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The strong chromatic index of a class of graphs☆

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Abstract

The strong chromatic index of a graph G is the minimum integer k such that the edge set of G can be partitioned into k induced matchings. Faudree et al. [R.J. Faudree, R.H. Schelp, A. Gyárfás, Zs. Tuza, The strong chromatic index of graphs, Ars Combin. 29B (1990) 205–211] proposed an open problem: If G is bipartite and if for each edge $xy \in E(G)$, $d(x) + d(y) \leq 5$, then $s\chi'(G) \leq 6$. Let H_0 be the graph obtained from a 5-cycle by adding a new vertex and joining it to two nonadjacent vertices of the 5-cycle. In this paper, we show that if G (not necessarily bipartite) is not isomorphic to H_0 and $d(x) + d(y) \leq 5$ for any edge xy of G then $s\chi'(G) \leq 6$. The proof of the result implies a linear time algorithm to produce a strong edge coloring using at most 6 colors for such graphs.

Keywords: Strong edge coloring; Strong chromatic index; Induced matching; Line graph

1. Introduction

Graphs in this paper are finite, undirected, without loops, but parallel edges are allowed.

Let $G = (V(G), E(G))$ be a graph. Denote by $L(G)$ the line graph of G. We use n to denote the vertex number of G and m the edge number of G. Let $\Delta(G)$ denote the maximum degree of G and $\Delta_L(G)$ the maximum edge degree of G (or equivalently the maximum degree of $L(G)$).

For two integers a and b with $a \leq b$, by $[a, b]$ we denote the set of integers $a, a+1, \ldots, b$.

Definition 1.1. Let e_1 and e_2 be any two edges of G. The distance between e_1 and e_2, $d(e_1, e_2)$, is defined as the distance between the corresponding two vertices in the line graph of G.

It is clear that if e_1 and e_2 are at distance 1 then they share at least one end-vertex, and if e_1 and e_2 are at distance 2 then they share no end-vertex and there exists another edge e' such that e' is at distance 1 from e_1 and e_2, respectively.

Definition 1.2. Given a positive integer k, a k-strong edge coloring of G is a mapping f from $E(G)$ to $[1, k]$ such that, for any two edges e and e' of G, $d(e, e') \leq 2$ implies $f(e) \neq f(e')$.

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It follows from the definition that each color class of a strong edge coloring of G is an induced matching in G.

Definition 1.3. The strong chromatic index, denoted by $s\chi'(G)$, is the minimum integer k such that G has a k-strong edge coloring.

The strong chromatic index of a graph was studied by Faudree et al. in [5]. It was conjectured (1985) by Erdős and Nešetřil that $s\chi'(G) \le 5\Delta^2/4$ if Δ is even and $\le 5\Delta^2/4 - \Delta/2 + 1/4$ if Δ is odd, where Δ is the maximum degree of G, see [10]. The conjecture is clearly true for $\Delta \le 2$. The case $\Delta = 3$ was settled independently by Andersen [1] and by Horák, Qing, and Trotter [8]. They showed that $s\chi'(G) \le 10$ for graphs with maximum degree 3. Horák [9] showed that $s\chi'(G) \le 23$ for graphs with maximum degree 4. And recently, Cranston [4] showed that $s\chi'(G) \le 22$ for graphs with maximum degree 4. The conjecture is unsolved for $\Delta \ge 4$.

In [7], Griggs and Yeh introduced the distance two labeling of a graph. Afterwards, this concept was studied extensively in the literature. The edge version of distance two labeling was first investigated by Georges and Mauro in [6]. We would like to indicate that the strong edge coloring is related to the edge version of distance two labeling. We first give the following definition.

Definition 1.4. Let j and k be two positive integers. An $L(j,k)$-edge-labeling of a graph G is an assignment of nonnegative integers to the edges of G such that the difference between labels of any two edges at distance 1 is at least j, and the difference between labels of any two edges that are at distance two apart is at least k. The minimum range of labels over all $L(j,k)$-edge-labelings of a graph G is called the $\lambda'_{j,k}$-number of G, denoted by $\lambda'_{j,k}(G)$.

From the above definitions, we know that, for any graph G, $s\chi'(G) = \lambda'_{1,1}(G) + 1$. The following theorem is proved in [3].

Theorem 1.1 (*[3]*). *Let G be a simple or multiple graph and let Δ_L be the maximum degree of its line graph. Suppose $\Delta_L \ge 2$. Except the case that G is a 5-cycle and $j = k$, we have $\lambda'_{j,k}(G) \le k\lfloor \Delta_L^2/2 \rfloor + j\Delta_L - 1$.*

The following corollary is an immediate consequence of Theorem 1.1.

Corollary 1.1 (*[3]*). *Let G be a graph with maximum degree $\Delta \ge 2$. Let Δ_L be the maximum degree of the line graph $L(G)$. If G is not isomorphic to a 5-cycle, then $s\chi'(G) \le \lfloor \Delta_L^2/2 \rfloor + \Delta_L \le 2\Delta^2 - 2\Delta$.*

This corollary provides an upper bound for $s\chi'(G)$ in terms of Δ_L.

When Δ is small the upper bound in Corollary 1.1 is close to the known bounds for $s\chi'(G)$. For example, the corollary gives the upper bounds 12 for $\Delta = 3$ and 24 for $\Delta = 4$. In particular, when $\Delta_L = 3$ the upper bound for $s\chi'(G)$ given by Corollary 1.1 is 7. This upper bound is the best possible. One can see this from the following defined graph H_0. Let H_0 denote the graph obtained from a 5-cycle by adding a new vertex and joining it to two nonadjacent vertices of the 5-cycle. It is easy to see that any two edges in H_0 are at distance at most 2 and H_0 has 7 edges. It follows that $s\chi'(H_0) = 7$.

In [5], Faudree et al. proposed an open problem: If G is bipartite and if for every edge $xy \in E(G)$, $d(x)+d(y) \le 5$, then $s\chi'(G) \le 6$. (If true, this is the best possible. If the bipartite condition is dropped, then $s\chi'(G) \le 7$ follows and this is the best possible.)

Corollary 1.1 shows that $s\chi'(G) \le 7$ for all graphs G with $d(x) + d(y) \le 5$ for each edge $xy \in E(G)$. In this paper, we shall prove a result which is stronger than that in the open problem proposed by Faudree et al. Actually we shall show that H_0 is the only graph with its strong chromatic index equal to 7 and with its maximum edge degree less than or equal to 3. Our main result is the following theorem.

Theorem 1.2. *Let G be a connected graph. If G is not isomorphic to H_0 and $d(x) + d(y) \le 5$ for any edge xy of G, then $s\chi'(G) \le 6$.*

We would like to indicate that the upper bound given by Theorem 1.2 is the best possible. Let H_1 denote the graph obtained from a 8-cycle $C = v_1v_2 \ldots v_8$ by adding two vertices v_1' and v_5' and joining v_1' to v_8, v_2 and v_5' to v_4, v_6. We show that $s\chi'(H_1) \ge 6$. If this is not true, suppose c is a 5-strong edge coloring of H_1. Then it is easy to see that $c(v_7v_8) = c(v_2v_3)$ and $c(v_6v_7) = c(v_3v_4)$. It follows that there are at most three colors for the 4-cycle $v_1v_2v_1'v_8$, a contradiction. Thus $s\chi'(H_1) \ge 6$. By Theorem 1.2, $s\chi'(H_1) = 6$. Also it is easy to see that $s\chi'(K_{2,3}) = 6$. It seems difficult to find a non-bipartite graph G with $s\chi'(G) = 6$ and $\Delta_L(G) = 3$.

2. Preliminaries

In order to use greedy algorithm to construct a partial strong edge coloring of a cubic graph, Andersen in [1] introduced a method of ordering the edges of a graph. In this section we first modify this method of ordering the edges of a graph and then show that a greedy algorithm, in an ordering of edges given by this method, will produce a partial strong edge coloring using at most 6 colors of a graph G with $\Delta_L(G) \leq 3$, where only a few particular edges may be left uncolored.

Definition 2.1. Suppose S is a subset of $V(G)$. For a vertex $v \in V(G)$, the distance from v to S, denoted by $d_S(v)$, is defined as $\min_{w \in S}\{d(v, w)\}$.

Definition 2.2. Suppose S is a subset of $V(G)$ and suppose that the maximum distance from a vertex of $V(G)$ to S is I. For $i = 0, 1, \ldots, I$, let $D_i = \{v \in V(G) | d(v, S) = i\}$. We define a mapping d_S from $E(G)$ to $[0, I]$ as: $d_S(e) = \min\{i | e \cap D_i \neq \emptyset,\ 0 \leq i \leq I\}$ for any edge $e \in E(G)$.

Note that if $d_S(e) > 0$ then there exists an edge e' sharing one end-vertex with e such that $d_S(e') < d_S(e)$.

Definition 2.3. Let S be a subset of $V(G)$ and let $R = (e_{k_1}, e_{k_2}, \ldots, e_{k_m})$ be an ordering of the edges of G. For any two integers i and j in $[1, m]$, if $k_i < k_j$ implies $d_S(e_{k_i}) \geq d_S(e_{k_j})$, then we say that the edge ordering R of G is compatible with the mapping d_S.

For any edge e in $E(G)$, let $N(e)$ denote the set of edges in $E(G)$ that are at distance at most 2 from e. In our coloring process, we shall frequently use $F(e)$ to denote the set of colors we have already assigned to the edges in $N(e)$, and by $A(e)$ the set of colors available for e at that moment.

Lemma 2.1. *Let S be any subset of $V(G)$. A greedy algorithm coloring the edges of G in an ordering R compatible with the mapping d_S, will produce a partial strong edge coloring using at most* 6 *colors of G, where only edges e with* $d_S(e) = 0$ *may be left uncolored.*

Proof. If e is an edge with $d_S(e) > 0$ then there exists an edge e' sharing one end-vertex with e such that $d_S(e') < d_S(e)$. Let x be the common end-vertex of e and e' and y the another end-vertex of e'. Then, at the stage of the greedy algorithm when e is to be colored, no edges incident with y has yet been colored. It follows from the structure of G that $|F(e)| < 6$. Thus e can be colored properly. ■

The following lemma was proved by Brualdi and Massey in [2]. It will be useful in the proof of our main result.

Lemma 2.2. *Let H be a bipartite graph with bipartition X, Y with no cycles of length* 4. *Let the maximum degree of a vertex of X be* 2 *and the maximum degree of a vertex of Y be Γ. Then $s\chi'(H) \leq 2\Gamma$.*

3. The proof of Theorem 1.2

Let C_n be a cycle of length n and P_n the path on n vertices. It is easy to see that $s\chi'(P_2) = 1$, $s\chi'(P_3) = 2$ and $s\chi'(P_n) = 3$ if $n \geq 4$. Also it is not difficult to prove that $s\chi'(C_5) = 5$, $s\chi'(C_n) = 3$ if $n = 3k$, and $s\chi'(C_n) = 4$ if $n = 3k + 1$ or $3k + 2$ $(n \neq 5)$.

Note that we assume that G is connected. If $\Delta(G) \leq 2$ then $s\chi'(G) \leq 5$ and the equality holds only when G is a 5-cycle. If $\Delta(G) = 4$ then G is isomorphic to $K_{1,4}$ and clearly $s\chi'(K_{1,4}) = 4$. Thus we only need to consider the graphs with $\Delta(G) = 3$. Since $d(x) + d(y) \leq 5$ for any edge xy of G, any two vertices of degree 3 are nonadjacent. The proof of Theorem 1.2 consists of a series of lemmas.

Lemma 3.1. *If G has a vertex of degree* 1 *then $s\chi'(G) \leq 6$.*

Proof. Let v_0 be a vertex of degree 1 and let e_0 be the edge incident with v_0. Put $S = \{v_0\}$. Then, by Lemma 2.1, all edges except e_0 can be colored properly. Since $|N(e_0)| \leq 4$, e_0 can also be colored properly. ■

From now on, we may assume that G has no vertex of degree 1.

Lemma 3.2. *If G has a cut-vertex then $s\chi'(G) \leq 6$.*

Proof. Due to the construction of the graph G and since we assume that G has not vertex of degree 1, if G has a cut-vertex then it has a cut-vertex of degree 2. Let v_0 be a cut-vertex of degree 2 and let $N(v_0) = \{v_1, v_2\}$. Let $e_1 = v_0v_1$ and $e_2 = v_0v_2$. Then $G - v_0$ has exactly two components, say G_1 and G_2. Let $G_1' = G_1 + e_1$ and $G_2' = G_2 + e_2$. By Lemma 3.1, $s\chi'(G_1') \leq 6$ and $s\chi'(G_2') \leq 6$. Let ϕ_i $(i = 1, 2)$ be a strong edge coloring of G_i' using at most 6 colors. Since we can permute the colors among the edges, we may, without loss of generality, assume that the set of colors that ϕ_1 assigns to edges incident with v_1 is disjoint from the set of colors ϕ_2 that assigns to the edges incident with v_2. Finally, by combining ϕ_1 and ϕ_2, we obtain a strong edge coloring of G using at most 6 colors. ■

The *girth* of a graph G, $g(G)$, is the length of a shortest cycle in G.

Lemma 3.3. *If $g(G) = 2$ or 3 then $s\chi'(G) \leq 6$.*

Proof. Let S be the set of vertices of a shortest cycle in G. As in the proof of Lemma 3.1, we can show that 6 colors are enough for the edges of G. ■

Lemma 3.4. *If $g(G) = 4$ then $s\chi'(G) \leq 6$.*

Proof. Let $C = v_1v_2v_3v_4v_1$ be a 4-cycle in G. Let $S = V(C)$. If there is at most one vertex on C having degree 3, then G has a cut-vertex and the lemma follows from Lemma 3.2. Thus there are exactly two nonadjacent vertices on C having degree 3. Without loss of generality, assume that $d(v_1) = d(v_3) = 3$. Let u_1 and u_3 be the vertices adjacent to v_1 and v_3 not in C, respectively. Let $e_1 = v_1v_2$, $e_2 = v_2v_3$, $e_3 = v_3v_4$, $e_4 = v_4v_1$, $f_1 = v_1u_1$ and $f_3 = v_3u_3$. If $u_1 = u_3$ then G is isomorphic to $K_{2,3}$ and it is clear that $s\chi'(K_{2,3}) = 6$. Thus we assume that $u_1 \neq u_3$. If $u_1u_3 \in E(G)$ then G is isomorphic to H_0. Therefore we assume that $u_1u_3 \notin E(G)$. Let another neighbor of u_1 be w_1 and another neighbor of u_3 be w_3.

We first use the greedy algorithm to color all edges e of G with $d_S(e) > 0$. Denote this partial strong edge coloring of G by ϕ. Now there are six edges left uncolored. We shall show that ϕ can be extended to the whole graph. Note that f_1 and f_3 are at distance 3. If $A(f_1) \cap A(f_3) \neq \emptyset$ then let $\phi(f_1) = \phi(f_3) \in A(f_1) \cap A(f_3)$. It is now easy to see that $|A(e_i)| \geq 4$ for $i = 1, 2, 3, 4$. And we can choose one color from $A(e_i)$ for each edge e_i and so get a strong edge coloring of G using at most 6 colors.

We now assume that $A(f_1) \cap A(f_3) = \emptyset$. If $w_1 = w_3$ then $A(f_1) = A(f_3)$, a contradiction. Thus $w_1 \neq w_3$. Let $t_1 = u_1w_1$ and $t_3 = u_3w_3$. Since $A(f_1) \cap A(f_3) = \emptyset$, $\phi(t_1) \in A(f_3)$ and $\phi(t_3) \in A(f_1)$. Let $\phi(f_1) = \phi(t_3)$ and $\phi(f_3) = \phi(t_1)$. Then $|A(e_i)| \geq 4$ for each $i = 1, 2, 3, 4$ and we can easily get a strong edge coloring of G using at most 6 colors. ■

Lemma 3.5. *If $g(G) = 5$ then $s\chi'(G) \leq 6$.*

Proof. Let $C = v_1v_2v_3v_4v_5v_1$ be a 5-cycle in G. Let $S = V(C)$. If there is at most one vertex on C having degree 3, then G has a cut-vertex and the lemma follows from Lemma 3.2. Thus there are exactly two nonadjacent vertices on C having degree 3. Without loss of generality, assume that $d(v_1) = d(v_3) = 3$. Let u_1 and u_3 be the vertices adjacent to v_1 and v_3 not in C, respectively. Let $e_1 = v_1v_2$, $e_2 = v_2v_3$, $e_3 = v_3v_4$, $e_4 = v_4v_5$, $e_5 = v_5v_1$, $f_1 = v_1u_1$ and $f_3 = v_3u_3$. Since G has no 4-cycle, we have $u_1 \neq u_3$. If $u_1u_3 \in E(G)$ then G has no other vertices and it is easy to see that $s\chi'(G) \leq 6$. Therefore we assume that $u_1u_3 \notin E(G)$. Let another neighbor of u_1 be w_1 and another neighbor of u_3 be w_3. Let $t_1 = u_1w_1$ and $t_3 = u_3w_3$.

We first use the greedy algorithm to color all edges e of G with $d_S(e) > 0$. Denote this partial strong edge coloring of G by ϕ. Now there are seven edges left uncolored. We shall show that ϕ can be extended to the whole graph. Note that f_1 and f_3 are at distance 3. If $A(f_1) \cap A(f_3) \neq \emptyset$ then let $\phi(f_1) = \phi(f_3) \in A(f_1) \cap A(f_3)$. It is now easy to see that $|A(e_i)| \geq 4$ for $i = 1, 2, 3, 5$ and $|A(e_4)| \geq 5$. Then we can greedily choose one color from $A(e_i)$ for each edge e_i in the order e_1, e_2, e_3, e_5, e_4 and so extend ϕ to the whole graph.

We now assume that $A(f_1) \cap A(f_3) = \emptyset$. If $w_1 = w_3$ then $A(f_1) = A(f_3)$, a contradiction. Thus $w_1 \neq w_3$. Let $a = \phi(t_1)$. Since $A(f_1) \cap A(f_3) = \emptyset$, $a \in A(f_3)$. Note that when the edge t_1 is to be colored we actually have at least two choices of the colors. Thus we can change the color for t_1. Then the color a is available for both f_1 and f_3. We are back to the case $A(f_1) \cap A(f_3) \neq \emptyset$ and the lemma follows. ■

Lemma 3.6. *If G has three vertices v_1, v_2 and v_3 of degree 2 such that $v_1v_2v_3$ is a path in G, then $s\chi'(G) \leq 6$.*

Proof. Let $S = \{v_2\}$. The greedy algorithm will produce a strong edge coloring of G using at most 6 colors. ■

Lemma 3.7. *If G has two adjacent vertices of degree 2 then $s\chi'(G) \leq 6$.*

Proof. By Lemmas 3.3–3.5, we may assume that the girth of G is at least 6. Let v_1 and v_2 be the two adjacent vertices of degree 2. Let u_1 and u_2 be the two vertices adjacent to v_1 and v_2, respectively. Let $e_1 = v_1v_2$, $f_1 = v_1u_1$, $f_2 = v_2u_2$. If u_1 or u_2 is of degree 2 then, by Lemma 3.6, $s\chi'(G) \leq 6$. Thus we assume that $d(u_1) = d(u_2) = 3$. Let $N(u_1) = \{v_1, w_1, w_2\}$ and $N(u_2) = \{v_2, w_3, w_4\}$. Since G has no cycle of length less than 6 and has no vertex of degree 1, w_1, w_2, w_3, w_4 are four different vertices of degree 2. Let $g_1 = u_1w_1$, $g_2 = u_1w_2$, $g_3 = u_2w_3$ and $g_4 = u_2w_4$.

Let G' be the graph obtained from G by deleting the vertices v_1, v_2, u_1 and adding the edge w_2u_2. Then w_1 is a vertex of degree 1 in G'. Let s_1 be the vertex adjacent to w_1 in G'. Clearly s_1 is a cut-vertex of G'. If G' is not connected then u_1 is a cut-vertex of G. By Lemma 3.2, $s\chi'(G) \leq 6$. Thus we assume that G' is connected. Clearly G' satisfies the conditions of Theorem 1.2. Since w_1 is a vertex of degree 1 in G', by Lemma 3.1, G' has a strong edge coloring using at most 6 colors. Since there are at most 4 edges at distance 1 or 2 from w_1s_1, it is not difficult to see that G' has a strong edge coloring ϕ using at most 6 colors with $\phi(w_1s_1) \neq \phi(w_2u_2)$.

Now we extend ϕ to the graph G. Color the two edges g_2 and f_2 with the same color $\phi(w_2u_2)$. And then color the remaining three edges in the order g_1, f_1, e_1 in a greedy way. It is not difficult to see that this will produce a strong edge coloring of G using at most 6 colors. ■

Let X be the set of vertices of degree 2 and Y the set of vertices of degree 3 in G. By Lemma 3.7, we may assume that any two vertices of degree 2 are nonadjacent. This implies that G is bipartite with bipartition X, Y. By Lemma 3.4, we may assume that G has no cycles of length 4. Theorem 1.2 now follows from Lemma 2.2.

Remark. Except the three graphs H_0, H_1, and $K_{2,3}$, we do not find other graphs G with $\Delta_L(G) = 3$ and $s\chi'(G) \geq 6$. It seems difficult to find such graphs.

In the proof of Theorem 1.2, the argument for the case that G has no adjacent degree 2 vertices employs a result from [2] which does not reflect a linear time algorithm to produce the desired strong edge coloring of G. In the next section, we shall first deal with this case in an algorithmic way and then describe a linear time algorithm that can produce a strong edge coloring using at most 6 colors of graphs G satisfying the conditions of Theorem 1.2.

4. The 6-strong edge coloring algorithm

The purpose of this section is to give a linear time algorithm that can produce a strong edge coloring using at most 6 colors of graphs G with $d(x) + d(y) \leq 5$ for any edge xy of G or report that G is isomorphic to H_0 or $\Delta_L(G) \geq 4$.

Let G be a graph with n vertices. We represent the graph G by listing all its vertices and, for each vertex, we attach with it the information about the edges incident to it together with the other end-vertices of the edges. It is easy to see that one can find the connected components of G in $O(n)$ time. And it spends at most $O(n)$ time to check whether G has some edge xy such that $d(x) + d(y) \geq 6$ (i.e. $\Delta_L(G) \geq 4$). If G satisfies the conditions in Theorem 1.2 then we can find a vertex of degree less than 3 in a constant time. If G satisfies the conditions in Theorem 1.2 then the number of edges is $O(n)$. Suppose S is a subset of $V(G)$. Then clearly we can get an edge ordering R of G which is compatible with d_S in $O(n)$ time and the greedy algorithm will produce a partial strong edge coloring of G in $O(n)$ time using at most 6 colors leaving only the edges with $d_S(e) = 0$ uncolored. Thus the proofs in the previous section actually imply a linear time algorithm to produce a strong edge coloring of G using at most 6 colors provided that we have found a vertex of degree 1, or a cycle of length $l \in \{2, 3, 4, 5\}$, or two adjacent vertices both of degree 2.

We first prove in an algorithmic way that $s\chi'(G) \leq 6$ if G has no adjacent degree 2 vertices and satisfies the conditions in Theorem 1.2.

Lemma 4.1. *Let G be a graph on n vertices which satisfies the conditions in Theorem 1.2. If $g(G) = 6$ and a 6-cycle $C = v_1v_2v_3v_4v_5v_6$ is given then we can obtain a strong edge coloring of G using at most 6 colors in $O(n)$ time.*

Proof. We assume that G is connected. If there are at most two degree 3 vertices on C then G has two adjacent degree 2 vertices. Using the method from Lemma 3.7, we can obtain a strong edge coloring of G using at most 6 colors in $O(n)$ time. Thus we assume that $d(v_1) = d(v_3) = d(v_5) = 3$ and let u_1, u_3, u_5 be the three vertices not on C adjacent to v_1, v_3, v_5, respectively. Since G has no cycles of length less than 6, u_1, u_3, u_5 are distinct and nonadjacent vertices. Let G' be the graph obtained from G by deleting $V(C)$ and adding an edge u_1u_3. Then G' has at most two connected components and each component has a vertex of degree 1 or two adjacent vertices of degree 2. By Lemmas 3.1 and 3.7, we can get a strong edge coloring ϕ of G' using at most 6 colors in $O(n)$ time. Next we shall extend ϕ to G.

Suppose $\phi(u_1u_3) = a$. Now let $\phi(v_1u_1) = \phi(v_3u_3) = a$. And color the edge v_5u_5 with a color from $A(v_5u_5)$ (note that $A(v_5u_5) \geq 3$). Then it is easy to see that $|A(v_2v_3)| \geq 4$ and $|A(v_5v_6)| \geq 3$. Thus $A(v_2v_3) \cap A(v_5v_6)$ is nonempty. Let $b \in A(v_2v_3) \cap A(v_5v_6)$. Since $a \notin A(v_2v_3)$, $b \neq a$. Now assign the color b to both v_2v_3 and v_5v_6. At this moment, we have $|A(v_1v_6)| \geq 2$, $|A(v_4v_5)| \geq 2$, $|A(v_3v_4)| \geq 2$, $|A(v_1v_2)| \geq 3$. By greedily coloring the remaining four edges in the order v_1v_6, v_4v_5, v_3v_4, v_1v_2, we get the strong edge coloring of G using at most 6 colors. The extension of ϕ to G takes a constant time. ■

Lemma 4.2. *Let G be a graph on n vertices which satisfies the conditions in Theorem 1.2. If $g(G) \geq 8$ and G has no adjacent vertices both of degree 2 then we can obtain a strong edge coloring using at most 6 colors of G in $O(n)$ time.*

Proof. We may assume that G is connected and has no vertex of degree 1. Let v_0 be a vertex of degree 2 and let $N(v_0) = \{u_1, u_2\}$. Then $d(u_1) = d(u_2) = 3$. Let $N(u_1) = \{v_0, w_1, w_2\}$ and $N(u_2) = \{v_0, w_3, w_4\}$. Then $d(w_i) = 2$ for $i = 1, 2, 3, 4$. Let s_i ($i \in \{1, 2, 3, 4\}$) be the vertex adjacent to w_i other than u_1 and u_2. Since $g(G) \geq 8$, all the vertices specified above are distinct. Then $d(s_i) = 3$ for $i = 1, 2, 3, 4$. Let $N(s_1) = \{w_1, t_1, t_2\}$, $N(s_2) = \{w_2, t_3, t_4\}$, $N(s_3) = \{w_3, t_5, t_6\}$, and $N(s_4) = \{w_4, t_7, t_8\}$. Since $g(G) \geq 8$, t_1, t_2, t_3, t_4 are four distinct vertices, and t_5, t_6, t_7, t_8 are also four distinct vertices. But it is possible that some vertex from $\{t_1, t_2, t_3, t_4\}$ is identified with some vertex from $\{t_5, t_6, t_7, t_8\}$.

Observation 1. Let f be a partial strong edge coloring of G using at most 6 colors with only v_0u_1 and v_0u_2 uncolored. (By letting $S = \{v_0\}$, such partial strong edge coloring can be constructed in linear time according to Lemma 2.1). We observe that if f satisfies one of the following three conditions then it can be extended to all edges of G.

Condition A: $|f(\{u_1w_1, u_1w_2, u_2w_3, u_2w_4\})| = 2$. Then $|A(v_0u_1)|, |A(v_0u_2)| \geq 2$.

Condition B: $|f(\{u_1w_1, u_1w_2, u_2w_3, u_2w_4\})| \leq 3$ and $f(\{w_1s_1, w_2s_2\}) \cap f(\{u_2w_3, u_2w_4\}) \neq \emptyset$ (or $f(\{w_3s_3, w_4s_4\}) \cap f(\{u_1w_1, u_1w_2\}) \neq \emptyset$). Then $|A(v_0u_1)| \geq 2$ and $|A(v_0u_2)| \geq 1$.

Condition C: $|f(\{u_1w_1, u_1w_2, u_2w_3, u_2w_4\})| \leq 3$ and $f(\{w_1s_1, w_2s_2\}) \cap A(v_0u_2) \neq \emptyset$ (or $f(\{w_3s_3, w_4s_4\}) \cap A(v_0u_1) \neq \emptyset$). Then assign v_0u_2 a color from $f(\{w_1s_1, w_2s_2\}) \cap A(v_0u_2)$ and v_0u_1 can be colored properly.

Observation 2. Let f be a partial strong edge coloring of G using at most 6 colors with exactly all edges incident to v_0, u_1, u_2 uncolored. If for some color a, $f(w_is_i) = a$ for all $i \in \{1, 2, 3, 4\}$ then f can be extended to all the edges of G.

Proof. Note that both $|A(u_1w_1)|$ and $|A(u_2w_3)|$ are equal to 3. Since $a \notin A(u_1w_1) \cup A(u_2w_3)$, $|A(u_1w_1) \cup A(u_2w_3)| \leq 5$. It follows that $A(u_1w_1) \cap A(u_2w_3) \neq \emptyset$. Let b be a color in $A(u_1w_1) \cap A(u_2w_3)$. After assigning the color b to both u_1w_1 and u_2w_3, we have $|A(v_0u_1)| = |A(v_0u_2)| = 4$ and $|A(u_1w_2)|, |A(u_2w_3)| \geq 2$. Now it is easy to see that we can properly color the remaining four edges. ■

Let G' be the graph obtained from G by deleting the three vertices v_0, u_1, u_2 and adding the two edges w_1w_4, w_2w_3. By Lemma 3.7, we can get a strong edge coloring using at most 6 colors f' of G' in linear time. If $f'(w_1w_4) \neq f'(w_2w_3)$ then, by letting $f(e) = f'(e)$ for $e \in E(G) \cap E(G')$, $f(u_1w_1) = f(u_2w_4) = f'(w_1w_4)$, and $f(u_1w_2) = f(u_2w_3) = f'(w_2w_3)$, we get a partial strong edge coloring f of G satisfying Condition A in Observation 1, and we are done. Thus we now assume that $f'(w_1w_4) = f'(w_2w_3) = a$.

If $|f'(\{w_1s_1, s_1t_1, s_1t_2, w_4s_4, s_4t_7, s_4t_8\})| < 5$ (or $|f'(\{w_2s_2, s_2t_3, s_2t_4, w_3s_3, s_3t_5, s_3t_6\})| < 5$) then we can redefine $f'(w_1w_4)$ (or $f'(w_2w_3)$) such that $f'(w_1w_4) \neq f'(w_2w_3)$ and we are back to the above case. Therefore we assume that $|f'(\{w_1s_1, s_1t_1, s_1t_2, w_4s_4, s_4t_7, s_4t_8\})| = 5$ and $|f'(\{w_2s_2, s_2t_3, s_2t_4, w_3s_3, s_3t_5, s_3t_6\})| = 5$. This implies that $|f'(\{w_1s_1, s_1t_1, s_1t_2\}) \cap f'(\{w_4s_4, s_4t_7, s_4t_8\})| = 1$. Note that $f'(w_1s_1) \neq f'(w_4s_4)$. Without loss of generality, we may assume that $f'(w_1s_1) = f'(s_4t_7)$ or $f'(s_1t_1) = f'(s_4t_7)$. Suppose the six colors used by f' are a, b, c, x, y, z. We distinguish two cases.

Case 1: $f'(s_1t_1) = f'(s_4t_7)$.

Without loss of generality, suppose $f'(s_1t_1) = f'(s_4t_7) = y$, $f'(w_1s_1) = x$, $f'(s_1t_2) = z$, $f'(w_4s_4) = b$, $f'(s_4t_8) = c$.

If $f'(w_2s_2) \neq b$ then let $f(e) = f'(e)$ for $e \in E(G) \cap E(G')$, $f(u_1w_1) = b$, $f(u_1w_2) = f(u_2w_3) = a$, $f(u_2w_4) \in \{x, z\} \setminus \{f'(w_3s_3)\}$. Clearly f is a partial strong edge coloring using at most 6 colors of G satisfying Condition B in Observation 1, we are done. Similarly if $f'(w_3s_3) \neq x$ then we can get a strong edge coloring of G using at most 6 colors. Thus we assume that $f'(w_1s_1) = f'(w_3s_3) = x$ and $f'(w_2s_2) = f'(w_4s_4) = b$.

For edges $e \in (E(G)\cap E(G'))\setminus\{w_is_i|i = 1, 2, 3, 4\}$, let $f(e) = f'(e)$. Then $|A(w_is_i)| \geq 2$ for each $i \in \{1, 2, 3, 4\}$. If $a \notin A(w_1s_1)$ then assign the color other than x in $A(w_1s_1)$ to the edge w_1s_1 and let $f(w_2s_2) = f(w_4s_4) = b$, $f(w_3s_3) = f(u_1w_1) = x$, $f(u_1w_2) = f(u_2w_3) = a$, $f(u_2w_4) = z$. Then f becomes a partial strong edge coloring of G using at most 6 colors satisfying Condition B in Observation 1 and we are done. Similarly, if $a \notin A(w_is_i)$ for some $i \in \{1, 2, 3, 4\}$, then we can extend f to the whole graph G. Thus we assume that $a \in A(w_is_i)$ for each $i \in \{1, 2, 3, 4\}$. Let $f(w_is_i) = a$ for each $i = 1, 2, 3, 4$. By Observation 2, we can extend f to the whole graph G.

Case 2: $f'(w_1s_1) = f'(s_4t_7)$.

Without loss of generality, suppose $f'(w_1s_1) = f'(s_4t_7) = x$, $f'(s_1t_1) = y$, $f'(s_1t_2) = z$, $f'(w_4s_4) = b$, $f'(s_4t_8) = c$.

If $f'(w_2s_2) \neq b$ then let $f(e) = f'(e)$ for $e \in E(G) \cap E(G')$, $f(u_1w_1) = b$, $f(u_1w_2) = f(u_2w_3) = a$. Since $|A(u_2w_4)| \geq 1$, by assigning a color from $A(u_2w_4)$ to u_2w_4, f becomes a partial strong edge coloring of G using at most 6 colors satisfying Condition B in Observation 1, and we are done. Thus we assume that $f'(w_2s_2) = b$.

If $f'(w_3s_3) \neq x$ then since $f'(w_2s_2) = b$ and $f'(w_2w_3) = a$ we have $f'(w_3s_3) \in \{c, y, z\}$. If $f'(w_3s_3) = y$ (or $f'(w_3s_3) = z$) then let $f(e) = f'(e)$ for $e \in E(G) \cap E(G')$, $f(u_1w_2) = f(u_2w_3) = a$, $f(u_1w_1) = c$, $f(u_2w_4) = z$. f satisfies Condition C in Observation 1. We are done. Now suppose $f'(w_3s_3) = c$. If $c \notin f'(\{s_2t_3, s_2t_4\})$ then let $f(e) = f'(e)$ for $e \in E(G) \cap E(G')$, $f(u_1w_1) = f(u_2w_3) = a$, $f(u_1w_2) = c$, $f(u_2w_4) = z$, $f(v_0u_1) = y$, $f(v_0u_2) = x$. Clearly f is a strong edge coloring using at most 6 colors of G. If $c \in f'(\{s_2t_3, s_2t_4\})$ then let $f(e) = f'(e)$ for $e \in E(G) \cap E(G')$, $f(u_1w_1) = f(u_2w_3) = a$. As $A(u_1w_2) \subseteq \{y, z\} = A(u_2w_4)$, we can assign u_1w_2 and u_2w_4 the same color. Then f is a partial strong edge coloring of G using at most 6 colors satisfying Condition A in Observation 1 and we are done. Therefore we now assume $f'(w_3s_3) = x$.

At this moment we have $f'(w_1s_1) = f'(w_3s_3) = x$ and $f'(w_2s_2) = f'(w_4s_4) = b$. As in the proof of Case 1, we can extend f to the whole graph G. ■

Combining Lemmas from this and the previous sections, we have the following theorem.

Theorem 4.1. *There is a linear time algorithm that finds a strong edge coloring using at most 6 colors for graphs satisfying the conditions in Theorem 1.2.*

Proof. We shall give a sketch of the desired algorithm. Suppose G is a graph as an input to the algorithm. The algorithm first checks if G satisfies the conditions in Theorem 1.2. If G satisfies the conditions in Theorem 1.2 then it goes on to identify all its connected components. For each connected component of G, the algorithm runs as follows.

(s1): Choose any vertex, say x, of G. If $d(x) = 3$ then find a neighbor, say v_0, of x (clearly $1 \leq d(v_0) \leq 2$); otherwise set $v_0 = x$ (we also have $1 \leq d(v_0) \leq 2$).

(s2): If $d(v_0) = 1$ then let $S = \{v_0\}$, as in the proof of Lemma 3.1, the algorithm will produce a strong edge coloring of G using at most 6 colors. Else $d(v_0) = 2$. Let u_1 and u_2 be the two neighbors of v_0. Go to (s3).

(s3): If $u_1 = u_2$, or $u_1u_2 \in E(G)$, or $d(u_1) = d(u_2) = 2$, then let $S = \{v_0, u_1, u_2\}$, construct the edge ordering of G compatible with the mapping d_S and greedily color all edges of G in the edge ordering constructed just now. Else if $d(u_1) = d(v_0) = 2$ or $d(u_2) = d(v_0) = 2$, then as in the proof of Lemma 3.7, the algorithm will produce a strong edge coloring of G using at most 6 colors. Else $d(u_1) = d(u_2) = 3$, go to (s4).

(s4): Let $S = \{v_0\}$. Find D_i for $i = 0, 1, 2, 3, 4$. If G has two adjacent vertices of degree less than or equal to 2 among $\bigcup_{i=0}^{4} D_i$ then as in the proof of Lemma 3.7, the coloring can be constructed in linear time. Else if G has a cycle of length less than 8 among $\bigcup_{i=0}^{4} D_i$, then let S be the set of vertices on that cycle, as in the proof of Lemma 3.3, or 3.4, or 4.1, the coloring can be constructed in linear time. (Note that if any two vertices of degree less than or equal to

2 in $\bigcup_{i=0}^{4} D_i$ are nonadjacent then $\bigcup_{i=0}^{4} D_i$ contains no odd cycle.) Else, as in the proof of Lemma 4.2, the coloring can be constructed in linear time.

Thus in total, one can get the required coloring in linear time. ■

References

[1] L.D. Andersen, The strong chromatic index of a cubic graph is at most 10, Discrete Math. 108 (1992) 231–252.

[2] R.A. Brualdi, J.Q. Massey, Incidence and strong edge colorings of graphs, Discrete Math. 122 (1993) 51–58.

[3] Q. Chen, W. Lin, $L(j,k)$-Labelings and $L(j,k)$-edge-Labelings of graphs, Ars Combin. 2007 (in press).

[4] D. Cranston, Strong edge-coloring of graphs with maximum degree 4 using 22 colors, Discrete Math. 306 (2006) 2772–2778.

[5] R.J. Faudree, R.H. Schelp, A. Gyárfás, Zs. Tuza, The strong chromatic index of graphs, Ars Combin. 29B (1990) 205–211.

[6] J.P. Georges, D.W. Mauro, Edge labelings with a condition at distance two, Ars Combin. 70 (2004) 109–128.

[7] J.R. Griggs, R.K. Yeh, Labelling graphs with a condition at distance 2, SIAM J. Discrete Math. 5 (1992) 586–595.

[8] P. Horák, H. Qing, W.T. Trotter, Induced matchings in cubic graphs, J. Graph Theory 17 (1993) 151–160.

[9] P. Horák, The strong chromatic index of graphs with maximum degree four, Contemp. Methods Graph Theory (1990) 399–403.

[10] T.R. Jensen, B. Toft, Graph Coloring Problems, Wiley Interscience, 1995.