JOURNAL OF APPROXIMATION THEORY 75, 248–265 (1993)

A Marsden Type Identity for Periodic Trigonometric Splines*

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Communicated by Sherman D. Riemenschneider

Received April 11, 1991; accepted in revised form August 3, 1992

An extension of Marsden's identity for periodic trigonometric splines is obtained by a bivariate approach to that space. A basis of these spaces, whose elements have minimal or quasi-minimal support, is studied. © 1993 Academic Press, Inc.

1. INTRODUCTION

In this paper we consider a partition of the plane by rays starting at the same point. We work with the space of splines formed by C^{k-1} functions which are homogeneous polynomials of degree less than or equal to k at each angular region. W. Dahmen [3] and W. Dahmen and C. A. Micchelli [4] gave a definition of multivariate truncated power. In this paper, we generalize this definition in the bivariate case, in the sense that the support of our basic functions need not be a convex region of the plane. This will also imply the loss of some properties of truncated powers. For instance, they are not necessarily positive.

We also obtain a local basis. The support of the basic functions contains a very reduced number of angular regions, which brings advantages in order to solve interpolation and least squares problems in this space, since the associated matrices have many zeros. The problem of obtaining a local basis whose functions have a reduced support in multivariate spaces of splines is far from being solved, except in the case where we reduce the smoothness of the splines (see, for example, [1]). In our case, the basic functions have minimal or quasi-minimal support (see 3.3 of [2]). This situation is very interesting in the frame of general multivariate spline spaces.

Finally, we obtain a Marsden type identity (cf. [6, 8]). These identities for multivariate spaces are very useful, for example, in quasi-interpolation

* Research partially supported by DGICYT under Grant PS90-0121.

0021-9045/93 $5.00

schemes, but to obtain them is difficult in multivariate spaces, as indicated on page 113 of [2]. Let us also remark that the space of bivariate homogeneous splines on radial partitions can be interpreted as the space of univariate periodic trigonometric splines. In fact, if f is a bivariate homogeneous spline, then the function $\varphi(\omega) := f(\cos\omega, \sin\omega)$ is a periodic trigonometric spline in $[0, 2\pi]$. Conversely, from each periodic trigonometric spline φ of order $k+1$ in $[0, 2\pi]$ we may obtain a bivariate homogeneous spline taking $\omega \in [0, 2\pi)$, the unique value such that

$$\cos\omega = \frac{x}{\sqrt{x^2+y^2}}, \qquad \sin\omega = \frac{y}{\sqrt{x^2+y^2}},$$

and then defining $f(x, y) := (\sqrt{x^2+y^2})^k\, \varphi(\omega)$.

In [5], a Marsden type identity was obtained for the space of (non-periodic) univariate trigonometric splines.

In Section 2 we introduce basic notations and preliminary results, obtaining the dimension of the space. Larry Schumaker studied the dimensions of these and other spaces (see [7]). In fact, most of the results of Section 2 are well known but we introduce the basic tools and notations which will be used in the sequel. In Section 3 we study the special case (i.e., the rays are contained in a number of lines less than or equal to $k+1$) and we find a basis with locally linearly independent splines having minimal or quasi-minimal support (Theorem 3.1). In Section 4, the general case (more than $k+1$ lines) is studied. The minimally or quasi-minimally supported functions are characterized (Theorem 4.3). We also describe a basis with locally linearly independent splines having minimal or quasi-minimal support (Theorem 4.4). Finally, a Marsden type identity is obtained (Theorem 4.6).

2. Basic Notations and Preliminary Results

Let us consider the vector space $\mathbb{R}^2$. A cone of $\mathbb{R}^2$ is a non-empty set $C \subseteq \mathbb{R}^2$ such that if $u \in C$ and $t \geqslant 0$, then $tu \in C$. Remark that the set $\{0\}$ is always a cone. We shall say that a cone is proper if it contains a non-zero vector.

An angular region of $\mathbb{R}^2$ is a proper cone of $\mathbb{R}^2$ which is also a connected set. A ray is a minimal angular region, in the sense that it contains no angular region except r itself. An angular region, which is not a ray is said to be a proper angular region.

Each ray r of $\mathbb{R}^2$ intersects the unit circle $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ only at one point $r \cap S = (\cos\omega, \sin\omega)$. Conversely, each point of the unit circle $(\cos\omega, \sin\omega)$ defines a unique ray of $\mathbb{R}^2$, $r = \{(t\cos\omega, t\sin w) \mid t \geqslant 0\}$.

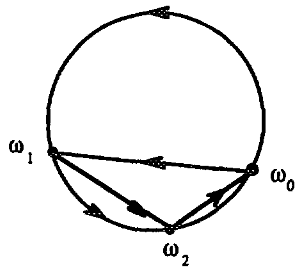

FIGURE 2.1

Let us define the following relationship between rays. A ray r_1 lies between r_0 and r_2 (see Fig. 2.1), $r_0 < r_1 < r_2$, if the triangle whose vertices are $r_0 \cap S = (\cos \omega_0, \sin \omega_0)$, $r_1 \cap S = (\cos \omega_1, \sin \omega_1)$, and $r_2 \cap S = (\cos \omega_2, \sin \omega_2)$ is positively oriented, that is,

$$\begin{vmatrix} 1 & 1 & 1 \\ \cos \omega_0 & \cos \omega_1 & \cos \omega_2 \\ \sin \omega_0 & \sin \omega_1 & \sin \omega_2 \end{vmatrix} > 0.$$

The previous definition allows us to define a closed ray interval

$$[r_0, r_1] = \{r_0\} \cup \{r \mid r_0 < r < r_1\} \cup \{r_1\}.$$

Analogously, one can define another kind of intervals. A ray-interval can also be seen as a set of points, and thus, it defines an angular region.

For a given ray r, we can also define its opposite ray,

$$-r := \{(-x, -y) \mid (x, y) \in r\}.$$

We shall also say that a set of rays is in *general position* if it contains no pair of opposite rays.

Let us consider next a partition of $\mathbb{R}^2$ in angular regions induced by an ordered set of different rays (see Fig. 2.2), $r_0 < r_1 < r_2 < \cdots < r_n$.

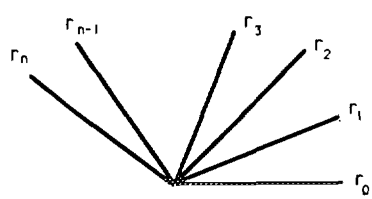

FIGURE 2.2

We shall use a cyclic notation by setting $r_{n+1}=r_0, ..., r_{n+k}=r_{k-1}, ...$ and $r_{-1}=r_n, ..., r_{-k}=r_{n+1-k}, ...$ Associated to each ray r_i, we have an angular value ω_i such that $(\cos\omega_i, \sin\omega_i)\in r$.

Let $\Pi_k(\mathbb{R}^2)$ be the vector space of all the bivariate polynomial functions of degree less than or equal to k. Let us define

$$S_k(r_0, ..., r_n)=\{f\in C^{k-1}(\mathbb{R}^2) \mid f|_{[r_i, r_{i+1}]}\in \Pi_k(\mathbb{R}^2),\ i=0, 1, ..., n\} \quad (2.1)$$

as the vector space of all bivariate splines generated by this partition and let $H_k(r_0, ..., r_n)$ be the subspace of homogeneous functions in $S_k(r_0, ..., r_n)$,

$$H_k(r_0, ..., r_n)=\{f\in S_k(r_0, ..., r_n) \mid f(tx, ty)=t^k f(x, y),\ t\geqslant 0\}. \quad (2.2)$$

DEFINITION 2.1. A function $f\in S_k(r_0, ..., r_n)$ is said to be globally supported if $\operatorname{supp} f=\mathbb{R}^2$. The functions of $S_k(r_0, ..., r_n)$ which are not globally supported are said to be locally supported.

Let us introduce the following space consisting of locally supported functions

$$L_k(r_0, ..., r_n)=\{f\in S_k(r_0, ..., r_n) \mid \operatorname{supp} f\subseteq [r_0, r_n)\}. \quad (2.3)$$

Given an element $f\in H_k(r_0, r_1, ..., r_n)$, we can denote by P_j the homogeneous polynomial which agrees with f in $[r_j, r_{j+1}]$, $j=0, ..., n$. It can be easily seen that the polynomials P_j satisfy

$$P_j(x, y)=P_n(x, y)+\sum_{i=0}^{j} d_i(\cos\omega_i\, y-\sin\omega_i\, x)^k, \qquad j=0, 1, ..., n. \quad (2.4)$$

From (2.4) we deduce that the polynomials P_i differ in a homogeneous polynomial. Using this fact, we can deduce the relationships

$$L_k(r_0, ..., r_n)\subseteq H_k(r_0, ..., r_n)\subseteq S_k(r_0, ..., r_n), \quad (2.5)$$

$$S_k(r_0, ..., r_n)=H_k(r_0, ..., r_n)\oplus \Pi_{k-1}(\mathbb{R}^2), \quad (2.6)$$

$$S_k(r_0, ..., r_n)=L_k(r_0, ..., r_n)\oplus \Pi_k(\mathbb{R}^2), \quad (2.7)$$

$$H_k(r_0, ..., r_n)=L_k(r_0, ..., r_n)\oplus \Pi_k^h(\mathbb{R}^2), \quad (2.8)$$

where $\Pi_k^h(\mathbb{R}^2)$ denotes the vector space of homogeneous polynomials of degree k.

That means that analyzing the structure of one of these spaces (dimension, a choice of a basis, etc.) gives information on the structure of the other spaces. We shall study mainly the space of homogeneous splines $H_k(r_0, ..., r_n)$, which is equivalent to the space of one-dimensional trigonometric periodic splines.

From (2.4), taking $j = n$ we also obtain

$$\sum_{i=0}^{n} d_i (\cos \omega_i y - \sin \omega_i x)^k = 0, \tag{2.9}$$

where expanding the powers and considering the coefficients of $x^j y^{k-j}$, we obtain

$$\sum_{i=0}^{n} \cos^{k-j} \omega_i \sin^j \omega_i \, d_i = 0, \qquad j = 0, 1, ..., k, \tag{2.10}$$

which can be expressed in matrix form as

$$A_k(\omega_0, \omega_1, ..., \omega_n) \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.11}$$

where $A_k(\omega_0, \omega_1, ..., \omega_n)$ is the $(k+1) \times (n+1)$ matrix given by

$$A_k(\omega_0, \omega_1, ..., \omega_n) = \begin{pmatrix} \cos^k \omega_0 & \cos^k \omega_1 & \cdots & \cos^k \omega_n \\ \cos^{k-1} \omega_0 \sin \omega_0 & \cos^{k-1} \omega_1 \sin \omega_1 & \cdots & \cos^{k-1} \omega_n \sin \omega_n \\ \cdots & \cdots & \cdots & \cdots \\ \sin^k \omega_0 & \sin^k \omega_1 & \cdots & \sin^k \omega_n \end{pmatrix}.$$

Thus, each element of $H_k(r_0, r_1, ..., r_n)$ can be defined by a homogeneous polynomial of degree k and a vector of constants d such that $A_k(\omega_0, \omega_1, ..., \omega_n)\, d = 0$. Therefore, we obtain the formula

$$\begin{aligned} \dim H_k(r_0, r_1, ..., r_n) &= k + 1 + \text{nullity } A_k(\omega_0, \omega_1, ..., \omega_n) \\ &= n + k + 2 - \text{rank } A_k(\omega_0, \omega_1, ..., \omega_n). \end{aligned} \tag{2.13}$$

By (2.8) we have $\dim H_k(r_0, r_1, ..., r_n) = k + 1 + \dim L_k(r_0, r_1, ..., r_n)$ and therefore

$$\dim L_k(r_0, r_1, ..., r_n) = n + 1 - \text{rank } A_k(\omega_0, \omega_1, ..., \omega_n). \tag{2.14}$$

We next analyze rank $A_k(\omega_0, \omega_1, ..., \omega_n)$ in order to give an expression for the dimension.

PROPOSITION 2.2. $\det A_k(\omega_0, \omega_1, ..., \omega_k) = \prod_{i,j=0,\, j>i}^{k} \sin(\omega_j - \omega_i)$.

Proof. If $\cos \omega_i \neq 0$ for all $i = 0, ..., k$

$$\det A_k(\omega_0, \omega_1, ..., \omega_k) = \prod_{j=0}^{k} \cos^k \omega_j \cdot \det(\tan^i \omega_j)_{i,j=0,...,k}. \tag{2.15}$$

Since the matrix $(\tan^i \omega_j)_{i,j=0,...,k}$ is of Vandermonde type, we obtain

$$\det(\tan^i \omega_j)_{i,j=0,...,k} = \prod_{\substack{i,j=0\\ j>i}}^{k} (\tan \omega_j - \tan \omega_i) \tag{2.16}$$

Now substitution of (2.16) in (2.15) provides the desired result. For the rest of the cases we can extend the formula by an argument of continuity, taking into account the fact that both sides of the formula are continuous functions of $\omega_1, ..., \omega_n$. ∎

The above proposition can be used, in the first place, to determine the rank of the matrix $A_k(\omega_0, \omega_1, ..., \omega_n)$.

DEFINITION 2.3. We define the rank of a set of rays $\operatorname{rank}(r_0, r_1, ..., r_n)$ as the minimal number of lines which contain all the rays, that is, the maximal number of rays among them which are in general position.

Remark that the number of pairs of opposite rays is $m = n + 1 - \operatorname{rank}(r_0, r_1, ..., r_n)$.

PROPOSITION 2.4. (1) *If* $\operatorname{rank}(r_0, r_1, ..., r_n) \leqslant k + 1$, then

$$\dim H_k(r_0, r_1, ..., r_n) = n + 1 + (k + 1 - \operatorname{rank}(r_0, r_1, ..., r_n)), \tag{2.17}$$

$$\dim L_k(r_0, r_1, ..., r_n) = n + 1 - \operatorname{rank}(r_0, r_1, ..., r_n). \tag{2.18}$$

(2) *If* $\operatorname{rank}(r_0, r_1, ..., r_n) \geqslant k + 1$, *then*

$$\dim H_k(r_0, r_1, ..., r_n) = n + 1, \tag{2.19}$$

$$\dim L_k(r_0, r_1, ..., r_n) = n - k. \tag{2.20}$$

Proof. We can use Proposition 2.2 in order to find regular submatrices of $A_k(\omega_0, \omega_1, ..., \omega_n)$ showing that

$$\operatorname{rank} A_k(\omega_0, \omega_1, ..., \omega_n) = \min(k + 1, \operatorname{rank}(r_0, r_1, ..., r_n)).$$

Now we can compute the dimensions using (2.13) and (2.14). ∎

COROLLARY 2.5. *The vector space* $L_k(r_0, r_1, ..., r_n) = 0$ *if and only if* $r_0, r_1, ..., r_n$ *are in general position and* $n \leqslant k$.

Proof. It is a consequence of imposing $\dim L_k(r_0, r_1, ..., r_n) = 0$ in Proposition 2.4. ∎

We try now to describe basic elements in $H_k(r_0, r_1, ..., r_n)$ with small support. These basic elements, called *multivariate truncated powers*, were introduced by Dahmen [3]. Our approach generalizes the approach of Dahmen and Micchelli [4] in the bivariate case, since we also consider functions supported on non-convex angular regions. We use in the following more suggestive univariate terminology and call the basic elements simply B-splines. Let us consider now the case of $k+2$ rays in general position. We know by (2.20) that

$$\dim L_k(r_0, r_1, ..., r_{k+1}) = 1. \tag{2.21}$$

A conveniently normalized function $B(x, y \mid r_0, r_1, ..., r_{k+1})$ of this space is called a B-spline. The rays $r_0, r_1, ..., r_{k+1}$ are said to be the *defining rays* of the *B*-spline.

The restriction of this function to each angular region $[r_j, r_{j+1}]$ is a polynomial (see Fig. 2.3) of the form

$$P_j(x, y) = \sum_{i=0}^{j} d_i (\cos \omega_i y - \sin \omega_i x)^k, \tag{2.22}$$

where $d = (d_i)_{i=0,1,...,k+1} \in \mathbb{R}^{k+1}$ is a solution of

$$A_k(\omega_0, \omega_1, ..., \omega_{k+1})\, d = 0. \tag{2.23}$$

A solution of the linear system (2.23) is given by

$$\begin{aligned} d_i &= (-1)^i \det A_k(\omega_0, \omega_1, ..., \omega_{i-1}, \omega_{i+1}, ..., \omega_k, \omega_{k+1}) \\ &= (-1)^i \prod_{\substack{j,l=0 \\ j,l \neq i;\, j>l}}^{k+1} \sin(\omega_j - \omega_l). \end{aligned} \tag{2.24}$$

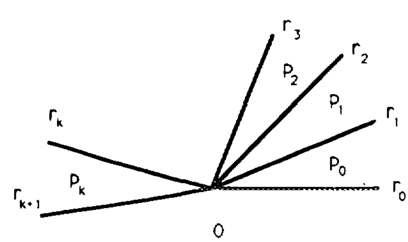

FIGURE 2.3

This choice produces our normalized B-spline with respect to the rays $r_0, r_1, ..., r_{k+1}$.

$$B(x, y \mid r_0, r_1, ..., r_k, r_{k+1}) := \sum_{i=0}^{k} (-1)^i \left(\prod_{\substack{j, l=0 \\ j, l \neq i;\, j > l}}^{k+1} \sin(\omega_j - \omega_l) \right) \times (\cos \omega_i\, y - \sin \omega_i\, x)^k_{[r_i, r_{k+1}]}, \qquad (2.25)$$

where $(\cos \omega_j\, y - \sin \omega_j\, x)^k_{[r_j, r_l]}$ is the corresponding truncated power

$$(\cos \omega_j\, y - \sin \omega_j\, x)^k_{[r_j, r_l]} := \begin{cases} (\cos \omega_j\, y - \sin \omega_j\, x)^k & \text{if } (x, y) \in [r_j, r_l] \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of $B(\cdot \mid r_0, r_1, ..., r_k, r_{k+1})$, we know that

$$\operatorname{supp} B(\cdot \mid r_0, r_1, ..., r_k, r_{k+1}) = [r_0, r_{k+1}],$$

because if $\operatorname{supp} B(\cdot \mid r_0, r_1, ..., r_k, r_{k+1}) \subset [r_0, r_{k+1}]$, there would be locally supported functions based on less than $k+2$ rays in general position, which is impossible by Corollary 2.5.

Let us consider now the case of two opposite rays r_0 and $-r_0$. We know by (2.18) that $\dim L_k(r_0, -r_0) = 1$. A non-zero function in this space is

$$(\cos \omega_0 y - \sin \omega_0 x)^k_+ = (\cos \omega_0\, y - \sin \omega_0\, x)^k_{[r_0, -r_0]}, \qquad (2.26)$$

whose support is the halfplane $[r_0, -r_0]$. Both rays r_0 and $-r_0$ are called the *defining rays* of this function.

We shall use functions (2.25) and (2.26) in order to construct the basic elements. We wish a property stronger than linear independence to be satisfied:

DEFINITION 2.6. We say that the non-zero functions $f_1, f_2, ..., f_n \in H_k(r_0, r_1, ..., r_n)$ are *locally linearly independent*, if for each angular region $[r_i, r_{i+1}]$, $i = 0, 1, ..., n$, the set of non-zero restrictions $f_j|_{[r_i, r_{i+1}]}$ for the indices j such that $\operatorname{supp} f_j \supseteq [r_i, r_{i+1}]$ is linearly independent.

Remark. We observe that local linear independence implies linear independence. Let $f_1, f_2, ..., f_n \in H_k(r_0, r_1, ..., r_n)$ be locally linearly independent functions such that $\sum_{j=1}^{n} t_j f_j = 0$. Since the $f_j \neq 0$, there exists an angular region $[r_i, r_{i+1}]$ such that $\operatorname{supp} f_j \supseteq [r_i, r_{i+1}]$ and thus considering the restriction to $[r_i, r_{i+1}]$ we conclude that $t_j = 0$ for each j.

DEFINITION 2.7. A non-zero function $f \in H_k(r_0, r_1, ..., r_n)$ is said to be *minimally supported* if for each non-zero $g \in H_k(r_0, r_1, ..., r_n)$

$$\operatorname{supp} g \subseteq \operatorname{supp} f \Rightarrow \operatorname{supp} g = \operatorname{supp} f.$$

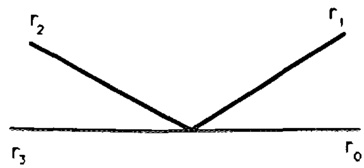

FIGURE 2.4

It would be very interesting if minimally supported functions generated the whole space $H_k(r_0, r_1, ..., r_n)$. But unfortunately this is not true in general, as shown in the following example.

EXAMPLE 2.8. Let us consider the ordered set of rays

$$r_0 < r_1 < r_2 < r_3,$$

where $r_3 = -r_0$, and the associated partition of the plane (See Fig. 2.4).

Let us analyze the structure of the space of homogeneous continuous piecewise linear functions $H_1(r_0, r_1, r_2, r_3)$. A function is minimally supported if and only if the support is one of the following angular regions: $[r_0, r_2]$, $[r_1, r_3]$, $[r_3, r_0]$. It can be readily seen that the minimally supported functions generate a subspace of dimension 3 while $\dim H_1(r_0, r_1, r_2, r_3) = 4$.

Let $M_k(r_0, r_1, ..., r_n)$ be the subspace of $H_k(r_0, r_1, ..., r_n)$ spanned by the minimally supported functions. If $M_k(r_0, r_1, ..., r_n) = H_k(r_0, r_1, ..., r_n)$, then there exists a basis of minimally supported functions.

DEFINITION 2.9. A function $f \in H_k(r_0, r_1, ..., r_n) \backslash M_k(r_0, r_1, ..., r_n)$ is said to have quasi-minimal support if for each $g \in H_k(r_0, r_1, ..., r_n)$ such that $\operatorname{supp} g \subset \operatorname{supp} f$ then $g \in M_k(r_0, r_1, ..., r_n)$.

The concept of quasi-minimally supported functions is very useful for the definition of basis in multivariate spline spaces (see [2]).

3. A BASIS IN THE SPECIAL CASE

In this section we deal with the special case of $\operatorname{rank}(r_0, r_1, ..., r_n) \leqslant k+1$. The number of pairs of opposite rays among $\{r_0, r_1, ..., r_n\}$ is $m = n + 1 - \operatorname{rank}(r_0, r_1, ..., r_n)$. We can find indices

$$0 \leqslant \alpha(1) < \alpha(2) < \cdots < \alpha(m) \leqslant n$$

such that there exists $\alpha'(i)$ with $0 \leqslant \alpha(i) < \alpha'(i) \leqslant n$, $r_{\alpha'(i)} = -r_{\alpha(i)}$ for $i = 1, ..., m$.

Let us consider the set of functions

$$\begin{aligned} &(\cos \omega_{\alpha(i)} y - \sin \omega_{\alpha(i)} x)_+^k, \\ &(\cos \omega_{\alpha'(i)} y - \sin \omega_{\alpha'(i)} x)_+^k, \qquad i = 1, ..., m. \end{aligned} \tag{3.1}$$

We shall show that they are linearly independent and then they form a basis of the space $L_k(r_0, r_1, ..., r_n)$. Let us observe that each pair of functions of (3.1) add up to the homogeneous polynomials

$$(\cos \omega_{\alpha(i)} y - \sin \omega_{\alpha(i)} x)^k. \tag{3.2}$$

In order to obtain a basis of $H_k(r_0, r_1, ..., r_n)$, we shall use the functions of (3.1) and also functions of the form

$$(\cos \bar{\omega}_j y - \sin \bar{\omega}_j x)^k, \qquad j = 1, ..., k+1-m, \tag{3.3}$$

where $\bar{\omega}_j$ are angular values associated to rays $\bar{r}_j$ such that

$$\operatorname{rank}\{r_{\alpha(0)}, ..., r_{\alpha(m)}, \bar{r}_1, ..., \bar{r}_{k+1-m}\} = k+1.$$

Using Proposition 2.2 it is straightforward to see that the functions of (3.2) and (3.3) form a basis of $\Pi_k^h(\mathbb{R}^2)$.

THEOREM 3.1. *If* $m = 0$ *then* $H_k(r_0, r_1, ..., r_n) = \Pi_k^h(\mathbb{R}^2)$, *a basis is given by the functions of* (3.3), *and these functions are minimally supported. If* $m > 0$ *the following properties hold*:

(1) *The 2m functions of* (3.1) *are minimally supported and they form a basis of* $M_k(r_0, r_1, ..., r_n)$.

(2) *If* $f \in H_k(r_0, r_1, ..., r_n)$ *and* $\operatorname{supp} f \neq \mathbb{R}^2$ *then* $f \in M_k(r_0, r_1, ..., r_n)$.

(3) *A basis* $\mathscr{B}$ *of* $H_k(r_0, r_1, ..., r_n)$ *is given by the 2m locally supported functions of* (3.1) (*minimally supported*) *and the* $k+1-m$ *globally supported functions of* (3.3) (*quasi-minimally supported*). *Moreover the functions of this basis are locally linearly independent.*

Proof. The case $m = 0$ follows from (2.8) and Corollary 2.5. In the following, we study the case $m \neq 0$.

First we show that the functions in $\mathscr{B}$ are locally linearly independent. Let us consider any angular region $[r_i, r_{i+1}]$, $i = 0, ..., n$. The restrictions of the functions in $\mathscr{B}$ to this angular region are either 0 or some of the given polynomials of (3.2) and (3.3), which are linearly independent. Then by (2.17) the functions in $\mathscr{B}$ form a basis of $H_k(r_0, r_1, ..., r_n)$.

By (2.18) the functions of (3.1) are of minimal support. Let f be a locally supported function; that is, the restriction of f to some angular region $[r_i, r_{i+1}]$ is zero. Then f is a linear combination of the functions in $\mathscr{B}$ and, by the local linear independence of the basic functions, the coefficients corresponding to the functions whose support contains $[r_i, r_{i+1}]$ are zero. Therefore f is a linear combinaton of the functions of (3.1) and property (2) holds. Furthermore, the minimally supported functions are locally supported and then are generated by the functions of (3.1). So, property (1) is confirmed.

Finally the functions of (3.3) are quasi-minimally supported because they are not in $M_k(r_0, r_1, ..., r_n)$ and any function with strictly smaller support, i.e., locally supported, is by property (2) in $M_k(r_0, r_1, ..., r_n)$. ∎

Remark. As a consequence of the previous theorem we obtain that it is possible to find a basis consisting only of minimally supported functions (that is, $M_k(r_0, r_1, ..., r_n) = H_k(r_0, r_1, ..., r_n)$) if and only if $m = 0$ or $m = k+1$. If $m = 0$ every function is a homogeneous polynomial and so globally (and minimally) supported whereas if $m = k+1$ the basis $\mathscr{B}$ consists of locally (and minimally) supported functions.

Let us observe that the construction of $\mathscr{B}$ only uses the pairs of opposite rays among $r_0, ..., r_n$. Therefore, the rest of the rays are not relevant for the structure of the space $H_k(r_0, r_1, ..., r_n)$.

COROLLARY 3.2. *The set of minimally supported functions*

$$(\cos \omega_{\alpha(i)} y - \sin \omega_{\alpha(i)} x)_+^k, \qquad i = 1, ..., m$$

is a basis of the vector space $L_k(r_0, r_1, ..., r_n)$.

Proof. The functions are in fact in $L_k(r_0, r_1, ..., r_n)$ and are minimally supported by Theorem 3.1(1). They are also locally linearly independent by Theorem 3.1(3) and, by (2.18), they form a basis. ∎

4. A LOCAL BASIS AND A MARSDEN TYPE IDENTITY IN THE GENERAL CASE

In this section, we deal with the general case of $\operatorname{rank}(r_0, r_1, ..., r_n) \geq k+2$. In order to give a characterization of minimally or quasi-minimally supported functions we introduce the following definitions:

DEFINITION 4.1. Let $f \in H_k(r_0, r_1, ..., r_n)$ with $\operatorname{supp} f = [r_i, r_l]$ be a non-zero locally supported function, that is $\operatorname{supp} f \neq \mathbb{R}^2$. The function f is said to be *left minimally supported* if, for each $g \in H_k(r_0, r_1, ..., r_n)$ with $\operatorname{supp} g = [r_i, r_j]$, we have $[r_i, r_l] \subseteq [r_i, r_j]$.

Analogously, the function f is said to be *right minimally supported* if, for each $g \in H_k(r_0, r_1, ..., r_n)$ with supp $g = [r_j, r_l]$, we have $[r_i, r_l] \subseteq [r_j, r_l]$.

From the definition, it follows that each minimally supported function is left and right minimally supported. As we shall see, this is also true for quasi-minimally supported functions.

In order to obtain left minimally supported functions we make the following construction.

Let us observe that

$$1 = \operatorname{rank}(r_i) \leqslant \operatorname{rank}(r_i, r_{i+1}) \leqslant \cdots \leqslant \operatorname{rank}(r_i, r_{i+1}, ..., r_{i+n}).$$

Since $\operatorname{rank}(r_i, r_{i+1}, ..., r_{i+n}) \geqslant k+2$ and at each step only one ray is added, there exist rays $r_{\beta_j(i)}$ with

$$i = \beta_0(i) < \beta_1(i) < \cdots < \beta_k(i) < \beta_{k+1}(i) \leqslant i+n$$

such that

$$\operatorname{rank}(r_i, r_{i+1}, ..., r_{\beta_j(i)-1}) = j,$$
$$\operatorname{rank}(r_i, r_{i+1}, ..., r_{\beta_j(i)-1}, r_{\beta_j(i)}) = j+1.$$

Let us remark that, by definition, $r_{\beta_j(i)}$ is not opposite to any ray among

$$r_i < r_{i+1} < \cdots < r_{\beta_j(i)-1}$$

and therefore, $r_i, r_{\beta_1(i)}, ..., r_{\beta_{k+1}(i)}$ are in general position.

Let us define for each index i, $\delta(i)$ to be the unique index $i < \delta(i) \leqslant \beta_{k+1}(i)$ given by

$$\delta(i) := \begin{cases} i' & \text{if there exists } i' \text{ with } i < i' \leqslant i+k+1,\ r_{i'} = -r_i, \\ \beta_{k+1}(i) & \text{otherwise.} \end{cases} \tag{4.1}$$

We prove in Theorem 4.3 that $[r_i, r_{\delta(i)}]$ are just the left minimal supports. The next proposition will be used to prove that the left minimal supports and right minimal supports coincide.

PROPOSITION 4.2. *The mapping*

$$\Delta : \{r_0, r_1, ..., r_n\} \to \{r_0, r_1, ..., r_n\}$$
$$r_i \mapsto r_{\delta(i)},$$

with $\delta(i)$ *defined by* (4.1), *is bijective.*

Proof. We know that

$$1 = \operatorname{rank}(r_l) \leqslant \operatorname{rank}(r_l, r_{l-1}) \leqslant \cdots \leqslant \operatorname{rank}(r_l, r_{l-1}, ..., r_{l-n}).$$

Since $\operatorname{rank}(r_l, r_{l-1}, ..., r_{l-n}) \geqslant k+2$ and at each step only one ray is added, then there exists an index i with $l > i \geqslant l-n$ such that

$$\operatorname{rank}(r_l, r_{l-1}, ..., r_{i+1}) = k+1,$$

$$\operatorname{rank}(r_l, r_{l-1}, ..., r_{i+1}, r_i) = k+2.$$

From the definition of i, we know that r_i is not the opposite ray of any of the rays $r_l > r_{l-1} > \cdots > r_{i+1}$ and thus $\delta(i) = \beta_{k+1}(i)$. We know that $r_{\beta_{k+1}(i)}$ is the first ray such that

$$\operatorname{rank}(r_i, r_{i+1}, ..., r_{\beta_{k+1}(i)-1}, r_{\beta_{k+1}(i)}) = k+2,$$

which implies that $\beta_{k+1}(i) \leqslant l$.

If $\beta_{k+1}(i) = l$, then we obtain the desired result: $\delta(i) = l$.

If $\beta_{k+1}(i) < l$, then r_l must be opposite to one ray $r_{l'}$ among $r_{i+1} < r_{i+2} < \cdots < r_{l-1}$, which means that the rays $r_{l'} < r_{l'+1} < \cdots < r_{l-1}$ are in general position and

$$\operatorname{rank}(r_{l'}, r_{l'+1}, ..., r_{l-1}) \leqslant \operatorname{rank}(r_{i+1}, r_{i+2}, ..., r_{l-1}) \leqslant k+1.$$

That is, $l \leqslant l' + k + 1$ and $r_{l'} = -r_l$, which implies that $\delta(l') = l$. Hence, we have proved that Δ is surjective and then bijective. ∎

In the following, we denote by $\delta^{-1}(l)$ the unique index i, such that $l-n \leqslant i < l$, $\delta(i) = l$.

THEOREM 4.3. *Let $f \in H_k(r_0, r_1, ..., r_n)$ and $\delta(i)$ be the index defined by* (4.1). *Then the following statements are equivalent*:

(1) $\operatorname{supp} f = [r_i, r_{\delta(i)}]$ *for some* i,

(2) f *is left minimally supported,*

(3) f *is right minimally supported,*

(4) f *is minimally or quasi-minimally supported.*

Moreover, f is minimally supported (*resp. quasi-minimally supported*) *if and only if* $\delta(i) \leqslant i+k+1$ (*resp.* $\delta(i) \geqslant i+k+2$).

Proof. (1) ⇒ (2), (3), (4). If $\delta(i) \leqslant i+k+1$ (that is, either $\delta(i) = i'$ with $r_{i'} = -r_i$ or $\delta(i) = \beta_{k+1}(i) = i+k+1$) by (2.18) or (2.20) we have $\dim L_k(r_i, r_{i+1}, ..., r_{\delta(i)}) = 1$, which implies that the functions supported on $[r_i, r_{\delta(i)}]$ are minimally supported and so, left and right minimally supported.

If $\delta(i) \geqslant i+k+2$ then, by (4.1), $\delta(i) = \beta_{k+1}(i)$ and

$$\operatorname{rank}(r_i, r_{i+1}, ..., r_{\beta_{k+1}(i)-1}, r_{\beta_{k+1}(i)}) = k+2,$$

$$\operatorname{rank}(r_i, r_{i+1}, ..., r_{\beta_{k+1}(i)-1}) = \operatorname{rank}(r_{i+1}, ..., r_{\beta_{k+1}(i)-1}, r_{\beta_{k+1}(i)}) = k+1$$

and

$$\operatorname{rank}(r_{i+1}, ..., r_{\beta_{k+1}(i)-1}) = k.$$

By (2.20), we obtain $\dim L_k(r_i, r_{i+1}, ..., r_{\delta(i)}) \geqslant 2$ and by (2.18) we can compute the dimensions of the local spaces and obtain

$$\begin{aligned} L_k(r_i, r_{i+1}, ..., r_{\delta(i)-1}) &= L_k(r_{i+1}, ..., r_{\delta(i)-1}) \\ &= L_k(r_{i+1}, ..., r_{\delta(i)-1}, r_{\delta(i)}), \end{aligned} \tag{4.2}$$

which implies that the functions supported on $[r_i, r_{\delta(i)}]$, are left and right minimally supported and that $L_k(r_{i+1}, ..., r_{\delta(i)-1})$ is a non-zero vector space properly contained in $L_k(r_i, ..., r_{\delta(i)})$. We know by Corollary 3.2 that $L_k(r_{i+1}, ..., r_{\delta(i)-1})$ is generated by a minimally supported basis and by (4.2), this space contains all the elements whose support is properly contained in $[r_i, r_{\delta(i)}]$. In other words $L_k(r_{i+1}, ..., r_{\delta(i)-1})$ is the space generated by the minimally supported functions in $L_k(r_i, ..., r_{\delta(i)})$ and thus, each function supported on $[r_i, r_{\delta(i)}]$ is quasi-minimally supported.

(4) ⇒ (2), (3). As we have pointed out, minimal support implies both left and right minimal support.

Now, let f be a quasi-minimally supported function with $\operatorname{supp} f = [r_i, r_l]$. Let g be a function with $\operatorname{supp} g = [r_i, r_j]$. If $[r_i, r_j] \subset [r_i, r_l]$, then $g \in M_k(r_0, r_1, ..., r_n)$. We also know that there exists a real constant $t \in \mathbb{R}$ such that

$$(f - tg)|_{[r_i, r_{i+1}]} = 0$$

and thus $\operatorname{supp}(f - tg) \subseteq [r_{i+1}, r_l]$, which implies that $f \in M_k(r_0, r_1, ..., r_n)$, contradicting the quasi-minimality. Thus $[r_i, r_j] \supseteq [r_i, r_l]$, which means that f is left minimally supported.

Analogously, it can be seen that f is right minimally supported.

(2) ⇒ (1). Let $[r_i, r_l]$ be a left minimal support. We showed in the first part of the proof that $[r_i, r_{\delta(i)}]$ is also a left minimal support. We conclude that $l = \delta(i)$.

(3) ⇒ (1). Let $[r_i, r_l]$ be a right minimal support. Since $[r_{\delta^{-1}(l)}, r_l]$ is also a left minimal support, $i = \delta^{-1}(l)$. ∎

Let us consider $r_0 < r_1 < \cdots < r_n$ and the problem of finding a basis of locally supported functions with minimal or quasi-minimal support. As we have seen, we have two different cases:

(1) $[r_i, r_{\delta(i)}]$ is a halfplane, that is, $r_{\delta(i)} = -r_i$, or equivalently $\delta(i) < \beta_{k+1}(i)$;

(2) otherwise $r_{\delta(i)} \neq -r_i$, that is, $\delta(i) = \beta_{k+1}(i)$.

For each ray r_i, we define left minimally supported function B_i with $\text{supp } B_i = [r_i, r_{\delta(i)}]$ based on the functions defined in (2.25) and (2.26).

$$B_i(x, y) := \begin{cases} (\cos \omega_i y - \sin \omega_i x)_+^k & \text{if } \delta(i) < \beta_{k+1}(i) \\ \dfrac{B(x, y \mid r_{\beta_0(i)}, r_{\beta_1(i)}, ..., r_{\beta_{k+1}(i)})}{\prod_{j, h=1, j>h}^{k+1} \sin(\omega_{\beta_j(i)} - \omega_{\beta_h(i)})} & \text{if } \delta(i) = \beta_{k+1}(i). \end{cases} \tag{4.3}$$

Let us observe that the factor $(\prod_{j, h=1, j>h}^{k+1} \sin(\omega_{\beta_j(i)} - \omega_{\beta_h(i)}))^{-1}$ introduced in the basic function $B(x, y \mid r_{\beta_0(i)}, r_{\beta_1(i)}, ..., r_{\beta_{k+1}(i)})$ has been chosen in order to have

$$B_i|_{[r_i, r_{i+1}]} = (\cos \omega_i y - \sin \omega_i x)^k, \qquad i = 0, 1, ..., n. \tag{4.4}$$

We introduce a formula based on the definition (4.3) which may be derived analogously to (2.4) but rotating clockwise:

$$B_i(x, y) = \sum_{h=j}^{\delta(i)} d_h (\cos \omega_h y - \sin \omega_h x)^k, \qquad \forall (x, y) \in [r_{j-1}, r_j]. \tag{4.5}$$

Let us remark that $d_{\delta(i)} \neq 0$ and $d_h = 0$ if r_h is not a defining ray.

THEOREM 4.4. *The locally supported functions B_i, $i = 0, 1, ..., n$, of* (4.3) *are locally linearly independent and they form a basis $\mathscr{B}$ of $H_k(r_0, r_1, ..., r_n)$. Moreover, these functions are minimally (if $\delta(i) \leqslant i + k + 1$) or quasi-minimally (if $\delta(i) \geqslant i + k + 2$) supported.*

Proof. By (2.19) and Theorem 4.3, it is sufficient to show that the functions in $\mathscr{B}$ are locally linearly independent.

In first place, we shall prove that

$$B_{\delta^{-1}(\beta_l(j))}, \qquad l = 0, 1, ..., k \tag{4.6}$$

are the unique functions in $\mathscr{B}$ such that $[r_{j-1}, r_j]$ is contained in their support.

If $r_{\delta^{-1}(\beta_l(j))}$ is not opposite to $r_{\beta_l(j)}$, then $\beta_l(j) = \beta_{k+1}(\delta^{-1}(\beta_l(j)))$ and so

$$k + 2 = \text{rank}(r_{\delta^{-1}(\beta_l(j))}, r_{\delta^{-1}(\beta_l(j))+1}, ..., r_{\beta_l(j)}) \geqslant \text{rank}(r_{j-1}, r_j, ..., r_{\beta_l(j)}),$$

which implies that $\delta^{-1}(\beta_l(j)) \leqslant j - 1$.

If $r_{\delta^{-1}(\beta_l(j))}$ is opposite to $r_{\beta_l(j)}$, then by definition of $\beta_l(j)$, no ray among $r_j, r_{j+1}, ..., r_{\beta_l(j)-1}$ is opposite to $r_{\beta_l(j)}$ and then $\delta^{-1}(\beta_l(j)) < j$.

We conclude in any case that

$$\text{supp } B_{\delta^{-1}(\beta_l(j))} = [r_{\delta^{-1}(\beta_l(j))}, r_{\beta_l(j)}] \supseteq [r_{j-1}, r_j], \qquad l = 0, ..., k. \tag{4.7}$$

Now let B_m be such that supp $B_m \supseteq [r_{j-1}, r_j]$ and so, $m \leqslant j-1 < \delta(m)$. We shall see that $\delta(m) = \beta_l(j)$, for some $l \in \{0, 1, ..., k\}$, and then B_m is included among the functions of (4.6). Since $m+1 \leqslant j \leqslant \delta(m)$, $[r_j, r_{\delta(m)}] \subseteq [r_{m+1}, r_{\delta(m)}]$. By Theorem 4.3, $[r_m, r_{\delta(m)}]$ is a right minimal support and so, dim $L_k(r_j, r_{j+1}, ..., r_{\delta(m)}) = 0$. By Corollary 2.5, $r_j, r_{j+1}, ..., r_{\delta(m)}$ are in general position and $\delta(m) - j \leqslant k$, and then

$$\beta_q(j) = j + q, \qquad q = 0, 1, ..., \delta(m) - j. \tag{4.8}$$

Taking $l = \delta(m) - j$ we obtain $\delta(m) = \beta_l(j)$.

On the other hand, we can use formula (4.5) and, rewriting the sub-indices as indicated by formula (4.8), we obtain for each $l \in \{0, 1, ..., k\}$

$$B_{\delta^{-1}(\beta_l(j))}(x, y) = \sum_{m=0}^{l} \alpha_m^l (\cos \omega_{\beta_m(j)} y - \sin \omega_{\beta_m(j)} x)^k$$
$$\forall\, (x, y) \in [r_{j-1}, r_j], \tag{4.9}$$

where $\alpha_l^l \neq 0$.

In consequence, if

$$\sum_{l=0}^{k} t_l B_{\delta^{-1}(\beta_l(j))}(x, y) = 0 \qquad \forall\, (x, y) \in [r_{j-1}, r_j],$$

then

$$0 = \sum_{l=0}^{k} t_l \left(\sum_{m=0}^{l} \alpha_m^l (\cos \omega_{\beta_m(j)} y - \sin \omega_{\beta_m(j)} x)^k \right)$$
$$= \sum_{m=0}^{k} \left(\sum_{l=m}^{k} t_l \alpha_m^l \right) (\cos \omega_{\beta_m(j)} y - \sin \omega_{\beta_m(j)} x)^k \qquad \forall\, (x, y) \in [r_{j-1}, r_j].$$

Since $r_j, r_{\beta_1(j)}, ..., r_{\beta_k(j)}$ are in general position, we have, by Proposition 2.2, det $A_k(\omega_j, \omega_{\beta_1(j)}, ..., \omega_{\beta_k(j)}) \neq 0$ and thus

$$\sum_{l=m}^{k} t_l \alpha_m^l = 0, \qquad m = 0, ..., k$$

and so, we may prove that $t_k, t_{k-1}, ..., t_1$ and t_0 are equal to zero. So, the local linear independence is proved. ▮

COROLLARY 4.5. *A basis for the local space $L_k(r_0, r_1, ..., r_n)$ is given by the $n-k$ functions*

$$B_i, \qquad i \notin \{\delta^{-1}(0), \delta^{-1}(\beta_1(0)), ..., \delta^{-1}(\beta_k(0))\}, \tag{4.10}$$

which are the functions in $\mathscr{B}$ whose support is contained in $[r_0, r_n]$.

Proof. Let us observe that in the proof of Theorem 4.4 it was shown that the functions $B_{\delta^{-1}(\beta_l(0))}$, $l=0, 1, ..., k$, are the unique functions in $\mathscr{B}$ whose support contains $[r_n, r_0]$. Taking $j=0$, we see that the functions of (4.10) are in $L_k(r_0, r_1, ..., r_n)$. By Theorem 4.4, they are locally linearly independent and by (2.20) they form a basis. ∎

Finally, we may obtain a Marsden type identity:

THEOREM 4.6. *For any* ω,

$$(\cos \omega \cdot y - \sin \omega \cdot x)^k = \sum_{j=0}^{n} \frac{\prod_{i=1}^{k} \sin(\omega_{\beta_i(j)} - \omega)}{\prod_{i=1}^{k} \sin(\omega_{\beta_i(j)} - \omega_j)} B_j(x, y).$$

Proof. Let $f(x, y) := (\cos \omega \cdot y - \sin \omega \cdot x)^k$. By Theorem 4.4 we may write

$$f = \sum_{i=0}^{n} \lambda_i B_i. \tag{4.11}$$

Let $j \in \{0, 1, ..., n\}$ be given. We want to find the coefficient λ_j.

Let us take $r_j, r_{\beta_1(j)}, ..., r_{\beta_k(j)}$. As they are in general position, by Proposition 2.2, $\det A_k(\omega_j, \omega_{\beta_1(j)}, ..., \omega_{\beta_k(j)}) \neq 0$ and there exist $c_0^{(j)}, c_1^{(j)}, ..., c_k^{(j)}$ such that

$$f(x, y) = \sum_{l=0}^{k} c_l^{(j)} (\cos \omega_{\beta_l(j)} y - \sin \omega_{\beta_l(j)} x)^k \tag{4.12}$$

The $c_0^{(j)}$ is given by

$$c_0^{(j)} = \frac{\prod_{i=1}^{k} \sin(\omega_{\beta_i(j)} - \omega)}{\prod_{i=1}^{k} \sin(\omega_{\beta_i(j)} - \omega_j)}.$$

By (4.4), $B_j|_{[r_j, r_{j+1}]} = (\cos \omega_j y - \sin \omega_j x)^k$ and then we obtain from (4.12)

$$f(x, y) = c_0^{(j)} B_j + \sum_{l=1}^{k} c_l^{(j)} (\cos \omega_{\beta_l(j)} y - \sin \omega_{\beta_l(j)} x)^k$$
$$\forall\, (x, y) \in [r_j, r_{j+1}]. \tag{4.13}$$

If we consider $B_{\delta^{-1}(\beta_1(j))}, ..., B_{\delta^{-1}(\beta_k(j))}$ we can deduce from (4.7) that $[r_j, r_{j+1}]$ is contained in their support and analogously to (4.9) we can derive the following formula for each $l \in \{1, ..., k\}$:

$$B_{\delta^{-1}(\beta_l(j))}(x, y) = \sum_{m=1}^{l} \alpha_m^l (\cos \omega_{\beta_m(j)} y - \sin \omega_{\beta_m(j)} x)^k$$
$$\forall\, (x, y) \in [r_j, r_{j+1}],$$

where $\alpha_l^l \neq 0$.

In consequence, we can express $(\cos \omega_{\beta_l(j)} y - \sin \omega_{\beta_l(j)} x)^k$, $l = 1, ..., k$, as a linear combination of $B_{\delta^{-1}(\beta_1(j))}, ..., B_{\delta^{-1}(\beta_k(j))}$ on $[r_j, r_{j+1}]$, which are functions of $\mathscr{B}$ different from B_j. Then by (4.13), we derive

$$f|_{[r_j, r_{j+1}]} = \left(c_0^{(j)} B_j + \sum_{\substack{i=0 \\ i \neq j}}^{n} t_i B_i \right) \Bigg|_{[r_j, r_{j+1}]}.$$

From the local linear independence of the B_i's (Theorem 4.4) and from the fact that supp $B_j \supseteq [r_j, r_{j+1}]$, we obtain by (4.11) that $\lambda_j = c_0^{(j)}$. ∎

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