On some linear combinations of hypergeneralized projectors

Jerzy K. Baksalary a, Oskar Maria Baksalary b, *, Jürgen Groß c, 1

a Faculty of Mathematics, Informatics and Econometrics, Zielona Góra University, ul. Podgórna 50, PL 65-246 Zielona Góra, Poland
b Institute of Physics, Adam Mickiewicz University, ul. Umultowska 85, PL 61-614 Poznań, Poland
c Department of Statistics, University of Dortmund, Dortmund, Germany

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Abstract

The concept of a hypergeneralized projector as a matrix H satisfying $H^2 = H^\dagger$, where $H^\dagger$ denotes the Moore–Penrose inverse of H, was introduced by Groß and Trenkler [Generalized and hypergeneralized projectors, Linear Algebra Appl. 264 (1997) 463–474]. In the present paper, the problem of when a linear combination $c_1 H_1 + c_2 H_2$ of two hypergeneralized projectors $H_1$, $H_2$ is also a hypergeneralized projector is considered. Although, a complete solution to this problem remains unknown, this article provides characterizations of situations in which $(c_1 H_1 + c_2 H_2)^2 = (c_1 H_1 + c_2 H_2)^\dagger$ derived under certain commutativity property imposed on matrices $H_1$ and $H_2$. The results obtained substantially generalize those given in the above mentioned paper by Groß and Trenkler.

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* Corresponding author.
E-mail addresses: baxx@amu.edu.pl (O.M. Baksalary), gross@statistik.uni-dortmund.de (J. Groß).
1 Postal address: Eierkampstr. 24, D-44225 Dortmund, Germany.

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1. Introduction and preliminaries

Let $C_{m,n}$ be the set of $m \times n$ complex matrices. The symbols $K^*$, $\mathcal{R}(K)$, $\mathcal{N}(K)$, and $r(K)$ will denote the conjugate transpose, range (column space), null space, and rank, respectively, of $K \in C_{m,n}$. Further, $K^\dagger$ will stand for the Moore–Penrose inverse of $K$, i.e., the unique matrix satisfying the four equations

$$KK^\dagger K = K, \quad K^\dagger KK^\dagger = K^\dagger, \quad KK^\dagger = (KK^\dagger)^*, \quad K^\dagger K = (K^\dagger K)^*,$$

(1.1)

and $I_n$ will be the identity matrix of order $n$. Moreover, $C_{n,n}^{\text{QP}}$, $C_{n,n}^{\text{EP}}$, and $C_{n,n}^{\text{HGP}}$ will denote the subsets of $C_{n,n}$ consisting of quadripotent matrices, EP (range-Hermitian) matrices, and hypergeneralized projectors, respectively, i.e.,

$$C_{n,n}^{\text{QP}} = \{K \in C_{n,n} : K^4 = K\},$$

$$C_{n,n}^{\text{EP}} = \{K \in C_{n,n} : \mathcal{R}(K) = \mathcal{R}(K^*)\} = \{K \in C_{n,n} : KK^\dagger = K^\dagger K\},$$

and

$$C_{n,n}^{\text{HGP}} = \{K \in C_{n,n} : K^2 = K^\dagger\}. \quad (1.2)$$

The concept of a hypergeneralized projector was introduced by Groß and Trenkler [8, p. 466], who provided also several properties of the class of matrices defined in (1.2). The characterization

$$C_{n,n}^{\text{HGP}} = C_{n,n}^{\text{QP}} \cap C_{n,n}^{\text{EP}}, \quad (1.3)$$

constituting part (a) $\iff$ (d) of Theorem 2 in [8], plays an essential role in the present paper (see also Theorem 3 in [3]). (More information on EP matrices can be found in Pearl [13].)

Baksalary and Baksalary [1], established a complete solution to the problem of when a linear combination of two different projectors (idempotent matrices) is also a projector by listing all situations in which nonzero complex numbers $c_1$, $c_2$ and nonzero complex matrices $P_1$, $P_2$ ($P_1 \neq P_2$) satisfying $P_i^2 = P_i$, $i = 1, 2$, form the matrix $P = c_1 P_1 + c_2 P_2$ such that $P^2 = P$. Recently, Baksalary and Baksalary [2] solved an analogous problem for generalized projectors (defined in [8, p. 465] as matrices $G$ satisfying $G^2 = G^*$) instead of projectors and other related problems were solved in [5,6]. Similar considerations concerning a linear combination

$$H = c_1 H_1 + c_2 H_2 \quad (1.4)$$

of $H_1, H_2 \in C_{n,n}^{\text{HGP}}$ with $c_1, c_2 \in \mathbb{C}$ are more complicated. This is a consequence of the fact that the derivation of necessary conditions for $H^2 = H^\dagger$ may depend on the formula for the Moore–Penrose inverse of a sum of matrices, developed in the general case by Hung and Markham [12, Theorem 1], which is not easy to handle. Nevertheless, although a complete solution to the problem of when the matrix $H$ of the form (1.4) is a hypergeneralized projector remains unknown, we provide interesting characterizations of the condition $H^2 = H^\dagger$, obtained under certain commutativity property imposed on matrices $H_1$ and $H_2$. 
It seems reasonable to exclude from subsequent considerations certain simple versions of a linear combination of the form \((1.4)\). Trivially, when \(H_1\) and \(H_2\) are both zero, then the zero matrix \(H\) belongs to \(\mathbb{C}_n^{\text{HGP}}\) for any \(c_1, c_2 \in \mathbb{C}\). Other simple situations may be identified in view of the following observation.

**Observation.** If \(A \in \mathbb{C}_n^{\text{HGP}}\) is nonzero and \(\alpha \in \mathbb{C}\), then a scalar multiple \(\alpha A\) belongs to \(\mathbb{C}_n^{\text{HGP}}\) if and only if \(\alpha = 0\) or \(\alpha^3 = 1\).

From Observation it clearly follows that when \(H_1 \neq 0 = H_2\), then \(H \in \mathbb{C}_n^{\text{HGP}}\) if and only if the scalar \(c_1\) related to nonzero \(H_1\) is either zero or a cubic root of unity. To exclude this situation (as well as its counterpart when \(H_1 = 0 \neq H_2\)) from further considerations we will hereafter assume that \(H_1\) and \(H_2\) are both nonzero. In fact, a similar situation occurs when one of the nonzero hypergeneralized projectors involved in \(H\) is a scalar multiple of the other, e.g., when \(H_2 = \alpha H_1\) for some nonzero \(\alpha \in \mathbb{C}\) satisfying \(\alpha^3 = 1\). Then \(H = (c_1 + c_2\alpha)H_1\) belongs to \(\mathbb{C}_n^{\text{HGP}}\) if only if the number \(c_1 + c_2\alpha\) is either zero or a cubic root of unity. Thus, in the subsequent considerations we will assume that nonzero \(H_1\) and \(H_2\) in the matrix \(H\) of the form \((1.4)\) are not scalar multiples of each other. Moreover, in what follows, \(c_1\) and \(c_2\) are assumed to be nonzero.

In the subsequent considerations we will refer to two concepts known in the literature. The first of them is the star-orthogonality introduced by Hestenes [11]. Let us recall that matrices \(A, B \in \mathbb{C}_{m,n}\) are said to be star-orthogonal, denoted by \(A \perp B\), whenever

\[
AB^* = 0 \quad \text{and} \quad A^*B = 0.
\]  
(1.5)
If \(A, B \in \mathbb{C}_n^{\text{EP}}\), then, in view of \((1.1)\), the two conditions in \((1.5)\) are equivalent. Moreover, in such a case

\[
A \perp B \iff AB = 0 \iff BA = 0.
\]  
(1.6)

The second concept of interest is the star partial ordering, introduced by Drazin [7]. Let us recall that for matrices \(A, B \in \mathbb{C}_{m,n}\), a matrix \(A\) is said to be below \(B\) with respect to the star partial ordering, denoted by \(A \preceq B\), whenever

\[
A^*A = A^*B \quad \text{and} \quad AA^* = BA^*.
\]  
(1.7)
Several alternatives to and/or modifications of conditions \((1.7)\) are known in the literature. In particular, from statement \((1.11)\) in Baksalary et al. [4] (see also [9]) it follows that if \(A \in \mathbb{C}_n^{\text{EP}}\), then for any \(B \in \mathbb{C}_{n,n}\),

\[
A \preceq B \iff AB = A^2 = BA,
\]  
(1.8)
cf. Lemma 1 in [8].

The notions of the star-orthogonality and the star partial ordering are mutually related, and for matrices \(A, B \in \mathbb{C}_{m,n}\) the relationship between them may be expressed as

\[
A \perp B \iff A^* \preceq A + B \iff B^* \preceq A + B.
\]
Some results concerning hypergeneralized projectors referring to the star-orthogonality and the star partial ordering were given by Groß and Trenkler [8]. They are recalled in the following lemma.

**Lemma.** For \( H_1, H_2 \in \mathbb{C}_{n}^{HGP} \), the following three statements hold:

(i) If \( H_1 \perp H_2 \), then \( H_1 + H_2 \in \mathbb{C}_{n}^{HGP} \),

(ii) If \( H_1 \preceq H_2 \), then \( H_2 - H_1 \in \mathbb{C}_{n}^{HGP} \),

(iii) For nonzero \( H_1 \) and \( \alpha \in \mathbb{C} \), the condition \( \alpha H_1 \preceq H_2 \) implies either \( \alpha = 0 \) or \( \alpha^3 = 1 \).

**Proof.** Statements (i) and (ii) were already given by Groß and Trenkler [8, pp. 471, 472], where the condition \( H_1 \perp H_2 \) was equivalently expressed as \( H_1 H_2 = H_2 H_1 = 0 \). For the proof of statement (iii) note that in view of (1.3), all hypergeneralized projectors are EP matrices and, consequently, so are all their scalar multiples. Thus, \( \alpha H_1 \in \mathbb{C}_{n}^{EP} \) and, on account of Theorem 3 in [8], it follows that the star partial ordering \( \alpha H_1 \preceq H_2 \) forces \( \alpha H_1 \) to belong to \( \mathbb{C}_{n}^{HGP} \). But, since \( H_1 \neq 0 \), Observation yields \( \alpha = 0 \) or \( \alpha^3 = 1 \). □

2. Results

Considerations concerning the problem of when a linear combination \( H = c_1 H_1 + c_2 H_2 \) of two hypergeneralized projectors \( H_1, H_2 \) is also a hypergeneralized projector involve the Moore–Penrose inverse of \( H \). Although the formula for the Moore–Penrose inverse of a sum of two matrices was provided by Hung and Markham [12, Theorem 1], the expression for \( (c_1 H_1 + c_2 H_2)^\dagger \) is in general case difficult to handle. However, by assuming that nonzero \( H_1, H_2 \in \mathbb{C}_{n}^{HGP} \) appearing in a linear combination of the form (1.4) satisfy

\[
H_1 H_2 = \eta_1 H_1^2 + \eta_2 H_2^2 = H_2 H_1 \tag{2.1}
\]

for some scalars \( \eta_1, \eta_2 \in \mathbb{C} \), the situation improves and still several interesting characterizations of \( H^2 = H^\dagger \) can be obtained. Observe, that condition (2.1) guarantees that the square of the matrix \( H \) of the form (1.4) can be expressed as a linear combination

\[
H^2 = (c_1^2 + 2c_1 c_2 \eta_1) H_1^2 + (c_2^2 + 2c_1 c_2 \eta_2) H_2^2. \tag{2.2}
\]

Hence, from \( H_1^2 = H_1^\dagger \) and \( H_2^2 = H_2^\dagger \) it is seen that when \( H^2 = H^\dagger \), then (2.1) ensures that \( (c_1 H_1 + c_2 H_2)^\dagger = \alpha_1 H_1^\dagger + \alpha_2 H_2^\dagger \) for some \( \alpha_1, \alpha_2 \in \mathbb{C} \).
In what follows we will separately consider two versions of condition (2.1), namely when (at least) one of the scalars $\eta_1$, $\eta_2$ is zero and when both of them are nonzero. Let us first assume that (2.1) holds with $\eta_2$, say, being equal to zero. In such a case condition (2.1) takes the form

$$H_1H_2 = \eta H_1^2 = H_2H_1$$  \tag{2.3}

for some $\eta \in \mathbb{C}$. If $\eta$ is nonzero, then, on account of (1.8), condition (2.3) is satisfied if and only if

$$\eta H_1 \leq H_2.$$  \tag{2.4}

In view of point (iii) of Lemma, relation (2.4) implies $\eta^3 = 1$. Thus, it follows that the choice of $\eta$ in (2.3) is limited to the set $\{0, 1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i\}$.

The following theorem provides necessary and sufficient conditions for a linear combination of the form (1.4), with $H_1, H_2$ satisfying (2.3), to be also a hypergeneralized projector.

**Theorem 1.** Let nonzero $H_1, H_2 \in \mathbb{C}_{n}^{\text{HGP}}$ be such that they are not scalar multiples of each other and satisfy $H_1H_2 = \eta H_1^2 = H_2H_1$ for some $\eta \in \{0, 1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i\}$. Then for nonzero $c_1, c_2 \in \mathbb{C}$, a linear combination $H = c_1H_1 + c_2H_2$ belongs to $\mathbb{C}_{n}^{\text{HGP}}$ if and only if

$$c_2^3 = 1$$  \tag{2.5}

holds along with either one of the conditions

$$c_1 + c_2\eta = 0 \quad \text{or} \quad (c_1 + c_2\eta)^3 = 1.$$  \tag{2.6}

**Proof.** The assumptions ensure that inclusion $\eta H_1 \in \mathbb{C}_{n}^{\text{HGP}}$ as well as condition (2.4) hold. Thus, on account of point (ii) of Lemma, it follows that

$$H_2 - \eta H_1 \in \mathbb{C}_{n}^{\text{HGP}},$$  \tag{2.7}

where the matrix $H_2 - \eta H_1$ cannot be the zero matrix. In addition, it is easy to verify that relation (2.4) implies

$$(c_1 + c_2\eta)H_1 \leq H.$$  \tag{2.8}

For the proof of the ‘only if’ part, let $H \in \mathbb{C}_{n}^{\text{HGP}}$. Then, in view of (2.8) and point (iii) of Lemma, either one of equalities in (2.6) must hold. In addition, on account of point (ii) of this lemma, condition (2.8) combined with (2.6), implies $H - (c_1 + c_2\eta)H_1 \in \mathbb{C}_{n}^{\text{HGP}}$. It is easily seen that

$$H - (c_1 + c_2\eta)H_1 = c_2(H_2 - \eta H_1)$$

and thus, in view of (2.7), assumption $c_2 \neq 0$, and Observation, condition (2.5) follows.
For the proof of the ‘if’ part, let (2.5) along with either of conditions in (2.6) be satisfied. Then the two matrices
\[ H_3 = (c_1 + c_2\eta)H_1 \quad \text{and} \quad H_4 = c_2(H_2 - \eta H_1) \]
both belong to \( C_n^{\text{HGP}} \) and \( H_3 \perp H_4 \). Consequently, from point (i) of Lemma it follows that \( H = H_3 + H_4 \in C_n^{\text{HGP}} \). The proof is complete. \( \square \)

The following two corollaries are obtained from Theorem 1 by substituting \( \eta = 0 \) and \( \eta = 1 \), respectively.

**Corollary 1.** Let nonzero \( H_1, H_2 \in C_n^{\text{HGP}} \) satisfy \( H_1 \perp H_2 \). Then for nonzero \( c_1, c_2 \in \mathbb{C} \), a linear combination \( H = c_1H_1 + c_2H_2 \) belongs to \( C_n^{\text{HGP}} \) if and only if \( c_3^2 = 1 \) and \( c_1^3 = 1 \).

**Proof.** Since \( H_1, H_2 \in C_n^{\text{HGP}} \) are EP matrices, it follows from (1.6) that the relationship \( H_1 \perp H_2 \) is satisfied if and only if \( H_1H_2 = 0 \) (\( = H_2H_1 \)). Consequently, matrices \( H_1 \) and \( H_2 \) are not scalar multiples of each other, for otherwise \( H_2 = \alpha H_1 \) for some nonzero \( \alpha \in \mathbb{C} \) would imply \( \alpha H_1^2 = 0 \), which would further lead to \( H_1 = 0 \), being in a contradiction with the assumptions. Hence, the assertion follows from Theorem 1 for \( \eta = 0 \). \( \square \)

Corollary 1 provides a complete characterization of all situations in which a linear combination of the form \( H = c_1H_1 + c_2H_2 \), with hypergeneralized projectors \( H_1, H_2 \) satisfying \( H_1 \perp H_2 \), is a hypergeneralized projector. This result generalizes the one given by Groß and Trenkler [8, p. 471], which covers only the case \( c_1 = 1, c_2 = 1 \).

**Corollary 2.** Let nonzero different \( H_1, H_2 \in C_n^{\text{HGP}} \) satisfy \( H_1 \preceq H_2 \). Then for nonzero \( c_1, c_2 \in \mathbb{C} \), a linear combination \( H = c_1H_1 + c_2H_2 \) belongs to \( C_n^{\text{HGP}} \) if and only if \( c_3^2 = 1 \) holds along with \( c_1 + c_2 = 0 \) or along with \( (c_1 + c_2)^3 = 1 \).

**Proof.** Since \( H_1 \in C_n^{\text{HGP}} \) is an EP matrix, it follows from (1.8) that relationship \( H_1 \preceq H_2 \) is satisfied if and only if \( H_1H_2 = H_1^2 = H_2H_1 \). In such a case matrices \( H_1 \) and \( H_2 \) cannot be scalar multiples of each other unless \( H_1 = H_2 \). This is a consequence of the fact that \( H_2 = \alpha H_1 \) for some nonzero \( \alpha \in \mathbb{C} \) would imply \( \alpha H_1^2 = \alpha H_2^2 \), which would further lead to \( (1 - \alpha)H_1^2 = 0 \), being satisfied only when \( \alpha = 1 \). Hence, the assertion follows from Theorem 1 for \( \eta = 1 \). \( \square \)

Corollary 2 provides a complete characterization of all situations in which a linear combination of the form \( H = c_1H_1 + c_2H_2 \), with hypergeneralized projectors \( H_1, H_2 \) satisfying \( H_1 \preceq H_2 \), is a hypergeneralized projector. This result generalizes the
one given by Groß and Trenkler [8, p. 472], which covers only the case $c_1 = -1$, $c_2 = 1$.

**Remark 1.** Under the assumptions of Theorem 1, there are exactly 36 different pairs $(c_1, c_2)$ composed of nonzero $c_1, c_2 \in \mathbb{C}$, for which a linear combination $H = c_1H_1 + c_2H_2$ belongs to $\mathbb{C}_n^{HGP}$.

The last corollary with respect to Theorem 1 is concerned with the rank of a linear combination $H = c_1H_1 + c_2H_2$ belonging to $\mathbb{C}_n^{HGP}$.

**Corollary 3.** Under the assumptions of Theorem 1, let $H = c_1H_1 + c_2H_2$ belong to $\mathbb{C}_n^{HGP}$, $c_i \neq 0$, $i = 1, 2$. Then $r(H) = r(H_2) - \delta r(H_1)$, where

$$
\delta = \begin{cases} 
1 & \text{if } \eta \neq 0 \text{ and } c_1 + c_2\eta = 0, \\
0 & \text{if } \eta \neq 0 \text{ and } (c_1 + c_2\eta)^3 = 1, \\
-1 & \text{if } \eta = 0.
\end{cases}
$$

**Proof.** Since the assumptions of Theorem 1 imply conditions (2.4) and (2.8), it follows that $r(H_2 - \eta H_1) = r(H_2) - r(\eta H_1)$ and $r[H - (c_1 + c_2\eta)H_1] = r(H) - r[(c_1 + c_2\eta)H_1]$; see e.g., equivalence (1.6) in [10]. Combining these two equalities leads to

$$r(H) = r(H_2) + r[(c_1 + c_2\eta)H_1] - r(\eta H_1),$$

and the assertion follows straightforwardly from Theorem 1. \hfill \Box

Let us now assume that both scalars $\eta_1$ and $\eta_2$ involved in (2.1) are nonzero and observe that postmultiplying this condition by the matrix $I_n - H_1^\dagger H_1$ leads to

$$
\eta_2H_2^2(I_n - H_1^\dagger H_1) = 0. \tag{2.9}
$$

Since $\eta_2 \neq 0$, it follows from (2.9) that $\mathcal{R}(I_n - H_1^\dagger H_1) \subseteq \mathcal{N}(H_2^2)$, where $\mathcal{R}(I_n - H_1^\dagger H_1) = \mathcal{N}(H_1)$ and $\mathcal{N}(H_2^2) = \mathcal{N}(H_2^3) = \mathcal{N}(H_2) = \mathcal{N}(H_2)$. Consequently, $\mathcal{N}(H_1) \subseteq \mathcal{N}(H_2)$ or, in other words, $\mathcal{R}(H_2) \subseteq \mathcal{R}(H_1)$. A similar reasoning referred to the equality obtained from (2.1) by postmultiplying by the matrix $I_n - H_2^\dagger H_2$, under the assumption $\eta_1 \neq 0$, leads to $\mathcal{R}(H_1) \subseteq \mathcal{R}(H_2)$. Hence it is clear that (2.1) ensures $\mathcal{R}(H_1) = \mathcal{R}(H_2)$. The last condition can equivalently be expressed as $H_1^\dagger = H_2^3$ or, taking into account that $H_i^3 = H_i^2$, $i = 1, 2$, as

$$
H_i^3 = H_i^2. \tag{2.10}
$$

Another observation is that utilizing (2.10), the equality obtained from (2.1) by premultiplying by $H_1$ and postmultiplying by $H_2$ yields

$$
H_1^2 = \eta_1 H_2^2 + \eta_2 H_1 H_2. \tag{2.11}
$$

Combining (2.11) with (2.1) leads to the condition $H_1^2 = \eta_1 H_2^2 + \eta_1 \eta_2 H_1^2 + \eta_2^2 H_2^2$, which can alternatively be expressed in the form
(1 - \eta_1 \eta_2)H_1^2 = (\eta_1 + \eta_2^2)H_2^2. \quad (2.12)

It follows from (2.12) that if one of the numbers 1 - \eta_1 \eta_2 or \eta_1 + \eta_2^2 was zero and the other was not, then \textbf{H}_1^2 = \textbf{0} or \textbf{H}_2^2 = \textbf{0}, leading to \textbf{H}_1 = \textbf{0} or \textbf{H}_2 = \textbf{0}, i.e., to an alternative which is in a contradiction with the assumptions. On the other hand, if both these numbers were nonzero, i.e., if \textbf{H}_1^2 = \alpha \textbf{H}_2^2 for some nonzero \alpha \in \mathbb{C}, then \textbf{H}_1^\dagger = \alpha \textbf{H}_2^\dagger and so \textbf{H}_1 = \alpha^\dagger \textbf{H}_2, where \alpha^\dagger = 1/\alpha if \alpha \neq 0 and \alpha^\dagger = 0 if \alpha = 0. Since such a situation was excluded from the considerations, it is clear that (2.12) implies

\[ 1 - \eta_1 \eta_2 = 0 \quad \text{and} \quad \eta_1 + \eta_2^2 = 0. \quad (2.13) \]

Conditions (2.13) are satisfied if and only if \eta_2^3 = -1 and \eta_1 = 1/\eta_2 = \tilde{\eta}_2, where \tilde{\eta}_2 denotes the conjugate of \eta_2. Taking the relations between \eta_1 and \eta_2 into account, condition (2.1) can be expressed as

\[ H_1H_2 = \tilde{\eta}H_1^2 + \eta H_2^2 = H_2H_1, \quad \text{where} \ \eta \in \left\{ -1, \frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i \right\}. \quad (2.14) \]

The next theorem provides necessary and sufficient conditions ensuring that a linear combination \textbf{H} = c_1\textbf{H}_1 + c_2\textbf{H}_2, with \textbf{H}_1, \textbf{H}_2 \in \mathbb{C}^{\text{HGP}} satisfying (2.14), belongs to \mathbb{C}^{\text{QP}}, thus giving a necessary condition for \textbf{H} \in \mathbb{C}^{\text{HGP}}.

**Theorem 2.** Let nonzero \textbf{H}_1, \textbf{H}_2 \in \mathbb{C}^{\text{HGP}} be such that they are not scalar multiples of each other and satisfy \textbf{H}_1\textbf{H}_2 = \tilde{\eta}\textbf{H}_1^2 + \eta\textbf{H}_2^2 = \textbf{H}_2\textbf{H}_1 for some \eta \in \left\{ -1, \frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i \right\}. Then for nonzero \eta_1, \eta_2 \in \mathbb{C}, a linear combination \textbf{H} = c_1\textbf{H}_1 + c_2\textbf{H}_2 belongs to \mathbb{C}^{\text{QP}} if and only if

\[ c_1^3 + 6c_1c_2^2\eta + 4c_2^3 = 1 \quad \text{and} \quad c_2^3 + 6c_1^2c_2\tilde{\eta} + 4c_1^3 = 1. \quad (2.15) \]

**Proof.** From (2.2) it is seen that with \eta_1 = \tilde{\eta} and \eta_2 = \eta, the square of the matrix \textbf{H} can be expressed as \textbf{H}^2 = \delta_1\textbf{H}_1^2 + \delta_2\textbf{H}_2^2, where

\[ \delta_1 = c_1^2 + 2c_1c_2\tilde{\eta} \quad \text{and} \quad \delta_2 = c_2^2 + 2c_1c_2\eta. \quad (2.16) \]

On account of the condition

\[ H_1^2H_2^2 = \tilde{\eta}H_2 + \eta H_1 = H_2^2H_1^2, \quad (2.17) \]

obtained from (2.14) with the use of (2.10), it follows that

\[ \textbf{H}^4 = \textbf{H}^2\textbf{H}^2 = (\delta_1^2 + 2\delta_1\delta_2\eta)\textbf{H}_1 + (\delta_2^2 + 2\delta_1\delta_2\tilde{\eta})\textbf{H}_2. \quad (2.18) \]

Since \textbf{H}_1 and \textbf{H}_2 are both nonzero and are not scalar multiples of each other, it follows from (2.18) that \textbf{H}^4 = \textbf{H} if and only if \delta_1^2 + 2\delta_1\delta_2\eta = c_1 and \delta_2^2 + 2\delta_1\delta_2\tilde{\eta} = c_2, which can equivalently be expressed as in (2.15). \qed
To characterize all linear combinations $H$ of the form (1.4), which under the assumptions of Theorem 2 belong to $C_n^{HGP}$, it requires to characterize matrices $H$ being simultaneously quadripotent and EP. In fact, if $H \in C_n^{OP}$, then clearly $H^2$ is the group inverse of $H$, and it satisfies $H^2 = H^4$ if and only if $H^3$ is Hermitian, which appears to be difficult to characterize much further. However, it can be shown that inclusion $H \in C_n^{EP}$ is satisfied when matrix $H$ has the maximal possible rank under the imposed assumptions. This is a consequence of the fact that (2.1) ensures $\mathcal{R}(H_1) = \mathcal{R}(H_1^*) = \mathcal{R}(H_2) = \mathcal{R}(H_2^*)$. Then clearly $\mathcal{R}(H) \subseteq \mathcal{R}(H_1)$ and $\mathcal{R}(H^*) \subseteq \mathcal{R}(H_1)$, and the rank of $H$ is maximal if it is equal to $r(H_1) = r(H_2)).$ But in such a case $\mathcal{R}(H) = \mathcal{R}(H_1)$ and $\mathcal{R}(H^*) = \mathcal{R}(H_1)$, showing that $H$ is range-Hermitian and thus EP.

**Theorem 3.** Let nonzero $H_1, H_2 \in C_n^{HGP}$ be such that they are not scalar multiples of each other and satisfy $H_1H_2 = \eta H_1^2 + \eta H_2 H_1$ for some $\eta \in \{-1, \frac{1}{2} - \frac{\sqrt{3}}{2} i, \frac{1}{2} + \frac{\sqrt{3}}{2} i\}$. Then for nonzero $c_1, c_2 \in \mathbb{C}$, a linear combination $H = c_1 H_1 + c_2 H_2$ has maximal rank and in addition belongs to $C_n^{HGP}$ if and only if

$$c_1^3 = -1 \quad \text{and} \quad c_1 \eta + c_2 = 0$$

(2.19)

or, equivalently,

$$c_2^3 = -1 \quad \text{and} \quad c_1 + c_2 \bar{\eta} = 0.$$  

(2.20)

**Proof.** Straightforward verification confirms that pairs of conditions (2.19) and (2.20) are equivalent. In view of the discussion preceding the theorem, it is to be shown that any of them is satisfied if and only if $H^4 = H$ and $\mathcal{R}(H_1) \subseteq \mathcal{R}(H)$, where the latter condition is equivalent to the assertion that $H$ has maximum possible rank (which ensures that $H \in C_n^{EP}$). From $H \in C_n^{OP}$ it follows that inclusion $\mathcal{R}(H_1) \subseteq \mathcal{R}(H)$ is equivalent to $H^3 H_1 = H_1$. Utilizing (2.10) and (2.17), the product $H^3 H_1$ can be expressed as

$$H^3 H_1 = (\delta_1 c_1 + \delta_2 c_2 + \delta_2 c_1 \eta) H_1 + (\delta_1 c_2 + \delta_2 c_1 \bar{\eta}) H_2.$$  

(2.21)

It is easily seen that conditions (2.19) ensure (2.15) and thus, in view of Theorem 2, imply $H^4 = H$. Moreover, substituting first (2.16) and then (2.19) to (2.21) leads to $H^3 H_1 = H_1$.

On the other hand, if $H^4 = H$ and $H^3 H_1 = H_1$ are both satisfied, then from (2.21) it follows that

$$\delta_1 c_1 + \delta_2 c_2 + \delta_2 c_1 \eta = 1 \quad \text{and} \quad \delta_1 c_2 + \delta_2 c_1 \bar{\eta} = 0,$$  

(2.22)

for otherwise nonzero $H_1$ and $H_2$ would be scalar multiples of each other. Substituting (2.16) to the latter of the conditions in (2.22) yields $c_1 + c_2 \bar{\eta} = 0$. This equality is equivalent to $c_1 \eta + c_2 = 0$ which substituted, along with (2.16), to the former condition in (2.22) leads to $c_1^3 = -1$. Thus, the necessity of (2.19) is established and the proof is complete.  \(\square\)
Remark 2. Under the assumptions of Theorem 3, there are exactly nine different pairs \((c_1, c_2)\) composed of nonzero \(c_1, c_2 \in \mathbb{C}\), for which a linear combination \(c_1H_1 + c_2H_2\) has maximal possible rank and in addition belongs to \(C_n^{HGP}\).

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