x) and w cannot have the same label since x and y have consecutive labels and x (resp., y) and w are adjacent. Therefore, each of the n vertices of G must get a label that is different from the n labels assigned to the vertices in $K_n - e$. Since there are a total of $2n-2$ available labels, there are only $(2n-2) - n = n-2$ different labels available for the n vertices of G. Hence, either there exist two different labels each used more than once in G, which would imply item (i), or there exists a label used more than twice in G, which would imply item (ii).

Conversely, let us assume that item (i) holds. If $n = 4$, then every vertex of G must have degree at most one in G, or equivalently, G has at most one edge or G is the union of two disjoint copies of P_2. Let u, z be the vertices in G so that xu, yz are edges in M. Given the structure of G, it is possible to name the other two vertices in G as v and w so that $\{u, v\}$ and $\{w, z\}$ are two disjoint pairs of far vertices in G. Table 6 contains a 5-labeling of $(K_4 - e) +_M G$, so by Lemma 2.4 we conclude that $\lambda((K_4 - e) +_M G) = 5$. Now suppose $n \geq 5$ and let $\{u, v\}$ and $\{w, z\}$ be the two disjoint pairs of far vertices in G. We will construct a $(2n-3)$-labeling of $(K_n - e) +_M G$ where x and y will be assigned consecutive labels, and each pair $\{u, v\}$ and $\{w, z\}$ will be assigned the same label, respectively. With a possible renaming of vertices, we may assume without loss of generality that we have the four possible cases A1–A4 based on the edges in M between the sets $\{x, y\}$ and $\{u, v, w, z\}$.

Case A1: $xu, yv \in M$ $(n \geq 5)$.
Case A2: $xu, yw \in M$ $(n \geq 5)$.
Case A3: $xu \in M$, y not adjacent to any vertex in $\{v, w, z\}$ $(n \geq 5)$.
Case A4: x and y not adjacent to any vertex in $\{u, v, w, z\}$ $(n \geq 6)$.

Table 6
A 5-labeling of $(K_4 - e) +_M G$ where $\{u, v\}$ and $\{w, z\}$ are two pairs of far vertices.

	x	y	
0	2	3	$\boxed{5}$
4	**4**	**1**	**1**
v	u	z	w

Table 7
A $(2n - 3)$-labeling of $(K_n - e) +_M G$ for case A1.

						x	y
0	2	4	6	…	$2n - 6$	$\mathbf{2n-4}$	$\boxed{\mathbf{2n-3}}$
$\mathbf{2n-5}$	$\mathbf{2n-5}$	1	3	…	$2n - 9$	$\mathbf{2n-7}$	$\mathbf{2n-7}$
w	z					u	v

Table 8
A $(2n - 3)$-labeling of $(K_n - e) +_M G$ for case A2.

					x	y
0	2	…	$2n - 8$	$2n - 6$	$\mathbf{2n-4}$	$\boxed{\mathbf{2n-3}}$
3	5	…	$\mathbf{2n-5}$	**1**	**1**	$\mathbf{2n-5}$
			z	v	u	w

Table 9
A $(2n - 3)$-labeling of $(K_n - e) +_M G$ for case A3.

						x	y
0	2	…	$2n - 10$	$2n - 8$	$2n - 6$	$\mathbf{2n-4}$	$\boxed{\mathbf{2n-3}}$
5	7	…	$\mathbf{2n-5}$	$\mathbf{2n-5}$	**1**	**1**	3
			w	z	v	u	

Table 10
A $(2n - 3)$-labeling of $(K_n - e) +_M G$ for case A4.

						x	y
0	2	4	6	8	…	$\mathbf{2n-4}$	$\boxed{\mathbf{2n-3}}$
$\mathbf{2n-5}$	$\mathbf{2n-5}$	**1**	**1**	3	…	$2n - 9$	$2n - 7$
u	v	w	z				

We provide $(2n - 3)$-labelings of $(K_n - e) +_M G$ for cases A1–A4 in Tables 7–10, respectively. To verify that these are $L(2, 1)$-labelings, we first note that $n \geq 5$ in cases A1–A3 and $n \geq 6$ in case A4. Next, we note that all labels used are different except for the labels for the pairs of far vertices $\{u, v\}$ and $\{w, z\}$ in G, and consequently also far in $(K_n - e) +_M G$. The labels within a row of each table are all non-consecutive except for the labels of x and y, which are permitted to be consecutive since these vertices are not adjacent. Moreover, the labels within a column of each table are at least 2 apart. Therefore, by Lemma 2.4, we conclude that $\lambda((K_n - e) +_M G) = 2n - 3$.

Assume that item (ii) holds but item (i) does not. Let r, s and t be the three pairwise far vertices in G and let q be a vertex in G other than r, s and t (recall that $n \geq 4$). Note that q must be close in G to each of the vertices r, s and t, otherwise item (i) would hold. Since r, s and t are pairwise far vertices in G, q can be adjacent to at most one of them. Without loss of generality, let us assume that q is not adjacent to r and s. Let r' (resp., s') be the vertex adjacent to r (resp., s) in a shortest path connecting q to r (resp., s). Clearly, r, s, t, r', s' and q are all different and therefore we must have $n \geq 6$. We will next construct a $(2n - 3)$-labeling of $(K_n - e) +_M G$ where x and y will be assigned consecutive labels, and r, s and t will be assigned the same label. With a possible renaming of vertices, we may assume without loss of generality that we have the four possible cases B1–B4 based on the edges in M between the sets $\{x, y\}$ and $\{r, s, t\}$.

Case B1: $xr, ys \in M$ $(n \geq 6)$.
Case B2: $xr \in M$ and y not adjacent to a vertex in $\{s, t\}$ $(n \geq 6)$.
Case B3: x and y not adjacent to any vertex in $\{r, s, t\}$ $(n = 6)$.
Case B4: x and y not adjacent to any vertex in $\{r, s, t\}$ $(n \geq 7)$.

Table 11
A $(2n-3)$-labeling of $(K_n-e)+_M G$ for case B1.

						x	y
0	2	4	6	…	$2n-6$	$\mathbf{2n-4}$	$\boxed{\mathbf{2n-3}}$
$2n-7$	$2n-5$	**1**	3	…	$2n-9$	**1**	**1**
		t				r	s

Table 12
A $(2n-3)$-labeling of $(K_n-e)+_M G$ for case B2.

							x	y
0	2	4	6	8	…	$2n-6$	$\mathbf{2n-4}$	$\boxed{\mathbf{2n-3}}$
$2n-9$	$2n-7$	**1**	**1**	3	…	$2n-11$	**1**	$2n-5$
		s	t				r	

Table 13
A $(2n-3)$-labeling of $(K_n-e)+_M G$ for case B3.

				x	y
0	2	4	6	**8**	$\boxed{\mathbf{9}}$
7	**7**	**7**	1	3	5
r	s	t			

Table 14
A $(2n-3)$-labeling of $(K_n-e)+_M G$ for case B4.

								x	y
0	2	4	6	8	10	…	$2n-6$	$\mathbf{2n-4}$	$\boxed{\mathbf{2n-3}}$
$2n-11$	$2n-9$	**1**	**1**	**1**	3	…	$2n-13$	$2n-7$	$2n-5$
		r	s	t					

Table 15
A 4-labeling of $(K_3-e)+_M G$ where G has less than two edges.

y		x
1	$\boxed{4}$	**2**
3	**0**	**0**
	v	u

We provide $(2n-3)$-labelings of $(K_n-e)+_M G$ for cases B1–B4 in Tables 11–14, respectively. All the labels used are different except for the labels for the pairwise far vertices r, s and t in G, so consequently far in $(K_n-e)+_M G$. The labels within a row of each table are all non-consecutive except for the labels of x and y, which are permitted to be consecutive since these vertices are not adjacent. Moreover, the labels used within a column of each table are at least 2 apart. Therefore, by Lemma 2.4, we conclude that $\lambda((K_n-e)+_M G)=2n-3$. □

We will now provide complete characterizations of graphs G where the upper bound in Lemma 2.4 is tight. Lemma 2.6 addresses the cases where $n=2$ and 3 and Theorem 2.7 addresses the cases where $n\geq 4$.

Lemma 2.6. *Let G be a graph on $2\leq n\leq 3$ vertices, e be an edge in K_n, and M be a perfect matching between $V(K_n)$ and $V(G)$. Then, $\lambda((K_n-e)+_M G)=2n-1$ if and only if G contains at least $n-1$ edges.*

Proof. If $n=2$, then $(K_2-e)+_M G=P_4$ if G contains an edge, and $(K_2-e)+_M G$ is two disjoint copies of P_2 otherwise. Since $\lambda(P_2)=2$ and $\lambda(P_4)=3$ by item (i) of Result 1.3, we have $\lambda((K_2-e)+_M G)=3$ if and only if G contains an edge.

If $n=3$, recall that $\lambda((K_3-e)+_M G)\geq 4$ from the discussion immediately preceding Theorem 2.5. If G contains less than two edges then there exist two far vertices u and v in G so that x and y are the ends of e, x and u are adjacent, and y and v are not adjacent. To verify this claim, let v be the vertex in G not adjacent to an end of e. Since G has at most one edge, there

Table 16
A $(2n-2)$-labeling of $(K_n - e) +_M G$ where z and w are far.

0	2	…	$2n-6$	$2n-4$	$\boxed{2n-2}$
3	5	…	$2n-3$	**1**	**1**
				z	w

Table 17
A $(2n-2)$-labeling of $(K_n - e) +_M G$ where u and r are not adjacent.

y					x
0	2	…	$2n-6$	$2n-4$	$\boxed{2n-2}$
3	5	…	$2n-3$	**1**	**0**
v				r	u

exists a vertex u in G not adjacent to v. In addition, u must be adjacent to one of the ends of e; name this end of e as x and the other end as y. The 4-labeling in Table 15 shows that $\lambda((K_3 - e) +_M G) = 4$. If G contains at least two edges and the vertex of degree 2 in $K_3 - e$ is adjacent to a vertex of degree 1 in G, then one can verify by inspection that $(K_3 - e) +_M G$ has diameter 2 and consequently has λ-number at least 5. On the other hand, if G contains at least two edges and the vertex of degree 2 in $K_3 - e$ is adjacent to a vertex of degree 2 in G, then $(K_3 - e) +_M G$ contains the Cartesian product $P_2 \times P_3$ and one can verify by inspection that this latter graph has λ-number 5. Hence, if G contains at least two edges, then $\lambda((K_3 - e) +_M G) \geq 5$ and the equality follows from the upper bound in Lemma 2.4. We can finally conclude that $\lambda((K_3 - e) +_M G) = 5$ if and only if G contains at least two edges. □

Theorem 2.7. *Let G be a graph on $n \geq 4$ vertices, e be an edge in K_n, and M be a perfect matching between $V(K_n)$ and $V(G)$. Let x and y be the ends of e, and let u and v be the ends in G of the edges in M incident to x and y, respectively. Therefore, $\lambda((K_n - e) +_M G) = 2n - 1$ if and only if either*

(i) *u and v are adjacent and* $\mathrm{diam}(G) \leq 2$*; or*
(ii) *u and v are not adjacent and have degree $n - 2$ in G.*

Proof. To prove the sufficiency, let us argue by contradiction that $\lambda((K_n - e) +_M G) \neq 2n - 1$ and that either item (i) or (ii) holds. In this case, from the upper bound in Lemma 2.4, we must have that $\lambda((K_n - e) +_M G) \leq 2n - 2$. Consider an arbitrary $(2n-2)$-labeling of $(K_n - e) +_M G$. In both items (i) and (ii), it is clear that $\mathrm{diam}(G) \leq 2$, hence the vertices of $G - \{u, v\}$ must be assigned $n - 2$ different labels. These labels are also different from the n different labels assigned to the vertices in $K_n - e$ since every vertex in $G - \{u, v\}$ is close to any vertex in $K_n - e$. Therefore, of the available $2n - 1$ labels, there exists only $(2n - 1) - n - (n - 2) = 1$ unused label, say k, that could be potentially assigned to u or v but not both since they are close vertices in G. In addition, the vertex u (resp., v) is close to every vertex except possibly for vertex y (resp., x), hence it must get a label different from any of the labels assigned to the other vertices except possibly for the label of y (resp., x). By symmetry, we may assume without loss of generality that k is not assigned to v, and consequently x and v must use the same label j. If item (i) holds, then u and v would be adjacent and x and v would be vertices at distance 2 with the same label j, a contradiction. If item (ii) holds, then u and v would not be adjacent, and every vertex different from x or v would be adjacent to x or v. So if $1 \leq j \leq 2n - 3$, then the labels $j - 1$ and $j + 1$ must be disallowed, contradicting the fact that there is only one unused label. Hence $j = 0$ or $j = 2n - 2$. We will examine the case $j = 0$ (the case $j = 2n - 2$ will follow similarly). The label $k = j + 1 = 1$ was not used and consequently the vertices u and y will have the same label $h \geq 2$. If $h \neq 2n - 2$ (resp., $h = 2n - 2$), then $h + 1$ (resp., $h - 1$) cannot be used by any other vertex since every vertex is either adjacent to u or to y contradicting the fact that the only unused label is $k = 1 \neq h + 1$ (resp., $\neq h - 1 = 2n - 3$). Hence we can conclude that $\lambda((K_n - e) +_M G) = 2n - 1$.

Let us now focus on the necessity and suppose $\lambda((K_n - e) +_M G) = 2n - 1$. First assume for contradiction that u and v are adjacent in G and $\mathrm{diam}(G) > 2$. Let z and w be two far vertices in G. Table 16 contains a $(2n-2)$-labeling of $(K_n - e) +_M G$ which would contradict $\lambda((K_n - e) +_M G) = 2n - 1$.

On the other hand, assume for contradiction that u and v are not adjacent in G and, without loss of generality, that u has degree at most $n - 3$ in G. Let r be a vertex in G different from v so that u and r are not adjacent. Table 17 contains a $(2n-2)$-labeling of $(K_n - e) +_M G$ which would contradict $\lambda((K_n - e) +_M G) = 2n - 1$. □

Now, by combining the results in Theorem 2.5, Lemma 2.6 and Theorem 2.7, we obtain complete descriptions of the graphs G so that the λ-number of a matched sum of a near-complete graph and G attains each one of the possible three values given in Lemma 2.4.

3. Concluding remarks

In this work we determined $\lambda(K_n +_M G)$ and $\lambda((K_n - e) +_M G)$ for any graph G, e an edge in K_n, and M a perfect matching between $V(K_n)$ and $V(G)$. To determine exact λ-numbers of matched sums of other pairs of graphs, a natural direction for further research would be to remove yet another edge f from K_n and attempt to characterize the graphs G for the possible values of $\lambda((K_n - \{e, f\}) +_M G)$. Indeed, we have shown that $2n - 6 \leq \lambda((K_n - \{e, f\}) +_M G) \leq 2n - 1$, similarly as in Lemma 2.4. It may be difficult, however, to find characterizations that are as compact as the ones provided in this article when moving to classify which graphs G attain each of the six possible values between $2n - 6$ and $2n - 1$. One can better appreciate the increased complexity of such determination problems as more edges are removed from K_n by recalling that even the $\lambda(C_n +_M C_n)$ has not been completely determined since this number was first investigated in 2002 [5].

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