Coincidence probability as a measure of the average phase-space density at freeze-out

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Abstract

It is pointed out that the average semi-inclusive particle phase-space density at freeze-out can be determined from the coincidence probability of the events observed in multiparticle production. The method of measurement is described and its accuracy examined.
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1. Recently, several methods were proposed which allow to estimate the average of the single-particle *inclusive* phase-space density produced in ultrarelativistic heavy ion collisions [1–5]. This quantity is useful in discussions of the equilibrated systems and therefore such measurements open possibilities to verify the expected presence of the thermal phase-space distribution at freeze-out [6] and/or search for more exotic phenomena [7].

In the present Letter we propose an extension of these studies by including, in addition, the *exclusive* and *semi-inclusive* M-particle phase-space densities. We show that their averages can be estimated from measured coincidence probabilities of the multiparticle events observed in high-energy collisions. The information one may gain from this approach is complementary to that obtained from single-particle inclusive measurements. In particular, it gives an insight into the correlation structure of the final state of the collision (including both dynamic and Bose–Einstein correlations), a feature which is ignored in the single-particle inclusive measurements described in [1–5]. Furthermore, one can show [8] that the average semi-inclusive phase-space densities are closely related to the second Renyi entropy [9] and thus their measurement allows to estimate a lower limit of the true (Shannon) entropy of the system (with-

out assuming the thermodynamic equilibrium). Needless to say, a comparison of the measured average semi-inclusive phase-space densities with expectations for the thermalized systems would be very interesting indeed.

As emphasized in [1,3], the phase-space distribution of particles produced in high-energy collisions is not a precisely defined quantity. Apart from the standard problems with the uncertainty principle, one has to take into account that particles may be produced at different times [1]. Following Bertsch [1], we shall ignore this problem and assume that all particles are created at the same time. That is to say, we are considering a time-average of the density [1,3].

To define the average semi-inclusive phase-space density, consider a collection of events in which exactly M particles were observed in a given region of the momentum space. We shall call them M-particle events (independently of how many particles were actually produced).[1] These events can be described by the normalized M particle phase-space distribution $W_M(X, K)$, with $X = X_1, \ldots, Z_M$, $K = K_x^{(1)}, \ldots, K_z^{(M)}$. The corresponding particle phase-space density is $D_M(X, K) =$

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[1] This is terminology often used in experimental description of multiparticle processes. The proper technical term is the exclusive distribution if all particles are observed, and semi-inclusive distribution if besides a given number of observed particles there is an unspecified number of other particles. This should not be confused with inclusive M-particle distributions.

$M W_M(X, K)$ and thus

$$\langle |D_M| \rangle = M \int dX \, dK \, |W_M(X, K)|^2 \equiv M(2\pi)^{-3M} C_M \qquad (1)$$

gives the average phase-space density of the M particle system.

It should be emphasized that, as is clear from this discussion, the phase-space density D_M, describing the semi-exclusive distribution, refers only to particles actually *measured* in a given experiment and in a given momentum region. It gives no direct information about the particles which are not registered in the detector. To obtain information on the phase-space density of *all* produced particles additional assumptions (e.g., thermodynamic equilibrium) are necessary.

Note that the average semi-inclusive phase-space density averaged over all particle multiplicities is simply obtained from (1), using

$$\langle \langle D \rangle \rangle = \sum_M P(M) \langle D_M \rangle \qquad (2)$$

where $P(M)$ is the multiplicity distribution. Therefore, from now on, to simplify the discussion, we shall only consider the case of a fixed multiplicity.

Our method is based on the observation that, for a rather wide class of models of particle production, the quantity C, defined in (1),[2] can be approximated by the measured coincidence probability C^{exp} of the events with M particles, defined as [10,11]

$$C^{\text{exp}} = \frac{N_2}{N(N-1)/2}, \qquad (3)$$

where N_2 is the number of the observed pairs of identical events and N is the total number of events. $N(N-1)/2$ is the total number of pairs of events.[3]

It is clear that, since the observed events are described by the particle momenta which are continuous variables, Eq. (3) is not directly applicable: a discretization is necessary. Then one can define the identical events as those which have the same population of the predefined bins and thus counting of coincidences becomes straightforward.[4] The counting of identical events obviously depends on the binning, so the procedure is ambiguous [10,11,13,15]. In order to obtain a viable estimate of the average particle density, we thus have to select the binning in such a way that the result of (3) is as close as possible to the exact value of C which, as seen from (1), gives directly the particle phase-space density.

In the present Letter we argue that for a fairly large class of physically sensible models, one can find an adequate binning procedure and thus to determine rather precisely $\langle D \rangle$ by measuring C^{exp}, i.e., by counting the number of pairs of identical events. The method turns out to be particularly suitable for large systems and thus may be useful in heavy ion collisions.

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[2] To simplify the formulae we shall from now on omit the index M in all quantities. Since we are discussing solely M-particle events, this should not lead to any confusion.

[3] Formula (3) was first suggested, in a different context, by Ma [12].

[4] A detailed description of this procedure was given in [13] and applied in [14].

In the next section the discretization procedure is described in some detail and the corresponding formulae for C^{exp} are written down. The phase-space density $\langle D \rangle$ and its relation to C^{exp} are discussed in Section 3. Our conclusions and outlook are given in the last section.

2. In this section we discuss how the discretization procedure affects the definition (3) of the coincidence probability. To this end we first express the C^{exp} given by (3) in terms of the momentum distribution of M particles $w(K) = w(K^{(1)}, \ldots, K^{(M)})$.

Consider a set of discretized events constructed by dividing the particle momentum space into J rectangular bins of volume

$$\omega_j = (\Delta_x \Delta_y \Delta_z)_j, \quad j = 1, \ldots, J. \qquad (4)$$

Then the probability to find a particle in bin ω_{j_1}, another one in bin ω_{j_2}, etc., is

$$P(j_1, j_2, \ldots, j_M) = \prod_{m=1}^M \omega_{j_m} \langle w(K_{j_1}^{(1)}, \ldots, K_{j_M}^{(M)}) \rangle \qquad (5)$$

where

$$\langle w(K_{j_1}^{(1)}, \ldots, K_{j_M}^{(M)}) \rangle$$
$$= \prod_{m=1}^M (\omega_{j_m})^{-1} \int_{\omega_{j_1}} dK^{(1)} \cdots \int_{\omega_{j_M}} dK^{(M)} \, w(K^{(1)}, \ldots, K^{(M)}). \qquad (6)$$

Note that the bins $\omega_{j_1}, \ldots, \omega_{j_M}$ do not have to be different.

Thus the coincidence probability as measured by the formula (3) is

$$C_M^{\text{exp}} = \sum_{j_1} \cdots \sum_{j_M} \left[P(j_1, j_2, \ldots, j_M) \right]^2$$
$$= \sum_{j_1} \cdots \sum_{j_M} \prod_{m=1}^M [\omega_{j_m}]^2 \left[\langle w(K_{j_1}^{(1)}, \ldots, K_{j_M}^M) \rangle \right]^2. \qquad (7)$$

The first equality follows from the observation that sampling a series of events is the Bernoulli process and thus probability to find, after N trials, $n_1, \ldots, n_J$ events in configurations $\{1\}, \ldots, \{J\}$ is

$$B(n_1, \ldots, n_J) = \frac{N!}{n_1! \cdots n_J!} (P_1)^{n_1} \cdots (P_J)^{n_J}. \qquad (8)$$

From this formula it is not difficult to see that

$$\langle N_2 \rangle \equiv \sum_{n_1, \ldots, n_J} \sum_{j=1}^J \frac{n_j(n_j - 1)}{2} B(n_1, \ldots, n_J)$$
$$= \frac{N(N-1)}{2} \sum_{j=1}^J (P_j)^2. \qquad (9)$$

The question now is: how to select the bins ω_j to obtain a result as close as possible to C giving the average value of the particle phase-space density $\langle D \rangle$ (cf. (1)). This is discussed in the next section.

3. To analyze the relation between $C^{\exp}$ and C we consider the M-particle phase-space distribution of the general form

$$W(X, K) = \frac{1}{(L_x L_y L_z)^M} G[X/L] w(K), \qquad (10)$$

with $X/L \equiv (X_1 - \bar{X}_1)/L_x, \ldots, (Z_M - \bar{Z}_M)/L_z$, $K = K_1$, $\ldots, K_M$. The function G satisfies the normalization conditions

$$\int d^{3M} u\, G(u) = 1 \rightarrow \int dX\, G(X/L) = (L_x L_y L_z)^M,$$

$$\int d^{3M} u\, u_i\, G(u) = 0 \rightarrow \langle X_i, Y_i, Z_i \rangle = \bar{X}_i, \bar{Y}_i, \bar{Z}_i,$$

$$\int d^{3M} u\, (u_i)^2 G(u) = 1 \rightarrow \langle (X_i - \bar{X}_i)^2, \ldots \rangle = L_x^2, \ldots. \quad (11)$$

The first condition insures that $w(K)$ is the observed (multidimensional) momentum distribution,[5] the second defines the central values of the particle distribution in configuration space and the third defines L_x, L_y, L_z as giving root mean square sizes of the distribution in configuration space. Both sizes and central positions may depend on the particle momenta.[6] The form of the function G describes the shape of the multiparticle distribution in configuration space.

Ansatz (10) for the time-averaged phase space density is satisfied in a large variety of models [16].

Using (10) we obtain from (1)

$$C = (2\pi)^{3M} \int d^{3M} K \left[w(K_1, \ldots, K_M) \right]^2$$
$$\times \int \frac{dX_1 \cdots dZ_M}{(L_x L_y L_z)^{2M}} \left[G(X/L) \right]^2$$
$$= (2\pi g)^{3M} \int d^{3M} K \left[w(K_1, \ldots, K_M) \right]^2 \frac{1}{(L_x L_y L_z)^M}, \quad (12)$$

with

$$g^{3M} = \int d^{3M} u \left[G(u) \right]^2. \qquad (13)$$

The constant g depends on the shape of particle distribution in configuration space. This dependence is, however, rather mild. For example, we obtain $g^{-1} = 2\sqrt{\pi}$ for Gaussians and $g^{-1} = 2\sqrt{3}$ for a rectangular box.

In the discretized form, (12) can be written as

$$C = (2\pi g)^{3M} \sum_{j_1, \ldots, j_M} \prod_{m=1}^{M} \frac{\omega_{j_m}}{(L_x L_y L_z)_{j_m}} \langle [w_{j_1, \ldots, j_M}]^2 \rangle, \qquad (14)$$

where

$$\langle [w_{j_1, \ldots, j_M}]^2 \rangle$$
$$= \prod_{m=1}^{M} (\omega_{j_m})^{-1}$$
$$\times \int_{\omega_{j_1}} dK^{(1)} \cdots \int_{\omega_{j_M}} dK^{(M)} \left\{ w \left(K^{(1)}, \ldots, K^{(M)} \right) \right\}^2, \quad (15)$$

and $(L_x L_y L_z)_{j_m}$ is a suitable average of $(L_x L_y L_z)$ over bin j_m.

Comparing (14) with (7) one sees that to have $C_M^{\exp}$ as close as possible to C, the best volume of the bins is

$$\omega_{j_m} = (\Delta_x \Delta_y \Delta_z)_{j_m} = \frac{(2\pi g)^3}{(L_x L_y L_z)_{j_m}}. \qquad (16)$$

One sees from this formula that ω depends crucially on the volume of the system in configuration space. One sees, furthermore, that with this choice of ω the coincidence probability $C^{\exp}$, determined by counting the number of pairs of identical events (cf. (3)), is related to C, giving directly the average particle phase-space density $\langle D \rangle$ (cf. (1) and (14)), by the formula

$$C = C^{\exp} \frac{\sum_{j_1, \ldots, j_M} \langle [w_{j_1, \ldots, j_M}]^2 \rangle}{\sum_{j_1, \ldots, j_M} [\langle w_{j_1, \ldots, j_M} \rangle]^2}. \qquad (17)$$

It is thus clear that the accuracy of determination of C increases with increasing volume of the system. Indeed, for a volume large enough, the bins defined by (16) are small and the ratio on the R.H.S. of (17) approaches unity. For smaller volumes the method is less accurate but one may try to estimate the correcting ratio from the (measured) single particle distribution.

4. Several comments are in order.

(i) One sees from (16) that the optimal size of the bin does not depend on the average position of the particles at freeze-out. This implies that the momentum–position correlations induced by the K-dependence of $\bar{X}$ do not influence significantly the measurement of the coincidence probability.

(ii) It is also seen from (16) that only the volume of the bin $\omega_{j_m} = (\Delta_x \Delta_y \Delta_z)_{j_m}$, but not its shape, matters in the determination of the optimal discretization. One can use this freedom to improve the accuracy of the measurement by taking bins large in the directions with weak momentum dependence and small in the direction where the momentum dependence is significant.

(iii) One may improve the accuracy of the measurement by estimating the ratio

$$\frac{\sum_{j_1, \ldots, j_M} \langle [w_{j_1, \ldots, j_M}]^2 \rangle}{\sum_{j_1, \ldots, j_M} [\langle w_{j_1, \ldots, j_M} \rangle]^2}.$$

This may be possible if the momentum distribution of particles is measured with good accuracy.

(iv) Our analysis can be applied to any part of the momentum space. This allows to measure the local particle density in momentum space, averaged over all configuration space. In case of strong momentum–position correlations, the selection of a given momentum region can induce, however, a selection of a corresponding region in configuration space.

(v) The accuracy of the measurement depends crucially on the correct estimate of the size of the system. Information from HBT measurements should allow to determine the parameters L_x, L_y, L_z and—at least in principle—also

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[5] By definition, $w(K) = \int dX\, W(X, K)$.
[6] They may be also different for different kinds of particles.

the shape[7] of the function $G(u)$ (some procedures are described in [2,4]). Therefore, good HBT data are essential for a successful application of the method.

(vi) The presented analysis of the discretization procedure can be generalized to higher order coincidence probabilities [17]. This opens the way to a determination of higher Renyi entropies [8] and then, by extrapolation, to obtain information on the Shannon entropy of the system [11].

In conclusion, we propose to estimate the phase-space density of particles produced in high-energy collisions by measuring the coincidence probability of the observed events. The accuracy of the determination of the coincidence probability by counting the number of the identical events [10,11] was analyzed for a large class of physically sensible models. It was shown that the accuracy improves with increasing volume of the system and, therefore, the method is particularly suitable for heavy ion collisions. A formula giving the optimal discretization method in terms of the size of the system in the configuration space (Eq. (16)) was derived.

Acknowledgements

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