Characterizing imprimitive partition designs of binary Hamming graphs

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Abstract

Let \( G = (V, E) \) be a binary Hamming graph (or the 1-skeleton of a hypercube). A partition design of \( G \) with adjacency matrix \( M = (m_{ij})_{1 \leq i, j \leq r} \) is defined as a partition \( \{Y_1, \ldots, Y_r\} \) of the vertex set \( V \) such that for every \( x \in Y_i \) we have that \(|\{y \in Y_j \mid (x, y) \in E\}| = m_{ij}\); this holds for \( 1 \leq i, j \leq r \).

Let \( Y \) be a partition design with adjacency matrix \( M \). For every \( t \geq 2 \) we construct a partition design \( Y_t \) with adjacency matrix \( tM \), and we describe when \( Y_t \) is the unique partition design with adjacency matrix \( tM \).

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1. Introduction

A graph is a set (its elements are called vertices) with a binary, symmetric and non-reflexive relation (this relation is called adjacency and will be denoted by \( \sim \)). If we study partitions of this set, it is natural to restrict to partitions that satisfy some property in terms of the relation. Otherwise we are omitting the fact that there is a relation for that set, i.e., we are omitting the fact that the elements of the set are vertices of a graph. Probably, the most natural restriction that includes enough examples to make it interesting is given in the next definition.

Definition 1.1. A partition design of a graph is a partition \( \{Y_1, \ldots, Y_r\} \) of the vertex set with the following regularity property. For every \( x \in Y_i \), the number of vertices \( y \in Y_j \) satisfying \( x \sim y \) is a constant \( m_{ij} \), independent of the choice of \( x \). This holds for each \( i \) and \( j \) in \( \{1, \ldots, r\} \). The matrix \( M = (m_{ij})_{1 \leq i, j \leq r} \) is called the adjacency matrix of the
partition design. We say that two partition designs are *equivalent* if there is an isomorphism of the graph that maps one partition into the other.

The distance between two vertices \( x, y \) of a graph is the number of edges of a shortest path from \( x \) to \( y \). The diameter of a graph is the maximum distance possible between two of its vertices. From the notion of distance we obtain another family of partitions: the distance partitions.

**Definition 1.2.** The *distance partition of a graph from the vertex* \( v \), denoted by \( D_m(v) \), is the partition \( \{ S_0(v), \ldots, S_m(v) \} \) where \( m \) is the diameter of the graph and \( S_i(v) \) is the set of vertices at distance \( i \) from \( v \).

Which graphs satisfy that their distance partitions are partition designs? The answer is distance regular graphs and distance biregular graphs (for the proof see [5]). Let us ask another question: which graphs satisfy that all their distance partitions are partition designs with the same adjacency matrix? The answer, this time, is the distance regular graphs. As a consequence, we may speak of *the* adjacency matrix of the distance partitions of a distance regular graph.

In the preface of the main reference for distance regular graphs, Brouwer, Cohen and Neumaier [1], we find the following quote:

One intriguing phenomenon is the fact that quite often arithmetical regularity properties of an object imply the uniqueness of the object, . . . . The main emphasis of this book is on describing the known distance regular graphs, on classifying and if possible, characterizing them.

It is interesting to find distance regular graphs that are characterized by the adjacency matrix of its distance partitions, i.e., distance regular graphs such that the adjacency matrix of its distance partitions does not coincide with the adjacency matrix of the distance partitions of any other distance regular graph.

In a given distance regular graph there are many more partition designs than the distance partitions. In this paper we only work with binary Hamming graphs (an important family of distance regular graphs) and we ask a question similar to the characterization of distance regular graphs: we want to find partition designs that are characterized by their adjacency matrix, i.e., partition designs such that the adjacency matrix does not coincide with the adjacency matrix of any other partition design of the same graph.

Before we continue we present the basic definitions related to binary Hamming graphs and we introduce the notion of an imprimitive partition design.

Let \( F = \{0, 1\} \) be the field with two elements. We will write the addition in this field with the symbol \( \oplus \). The multiplication will be as usual. We will denote the \( m \)-dimensional vector space over \( F \) by \( F^m \) and again for addition we will use \( \oplus \). The reason for this notation is that we do not want to become confused between the addition in this vector space and the addition in the vector space that we will introduce in Section 2. The vector space \( F^m \) has a natural inner product: \( \forall x, y \in F^m, \langle x, y \rangle = x_1 y_1 \oplus \cdots \oplus x_m y_m \). The number of non-zero coordinates of a vector \( x \) in \( F^m \) is called the *weight* of the vector and is denoted by \( w(x) \).
Let $m \in \mathbb{N}$. The binary Hamming graph of diameter $m$, $H(m)$, has as vertices the elements of $\mathbb{F}^m$. The edges join vertices at distance one, where the distance between two vertices is the number of different coordinates. Since the binary Hamming graph $H(m)$ is distance regular, the distance partition from any vertex is a partition design, and the adjacency matrix of these partition designs is

$$M = \begin{pmatrix}
0 & m \\
1 & 0 & m-1 \\
2 & 0 & \ddots \\
\vdots & 0 & 1 \\
m & 0
\end{pmatrix}. \quad (1)
$$

All distance partitions of $H(m)$ are equivalent. So we may refer to the distance partition of $H(m)$. A representative of this equivalence class we take the distance partition of $H(m)$ from the zero-vector vertex and it will be denoted by $D_m$. The subsets of $D_m$ will be referred to as $S_0, \ldots, S_m$. These $S_i$ can also be defined as the set of vectors of $\mathbb{F}^m$ with weight $i$, for all $0 \leq i \leq d$. In particular, $S_1 = \{e_1, \ldots, e_m\}$ is the natural basis of $\mathbb{F}^m$.

Note that $x, y \in \mathbb{F}^m$ are at distance one iff $x \oplus y \in S_1$, and that

$$S_1(x) = \{x \oplus e_1, \ldots, x \oplus e_m\}$$

is the set of vertices at distance one from $x$. \quad (2)

To find all partition designs that are characterized by their adjacency matrix seems a very difficult problem. Instead of considering all partition designs, we restrict ourselves to the imprimitive partition designs. To define imprimitive partition designs we need the following construction.

**Definition 1.3.** Let $Y \subset \mathbb{F}^m$ and $t$ be any natural number. Then, $Y$ to the power $t$, denoted by $Y^t$, is defined as the following subset of $\mathbb{F}^{tm}$

$$Y^t := \{(x_1, \ldots, x_{tm}) \in \mathbb{F}^{tm} | (x_1 \oplus \cdots \oplus x_t, x_{t+1} \oplus \cdots \oplus x_{2t}, \ldots, x_{(m-1)t+1} \oplus \cdots \oplus x_{mt}) \in Y \}.$$

Let $\mathcal{Y} = \{Y_1, \ldots, Y_r\}$ be a partition of $\mathbb{F}^m$ and let $t$ be any natural number. Then, $\mathcal{Y}^t$ to the power $t$, and denoted by $\mathcal{Y}^t$, is defined as $\mathcal{Y}^t := \{Y_1^t, \ldots, Y_r^t\}$. Clearly $\mathcal{Y}^t$ is a partition of $\mathbb{F}^{tm}$.

**Proposition 1.1.** If $\mathcal{Y}$ is a partition design of $\mathbb{F}^m$ with adjacency matrix $M$, then $\mathcal{Y}^t$ is a partition design of $\mathbb{F}^{tm}$ with adjacency matrix $tM$.

**Proof.** Let $M = (m_{ij})_{1 \leq i, j \leq r}$ be the adjacency matrix of $\mathcal{Y} = \{Y_1, \ldots, Y_r\}$. From the definition of $m_{ij}$ and (2) we obtain that for all $x \in Y_i$

$$m_{ij} = |\{y \in Y_j | y \in S_1(x)\}| = |\{y \in S_1(x) | y \in Y_j\}| = |\{k \in \{1, \ldots, m\} | x \oplus e_k \in Y_j\}|.$$

Let $x = (x_1, \ldots, x_{tm}) \in Y_i^t$ and let us define

$$z := (x_1 \oplus \cdots \oplus x_t, x_{t+1} \oplus \cdots \oplus x_{2t}, \ldots, x_{(m-1)t+1} \oplus \cdots \oplus x_{mt}) \in \mathbb{F}^m.$$
By the definition of $Y'_t$ we have that $z \in Y_i$. For each $k \in \{1, \ldots, m\}$ such that $z \oplus e_k \in Y_j$ there are $t$ vertices at distance one from $x$ that belong to $Y'_j$, namely $x \oplus e_{(k-1)t+1}, x \oplus e_{(k-1)t+2}, \ldots, x \oplus e_{kt}$. Like this we obtain all vertices at distance one from $x$ that belong to $Y'_j$. This number of vertices is $tm_{ij}$ and it does not depend on the choice of $x \in Y'_i$. This shows that $\{Y'_1, \ldots, Y'_r\}$ is a partition design with adjacency matrix $tM$. □

**Definition 1.4.** A partition design $\mathcal{Y}$ of a binary Hamming graph is *imprimitive* if there exists a partition design $\mathcal{Z}$ and a natural number $t \geq 2$ such that $\mathcal{Y} = \mathcal{Z}^t$.

Our main theorem states which imprimitive partition designs are characterized by their adjacency matrix. Now we describe the partition designs that appear in this theorem.

(i) The **trivial partition of $H(m)$** is denoted by $T_m$ and is the partition with only one subset containing all vectors of $\mathbb{F}_m$. It is a partition design with adjacency matrix $(m)$.

(ii) The **even–odd partition of $H(m)$ from the vertex $v$** is defined as

\[
\left\{ \bigcup_{i \text{ even}} S_i(v), \bigcup_{i \text{ odd}} S_i(v) \right\}.
\]

All even–odd partitions of $H(m)$ are equal. The even–odd partition of $H(m)$, denoted by $E_m$, is a partition design with adjacency matrix $\left( \begin{array}{c} 0 & m \\ m & 0 \end{array} \right)$.

(iii) The **folded distance partition of $H(m)$ from the vertex $v$**, denoted by $H_m(v)$, is defined as

\[
\left\{ S_0(v) \cup S_m(v), S_1(v) \cup S_{m-1}(v), \ldots, S_{\lfloor m/2 \rfloor}(v) \cup S_{\lceil m/2 \rceil}(v) \right\}.
\]

The folded distance partition of $H(m)$ from any vertex $v$ is a partition design with adjacency matrix

\[
\begin{pmatrix}
0 & m \\
1 & 0 & m - 1 \\
2 & 0 & \ddots \\
\vdots & \ddots & \ddots & m + 3 \\
\frac{m-1}{2} & \frac{m+1}{2} & \ddots & \ddots \\
\frac{m}{2} - 1 & 0 & \frac{m}{2} + 1 & m & 0
\end{pmatrix}
\] (3)

when $m$ is an odd number, and

\[
\begin{pmatrix}
0 & m \\
1 & 0 & m - 1 \\
2 & 0 & \ddots \\
\vdots & \ddots & \ddots & m + 2 \\
\frac{m}{2} - 1 & 0 & \frac{m}{2} + 1 & m & 0
\end{pmatrix}
\] (4)

when $m$ is an even number.
All folded distance partitions of $H(m)$ are equivalent. So we may refer to the folded distance partition of $H(m)$, and as representative we will choose the folded distance partition of $H(m)$ from the zero-vector and it will be denoted by $H_m$.

Finally, we state the main theorem of this paper.

**Theorem 1.1.** The only imprimitive partition designs of binary Hamming graphs that are characterized by their adjacency matrix are

1. trivial partitions: $T_m$ for $m \geq 2$;
2. even–odd partitions: $E_m$ for $m \geq 2$;
3. squares of distance partitions: $D^2_m$ for $m \geq 2$;
4. squares of folded distance partitions: $H^2_m$ for $m \geq 3$, except $m = 4$.

In the literature we find several terms which are equivalent to the term partition design. Partition designs can be found in [3]. But the term equitable partitions is probably more popular and can be found in the papers by Godsil (see for example [4]). But also the term regular partitions is in use (see Appendix A.4 in [1]). The case of partition designs in the Hamming graph has been treated in detail in [2].

The study of partition designs in Hamming graphs is motivated by the fact that they produce orthogonal arrays and completely regular codes (see the definitions in [6], and see how to obtain completely regular codes from partition designs in [7]).

## 2. The group algebra

The Hamming graph has a rich underlying algebraic structure, we will try to take advantage of it. The best way is through the group algebra.

**Definition 2.1.** Let $\mathbb{C}^F^m$ be the complex vector space of all formal complex linear combinations of vectors of $F^m$:

$$\mathbb{C}^F^m = \left\{ \sum_{x \in F^m} \lambda_x x \mid \lambda_x \in \mathbb{C} \right\}.$$ 

The operation $\boxplus$ of $F^m$ extends linearly to the convolutional product in $\mathbb{C}^F^m$:

$$\left( \sum_{y \in F^m} \lambda_y y \right) \boxplus \left( \sum_{z \in F^m} \mu_z z \right) = \sum_{x \in F^m} \sum_{y \in F^m} \lambda_y \mu_z \left( y \boxplus z \right) x,$$

for any two $\sum_{x \in F^m} \lambda_x x, \sum_{x \in F^m} \mu_x x \in \mathbb{C}^F^m$. The convolutional product makes $\mathbb{C}^F^m$ a commutative algebra. The algebra is commutative because the operation $\boxplus$ in $F^m$ is commutative. This algebra is referred to as the group algebra.

**Notation.** If $Y \subseteq F^m$ then $Y$ will denote the element $\sum_{x \in Y} x$ in $\mathbb{C}^F^m$. We mention again that $\sum$ and $+$ are formal sums. On the other hand, the operation $\boxplus$ can be performed by adding elements in $F^m$. By $(Y_1, \ldots, Y_r)$ we denote the following vector subspace:

$$\{ \lambda_1 Y_1 + \cdots + \lambda_r Y_r \in \mathbb{C}^F^m \mid \lambda_1, \ldots, \lambda_r \in \mathbb{C} \} \subseteq \mathbb{C}^F^m.$$
There is a second product of elements of $\mathbb{C}P^m$.

**Definition 2.2.** The Hadamard product of the elements $\sum_{x \in \mathbb{F}^m} \lambda_x x$, $\sum_{x \in \mathbb{F}^m} \mu_x x \in \mathbb{C}P^m$ is

$$
\left( \sum_{x \in \mathbb{F}^m} \lambda_x x \right) \odot \left( \sum_{x \in \mathbb{F}^m} \mu_x x \right) = \sum_{x \in \mathbb{F}^m} (\lambda_x \mu_x) x.
$$

Substituting in **Definition 2.1** the convolution product by the Hadamard product we get another algebra: the Hadamard algebra.

The Hadamard algebra is important because it gives us a way to translate the notion of partition of a set into an algebraic language. This is done by the following result.

**Proposition 2.1** (Godsil). There is a one-to-one correspondence between the Hadamard subalgebras of $\mathbb{C}P^m$ that contain the all-one vector (i.e., the vector $\sum_{x \in \mathbb{F}^m} x$ that belongs to $\mathbb{C}P^m$) and the partitions of $\mathbb{F}^m$.

The definition of partition design given in Section 1 was translated by Simonis [8] into the language of the group algebra. This is the subject of the following proposition.

**Proposition 2.2.** The partition $\{Y_1, \ldots, Y_r\}$ is a partition design of $H(m)$ iff

$$
(Y_1, \ldots, Y_r) \boxplus S_1 \subset (Y_1, \ldots, Y_r).
$$

### 3. Fourier transform

In **Proposition 2.2** we have seen the equivalence between the combinatorial notion of a partition design in $\mathbb{F}^m$ and an algebraic structure in $\mathbb{C}P^m$. In this section we will see that the image of this algebraic structure through the Fourier transform is another that will improve our understanding of the structure of partition designs. In order to define the Fourier transform we need the notion of group characters.

**Definition 3.1.** A character of a finite additive group $G$ is a homomorphism $\lambda$ from $G$ to $\mathbb{C}\setminus\{0\}$, the multiplicative group of non-zero complex numbers. The set of characters of a finite Abelian group $G$ will be denoted by $G^*$ and forms a group with the product $\lambda_1 \boxplus \lambda_2$ of $\lambda_1, \lambda_2 \in G^*$ defined by $(\lambda_1 \boxplus \lambda_2)(x) := \lambda_1(x) \lambda_2(x)$ for all $x \in G$.

The use of $\boxplus$ for characters is motivated by the fact that we want to define a group algebra over $(\mathbb{F}^m)^*$ in the same way as we did for $\mathbb{F}^m$, and we want to use the same notation for the convolutional product in both group algebras.

The groups $G$ and $G^*$ are isomorphic (see for instance, Proposition 2.10.7 in [1]). In the case that $G = \mathbb{F}^m$ then we can define the isomorphism as follows: the image of $x \in \mathbb{F}^m$ is $x^*$ with $x^*(y) = (-1)^{(x,y)}$ for all $y \in \mathbb{F}^m$. The group algebra of $(\mathbb{F}^m)^*$ is defined as in **Definition 2.1** and we will denote it by $\mathbb{C}(\mathbb{F}^m)^*$. Given $U = \sum_{x \in \mathbb{F}^m} \lambda_x x \in \mathbb{C}\mathbb{F}^m$, by $U^*$ we denote the element $\sum_{x \in \mathbb{F}^m} \lambda_x x^* \in \mathbb{C}(\mathbb{F}^m)^*$. 
Note. Since we are going to use frequently that \( x^* \boxplus y^* = (x \boxplus y)^* \) for all \( x, y \in \mathbb{P}^m \), we include its proof. For all \( z \in \mathbb{P}^m \) we have that

\[
(x^* \boxplus y^*)(z) = x^*(z)y^*(z) = (-1)^{\langle x, z \rangle}(-1)^{\langle y, z \rangle} = (-1)^{\langle x \boxplus y, z \rangle} = (x \boxplus y)^*(z).
\]

Thus \( x^* \boxplus y^* \) and \( (x \boxplus y)^* \) act in the same way over all elements of \( \mathbb{P}^m \), so they are the same element of \( (\mathbb{P}^m)^* \).

Now we are ready to give the definition of the Fourier transform.

**Definition 3.2.** The Fourier transform \( \mathcal{F} : \mathbb{C}\mathbb{P}^m \longrightarrow \mathbb{C}(\mathbb{P}^m)^* \) is the linear extension of the mapping

\[
x \mapsto \sum_{y \in \mathbb{P}^m} y^*(x) = \sum_{y \in \mathbb{P}^m} (-1)^{\langle y, x \rangle} y^* \quad \text{for all } x \in \mathbb{P}^m.
\]

**Observation.** Each \( x \in \mathbb{P}^m \) specifies a character \( x^{**} \) of \((\mathbb{P}^m)^*\), which is defined by \( x^{**}(y^*) = y^*(x) \), for all \( y^* \in (\mathbb{P}^m)^* \). And since \( \mathbb{P}^m \) and \((\mathbb{P}^m)^*\) are isomorphic there is no sense to distinguish \( x \) from \( x^{**} \).

We introduce an essential ingredient of this paper: concatenation. After that we give four classical properties of the Fourier transform, one property of the concatenation, and a last one that states that concatenation and the Fourier transform commute.

**Definition 3.3.** Let \( x \in \mathbb{P}^m_1 \) and \( y \in \mathbb{P}^m_2 \). By \((x; y)\) we denote the element of \( \mathbb{P}^{m_1+m_2} \) obtained by the concatenation of \( x \) and \( y \). Concatenation is extended linearly to the group algebra: let \( \sum_{x \in \mathbb{P}^m_1} \lambda_x x \in \mathbb{C}\mathbb{P}^m_1 \) and \( \sum_{y \in \mathbb{P}^m_2} \mu_y y \in \mathbb{C}\mathbb{P}^m_2 \), then

\[
\left( \sum_{x \in \mathbb{P}^m_1} \lambda_x x; \sum_{y \in \mathbb{P}^m_2} \mu_y y \right) := \sum_{x \in \mathbb{P}^{m_1}} \sum_{y \in \mathbb{P}^{m_2}} \lambda_x \mu_y (x; y) \in \mathbb{C}\mathbb{P}^{m_1+m_2}.
\]

This will work in the same way for elements of \( \mathbb{C}(\mathbb{P}^m)^* \), where \((x^*; y^*) = (x; y)^*\).

**Proposition 3.1.** Let \( U, V, U', V' \in \mathbb{C}\mathbb{P}^m \).

(i) The Fourier transform is an isomorphism of vector spaces.

(ii) \( \mathcal{F} \) is an algebra isomorphism between the group algebra and the Hadamard algebra:

\[
\mathcal{F}(U \boxplus V) = \mathcal{F}(U) \circ \mathcal{F}(V) \quad \text{and} \quad \mathcal{F}((0, \ldots, 0)) = \sum_{x \in \mathbb{P}^m} x^*.
\]

(iii) \( \mathcal{F}/2^m = (1/2^m)\mathcal{F} \) is an algebra isomorphism between the Hadamard algebra and the group algebra:

\[
\frac{\mathcal{F}}{2^m}(U \circ V) = \frac{\mathcal{F}}{2^m}(U) \boxplus \frac{\mathcal{F}}{2^m}(V) \quad \text{and} \quad \frac{\mathcal{F}}{2^m}\left( \sum_{x \in \mathbb{P}^m} x \right) = (0, \ldots, 0).
\]

(iv) \( \frac{\mathcal{F}}{2^m}(\mathcal{F}(U)) = U \).

(v) \((U; V) \boxplus (U'; V') = (U \boxplus U'; V \boxplus V')\).

(vi) \( \mathcal{F}((U; V)) = (\mathcal{F}(U); \mathcal{F}(V)) \).
Proof. The first four are classical results. The fifth claim has a straightforward proof and the last claim is a consequence of the linearity of $\mathcal{F}$, the bilinearity of $(\cdot;\cdot)$, and the fact that for all $x, y \in \mathbb{F}^m$

$$\mathcal{F}(x; y) = \sum_{z \in \mathbb{F}^m} (-1)^{(z;x;y)}z^* = \sum_{z_1 \in \mathbb{F}^m} \sum_{z_2 \in \mathbb{F}^m} (-1)^{(z_1;z_2)(z_1;x)(z_2;y)}(z_1;z_2)^*$$

$$= \sum_{z_1 \in \mathbb{F}^m} \sum_{z_2 \in \mathbb{F}^m} (-1)^{(z_1;x)}(-1)^{(z_2;x)}(z_1^*; z_2^*)$$

$$= \sum_{z_1 \in \mathbb{F}^m} (-1)^{(z_1;x)}(z_1^*; \mathcal{F}(y)) = (\mathcal{F}(x); \mathcal{F}(y)). \quad \square$$

Note that in (iv) we assumed that $\mathcal{F}(V^*) = (\mathcal{F}(V))^*$ for all $V \in \mathbb{C}^m$ and that $U^{**} = U$; so $\mathcal{F}/\mathbb{C}^m$ is also considered an isomorphism between $\mathbb{C}(\mathbb{F}^m)^*$ and $\mathbb{C}^m$.

As a consequence of Propositions 2.2 and 3.1 we have the following corollary.

Corollary 3.1. Let $\{Y_1, \ldots, Y_r\}$ be a partition design of $H(m)$, then $\mathcal{F}(\langle Y_1, \ldots, Y_r \rangle)$ is an $r$-dimensional vector subspace of $\mathbb{C}(\mathbb{F}^m)^*$ such that it is closed under the convolution product and

$$\mathcal{F}(\langle Y_1, \ldots, Y_r \rangle) \circ S_i^* \subset \mathcal{F}(\langle Y_1, \ldots, Y_r \rangle) \quad \text{for } i = 0, \ldots, m. \quad (7)$$

Proof. Proposition 3.1 states that $\mathcal{F}$ is an isomorphism of vector spaces, then the image of the $r$-dimensional vector subspace $\langle Y_1, \ldots, Y_r \rangle$ is an $r$-dimensional vector subspace. Let $\sum_{i=1}^r \lambda_i Y_i, \sum_{i=1}^r \mu_i Y_i \in \langle Y_1, \ldots, Y_r \rangle$, then using again Proposition 3.1 we obtain that

$$\mathcal{F}\left(\sum_{i=1}^r \lambda_i Y_i\right) \oplus \mathcal{F}\left(\sum_{i=1}^r \mu_i Y_i\right) = 2^m \mathcal{F}\left((\sum_{i=1}^r \lambda_i Y_i) \circ (\sum_{i=1}^r \mu_i Y_i)\right)$$

$$= 2^m \mathcal{F}\left(\sum_{i=1}^r \lambda_i \mu_i Y_i\right) \in \mathcal{F}(\langle Y_1, \ldots, Y_r \rangle).$$

It remains to prove (7). Before we continue, we need to evaluate $\mathcal{F}(S_1)$:

$$\mathcal{F}(S_1) = \sum_{i=1}^m \mathcal{F}(e_i) = \sum_{i=1}^m \left(\sum_{x \in \mathbb{F}^m} (-1)^{(e_i;x)}x^*\right) = \sum_{x \in \mathbb{F}^m} \left(\sum_{i=1}^m (-1)^{(e_i;x)}\right)x^*$$

$$= \sum_{x \in \mathbb{F}^m} (m - 2w(x))x^* = \sum_{i=0}^m \sum_{x \in S_i} (m - 2w(x))x^* = \sum_{i=0}^m (m - 2i)S_i^*. \quad (8)$$

By Proposition 2.2 and using induction we obtain that for any $t \in \mathbb{N}$,

$$\langle Y_1, \ldots, Y_r \rangle \supset \langle Y_1, \ldots, Y_r \rangle \oplus \left(S_1 \oplus \cdots \oplus S_t\right).$$

Applying the Fourier transform on both sides we obtain that
\[
\mathcal{F}(\langle Y_1, \ldots, Y_r \rangle) \supset \mathcal{F}(\sum_{i=1}^{r} w_i Y_i) \oplus (S_1 \oplus \cdots \oplus S_1)
\]

\[
= \mathcal{F}(\langle Y_1, \ldots, Y_r \rangle) \circ \mathcal{F}(S_1) \circ \cdots \circ \mathcal{F}(S_1)
\]

\[
= \mathcal{F}(\langle Y_1, \ldots, Y_r \rangle) \circ \sum_{i=0}^{m} (m-2i)S_i^* \circ \cdots \circ \sum_{i=0}^{m} (m-2i)S_i^*
\]

\[
= \mathcal{F}(\langle Y_1, \ldots, Y_r \rangle) \circ \sum_{i=0}^{m} (m-2i)S_i^*
\]

for all \( t \in \mathbb{N} \). This formula is also true for \( t = 0 \) since \( \sum_{i=0}^{m} S_i^* = \sum_{x \in \mathbb{F}_m} x^* \) is the unity for the Hadamard multiplication. We obtain that \( \mathcal{F}(\langle Y_1, \ldots, Y_r \rangle) \) multiplied, in a Hadamard way, by any element of

\[
\left\{ \sum_{i=0}^{m} (m-2i)S_i^*, \sum_{i=0}^{m} (m-2i)S_i^*, \ldots, \sum_{i=0}^{m} (m-2i)S_i^* \right\}
\]

is contained in \( \mathcal{F}(\langle Y_1, \ldots, Y_r \rangle) \). This and the fact that

\[
\left\{ \sum_{i=0}^{m} (m-2i)S_i^*, \sum_{i=0}^{m} (m-2i)S_i^*, \ldots, \sum_{i=0}^{m} (m-2i)S_i^* \right\} = (S_0^*, S_1^*, \ldots, S_m^*)
\]

proves (7). Eq. (9) is a consequence of Vandermonde. □

Now we will relate the adjacency matrix \( M \) of a partition design to the algebraic structures in \( CF^m \) and \( C(F^m)^* \). The next result, except for the last formula, is a rephrasing in the language of group algebras of [4, Lemma 2.2 in Chapter 5].

**Proposition 3.2.** Let \( \{ Y_1, \ldots, Y_r \} \) be a partition design of \( H(m) \) and \( M = (m_{ij})_{1 \leq i, j \leq r} \) its adjacency matrix. The vector \( w = (w_1, \ldots, w_r) \) is an eigenvector of \( M \) with eigenvalue \( \theta \), if and only if

\[
S_1 \oplus \left( \sum_{i=1}^{r} w_i Y_i \right) = \theta \left( \sum_{i=1}^{r} w_i Y_i \right)
\]

Let \( w = (w_1, \ldots, w_r) \) be an eigenvector of \( M \) with eigenvalue \( \theta \), then \( (m - \theta)/2 \in \{0, 1, \ldots, m\} \) and

\[
S_{w=\theta} \circ \mathcal{F} \left( \sum_{i=1}^{r} w_i Y_i \right) = \mathcal{F} \left( \sum_{i=1}^{r} w_i Y_i \right).
\]

**Proof.** We will only proof the second part of the Proposition. Applying the Fourier transform to (10) (Eq. (10) is verified since \( w \) is an eigenvector of \( M \) with eigenvalue \( \theta \)
and using (8) we obtain that:

$$\theta \mathcal{F} \left( \sum_{i=1}^{r} w_i Y_i \right) = \mathcal{F} \left( S_1 \oplus \left( \sum_{i=1}^{r} w_i Y_i \right) \right) = \mathcal{F}(S_1) \circ \mathcal{F} \left( \sum_{i=1}^{r} w_i Y_i \right)$$

$$= \left( \sum_{i=0}^{m} (m - 2i) S_i^* \right) \circ \mathcal{F} \left( \sum_{i=1}^{r} w_i Y_i \right).$$

(11)

Taking into account that $S_i^* \circ S_j^* = \delta_{ij} S_i^*$, and multiplying, in a Hadamard way, both sides of (11) by $S_i^*$, we get

$$\theta S_i^* \circ \mathcal{F} \left( \sum_{i=1}^{r} w_i Y_i \right) = (m - 2j) S_i^* \circ \mathcal{F} \left( \sum_{i=1}^{r} w_i Y_i \right).$$

This can only be true if $\theta = m - 2j$ or if $S_j^* \circ \mathcal{F}(\sum_{i=1}^{r} w_i Y_i) = 0$. Let us suppose that there is no $j \in \{0, 1, \ldots, m\}$ such that $m - 2j = \theta$. That is, $(m - \theta)/2 \notin \{0, 1, \ldots, m\}$ and $S_j^* \circ \mathcal{F}(\sum_{i=1}^{r} w_i Y_i) = 0$ for all $j \neq (m - \theta)/2$. Finally, we obtain that

$$\mathcal{F} \left( \sum_{i=1}^{r} w_i Y_i \right) = \left( \sum_{x \in \mathbb{F}^m} x^* \right) \circ \mathcal{F} \left( \sum_{i=1}^{r} w_i Y_i \right) = \sum_{j=0}^{m} S_j^* \circ \mathcal{F} \left( \sum_{i=1}^{r} w_i Y_i \right) = S_{m-m/2} \circ \mathcal{F} \left( \sum_{i=1}^{r} w_i Y_i \right).$$

Let us say more things about $M$.

Counting in two different ways the pairs $(x, y) \in Y_i \times Y_j$ with $x$ adjacent to $y$, we obtain the following property

$$|Y_i| m_{ij} = |Y_j| m_{ji}.$$

As a result, $M$ is equivalent to a symmetric real matrix under a diagonal similarity transformation (defined by the square roots of the cardinalities of $Y_i$). This implies that $M$ is a diagonalizable matrix.

Given a partition design $\{Y_1, \ldots, Y_r\}$, we describe an alternative base for $<Y_1, \ldots, Y_r>$ that uses the eigenvectors of $M$.

**Definition 3.4.** Let $\{Y_1, \ldots, Y_r\}$ be a partition design of $H(m)$ and $M$ its adjacency matrix. Let $\{u_1, \ldots, u_r\}$ be a base of $\mathbb{R}^r$ formed by eigenvectors of $M$ such that $\theta_1 \geq \cdots \geq \theta_r$, ...
are their respective eigenvalues. Let us denote by \((u_{i1}, \ldots, u_{ir})\) the coordinates of the eigenvector \(u_i\) and define

\[ U_i := \sum_{j=1}^{r} u_{ij} Y_j \in \mathbb{C}^m \quad \text{for all } i \in \{1, \ldots, r\}. \tag{12} \]

As the entries of any row of \(M\) add up to \(m\), the all-one vector is always an eigenvector with eigenvalue \(m\). By Proposition 3.2, \(m\) is the maximum eigenvalue of \(M\) possible, so \(\theta_1 = m\) and we may assume that \(u_1 = (1, \ldots, 1)\). By Proposition 3.2, \(-m\) is the minimum eigenvalue of \(M\) possible, so \(\theta_r \geq -m\). In summary,

\[ m = \theta_1 \geq \cdot \cdot \cdot \geq \theta_r \geq -m. \tag{13} \]

Using the above notation and Proposition 3.2 we obtain that

\[ S_{m-\theta_i} \circ \mathcal{F}(U_i) = \mathcal{F}(U_i) \quad \text{for all } i \in \{1, \ldots, r\}. \tag{14} \]

The following Proposition is equivalent to (14).

**Proposition 3.3.** Let \(\{Y_1, \ldots, Y_r\}\) be a partition design of \(H(m)\) and \(M\) its adjacency matrix. Then

there exist \(\alpha_{i,x} \in \mathbb{R}\) such that \(\mathcal{F}(U_i) = \sum_{x \in S_{m-\theta_i}} \alpha_{i,x} x^*\) for all \(i \in \{1, \ldots, r\}\).

**Proof.** By the definition of the Fourier transform and since \(U_i\) has real coefficients, there are \(\alpha_{i,x} \in \mathbb{R}\) such that \(\mathcal{F}(U_i) = \sum_{x \in \mathbb{R}^m} \alpha_{i,x} x^*\). Finally, by Eq. (14) we obtain that

\[ \mathcal{F}(U_i) = S_{m-\theta_i} \circ \mathcal{F}(U_i) = S_{m-\theta_i} \circ \sum_{x \in \mathbb{R}^m} \alpha_{i,x} x^* = \sum_{x \in S_{m-\theta_i}} \alpha_{i,x} x^*. \quad \square \]

We have two bases for \(\langle Y_1, \ldots, Y_r \rangle: \{Y_1, \ldots, Y_r\}\) and \(\{U_1, \ldots, U_r\}\). Clearly the first one is the most natural. However, for the vector subspace \(\mathcal{F}\langle Y_1, \ldots, Y_r \rangle\), the base \(\{\mathcal{F}(Y_1), \ldots, \mathcal{F}(Y_r)\}\) is not convenient. By Proposition 3.3, the base \(\{\mathcal{F}(U_1), \ldots, \mathcal{F}(U_r)\}\) is much more useful.

### 3.1. Equivalent partition designs

Given a permutation \(\sigma\) on \(\{1, \ldots, m\}\), we also consider \(\sigma\) to be a permutation on \(\mathbb{R}^m\) that acts in the following way: \(\sigma((x_1, \ldots, x_m)) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(m)})\). And \(\sigma\), as a permutation on \(\mathbb{R}^m\), extends linearly to an isomorphism between \(\mathbb{C}^{\mathbb{R}^m}\) to \(\mathbb{C}^{\mathbb{R}^m}\) in a natural way.

Let us apply this notation to the subject of equivalent partition designs. The partition designs \(\{Y_1, \ldots, Y_r\}\) and \(\{Y'_1, \ldots, Y'_r\}\) are equivalent if and only if there is a permutation...
σ and a vector $x \in \mathbb{F}^m$ such that $Y'_i = \{x \oplus \sigma(y)\}_{y \in Y_i}$ for all $i \in \{1, \ldots, r\}$. Then

$$Y'_i = \sum_{y \in Y'_i} y = \sum_{y \in Y_i} (x \oplus \sigma(y)) = x \oplus \sigma \left( \sum_{y \in Y_i} y \right) = x \oplus \sigma(Y_i).$$

Let $\{U_1, \ldots, U_r\}$ be the base of $\langle Y_1, \ldots, Y_r \rangle$ defined by (12) using an arbitrary base $\{u_1, \ldots, u_r\}$ of eigenvectors of $M$. If $\{U'_1, \ldots, U'_r\}$ is the base of $\langle Y'_1, \ldots, Y'_r \rangle = \langle x \oplus \sigma(Y_1), \ldots, x \oplus \sigma(Y_r) \rangle$ defined by (12) using $\{u_1, \ldots, u_r\}$, then

$$U'_i = \sum_{j=1}^r u_{ij} (x \oplus \sigma(Y_j)) = x \oplus \sigma \left( \sum_{j=1}^r u_{ij} Y_j \right) = x \oplus \sigma(U_i). \quad (15)$$

Before we continue we must show that $\mathcal{F}$ and $\sigma$ commute. It is enough to show this for a base of $\mathbb{C}^m$:

$$\mathcal{F}(\sigma(z)) = \sum_{y \in \mathbb{F}^m} (-1)^{\langle y, \sigma(z) \rangle} y^* = \sum_{y \in \mathbb{F}^m} (-1)^{\langle \sigma(y), \sigma(z) \rangle} \sigma(y)^* = \sum_{y \in \mathbb{F}^m} (-1)^{\langle y, z \rangle} \sigma(y)^*$$

$$= \sigma \left( \sum_{y \in \mathbb{F}^m} (-1)^{\langle y, z \rangle} y^* \right) = \sigma(\mathcal{F}(z)) \quad \text{for all } z \in \mathbb{F}^m.$$

Let $\mathcal{F}(U_i) = \sum_{y \in \mathbb{F}^m} \alpha_{i,y} y^*$ and $\mathcal{F}(U'_i) = \sum_{y \in \mathbb{F}^m} \alpha'_{i,y} y^*$. Applying the Fourier transform to (15) we obtain that

$$\sum_{y \in \mathbb{F}^m} \alpha'_{i,y} y^* = \mathcal{F}(U'_i) = \mathcal{F}(x \oplus \sigma(U_i)) = \mathcal{F}(x) \circ \mathcal{F}(\sigma(U_i))$$

$$= \left( \sum_{y \in \mathbb{F}^m} (-1)^{\langle y, x \rangle} y^* \right) \circ \sigma(\mathcal{F}(U_i))$$

$$= \left( \sum_{y \in \mathbb{F}^m} (-1)^{\langle y, x \rangle} y^* \right) \circ \sum_{y \in \mathbb{F}^m} \alpha_{i,\sigma^{-1}(y)} y^*$$

$$= \sum_{y \in \mathbb{F}^m} (-1)^{\langle y, x \rangle} \alpha_{i,\sigma^{-1}(y)} y^*. \quad (16)$$

That is,

$$\alpha'_{i,y} = (-1)^{\langle y, x \rangle} \alpha_{i,\sigma^{-1}(y)} \quad \text{for all } x, y \in \mathbb{F}^m. \quad (17)$$

3.2. Invariants for partition designs

In this section we define two invariants of partition designs. One is rougher than the other. Both invariants are one of the basic ingredients in the proof of the main result.
We will need the following homomorphism. For all \( x \in \mathbb{F}_m \), let \( \pi_x : \mathbb{C}(\mathbb{F}_m)^* \to \mathbb{C} \) be the homomorphism that gives the coefficient of \( x^* \), that is,

\[
\pi_x \left( \sum_{z \in \mathbb{F}_m} \lambda_z z^* \right) = \lambda_x.
\]

**Definition 3.5.** The support of a partition design \( \mathcal{Y} = \{Y_1, \ldots, Y_r\} \) is defined as

\[
\text{SUPP}(\mathcal{Y}) := \{ z \in \mathbb{F}_m | \pi_z(\mathcal{F}(Y_1, \ldots, Y_r)) \neq 0 \}.
\]

Given a subset \( C \) of \( \mathbb{F}_m \), the distribution vector of \( C \) is the vector \( A_C = (a_0, \ldots, a_m) \), where

\[
a_i = \frac{1}{|C|} \left| \{(x, y) \in C \times C | w(x \oplus y) = i \} \right|.
\]

For all \( x \in \mathbb{F}_m \) and all permutations \( \sigma \) on \( m \) coordinates we see that

\[
\text{SUPP}((x \oplus \sigma(Y_1), \ldots, x \oplus \sigma(Y_r)))
\]

\[
= [ z \in \mathbb{F}_m | \pi_z(\mathcal{F}(x \oplus \sigma(Y_1), \ldots, x \oplus \sigma(Y_r))) \neq 0 ]
\]

\[
= [ z \in \mathbb{F}_m | \pi_z(\mathcal{F}(x \circ \sigma(\mathcal{F}(Y_1, \ldots, Y_r)))) \neq 0 ]
\]

\[
= [ z \in \mathbb{F}_m | \pi_z(\mathcal{F}(\mathcal{F}(Y_1, \ldots, Y_r))) \neq 0 ]
\]

\[
= [ z \in \mathbb{F}_m | \pi_{\sigma^{-1}(z)}(\mathcal{F}(Y_1, \ldots, Y_r)) \neq 0 ]
\]

\[
= \sigma(\text{SUPP}((Y_1, \ldots, Y_r))).
\]

A classical result of the distribution vectors states that if \( C \) and \( C' \) are equivalent sets (i.e., there is an isometry that maps one onto the other) then \( A_C = A_{C'} \). Thus

\[
A_{\text{SUPP}((x \oplus \sigma(Y_1), \ldots, x \oplus \sigma(Y_r)))} = A_{\sigma(\text{SUPP}((Y_1, \ldots, Y_r)))} = A_{\text{SUPP}((Y_1, \ldots, Y_r))}.
\]

So the vector distribution of the support of equivalent partition designs is the same. We have proved that \( \gamma((Y_1, \ldots, Y_r)) := A_{\text{SUPP}((Y_1, \ldots, Y_r))} \) is an invariant for partition designs.

Another classical result of the distribution vectors states that \( |C| = a_0 + a_1 + \cdots + a_m \) where \( A_C = (a_0, a_1, \ldots, a_m) \). So the cardinality of the support of \( \{Y_1, \ldots, Y_r\} \) is equal to the sum of the coordinates of the vector \( \gamma((Y_1, \ldots, Y_r)) \). In conclusion, the cardinality of the support of a partition design is also an invariant for partition designs, but rougher than \( \gamma \).

4. The twisting of partition designs

Given a partition design with adjacency matrix \( M \) and given a homomorphism \( \varphi \) with certain properties, the \( \varphi \)-twisting construction builds another partition design with adjacency matrix \( tM \) for some \( t \in \mathbb{N} \). In this paper we will only use two types of twistings. The first one, the \( \tau^t \)-twisting, will be introduced at the end of this section, and the second one, the \( \tau_{pq}^t \)-twisting, in Section 5.

The twisting construction is based in the following theorem.
Theorem 4.1. Let \( \{Y_1, \ldots, Y_r\} \) be a partition design of \( H(m) \) with adjacency matrix \( M \). Following Definition 3.4, we find a base \( \{U_1, \ldots, U_r\} \) of \( \langle Y_1, \ldots, Y_r \rangle \) using a base \( \{u_1, \ldots, u_r\} \) formed by eigenvectors of \( M \). Let \( t \in \mathbb{N} \) and let

\[
\varphi : \langle \mathcal{F}(U_1), \ldots, \mathcal{F}(U_r) \rangle \rightarrow \mathbb{C}^{(p^tm)^*}
\]

be a monomorphism of group algebras, such that

\[
S_{\psi, t}^{(m)} \circ \varphi(\mathcal{F}(U_i)) = \varphi(\mathcal{F}(U_i)) \quad \text{for all } i \in \{1, \ldots, r\}. \tag{18}
\]

Then \( \mathcal{F}(\varphi(Y_i)) = Z_i \) for all \( i \in \{1, \ldots, r\} \), where \( \{Z_1, \ldots, Z_r\} \) is a partition design of \( H(tm) \) with adjacency matrix \( tM \).

Proof. We consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Hadamard Algebras} & \langle Y_1, \ldots, Y_r \rangle & \rightarrow \mathbb{C}^{(p^tm)} \\
\mathcal{F} & \downarrow & \varphi \\
\text{Group Algebras} & \langle \mathcal{F}(U_1), \ldots, \mathcal{F}(U_r) \rangle & \rightarrow \mathbb{C}^{(p^tm)^*}
\end{array}
\]

(19)

where \( \langle Y_1, \ldots, Y_r \rangle \) and \( \mathbb{C}^{(p^tm)} \) are Hadamard algebras; \( \langle \mathcal{F}(U_1), \ldots, \mathcal{F}(U_r) \rangle \) and \( \mathbb{C}^{(p^tm)^*} \) are group algebras; \( \mathcal{F}/2^m \) and \( \mathcal{F}/2^{tm} \) are algebra isomorphism; and \( \tilde{\varphi} \) is the composition of \( \mathcal{F}/2^m \), \( \varphi \), and the inverse of \( \mathcal{F}/2^{tm} \). Note that, by Proposition 3.1, the inverse of \( \mathcal{F}/2^{tm} \) is \( \mathcal{F} \).

Let us split the proof in several steps.

(a) \( \tilde{\varphi} \) is a monomorphism of Hadamard algebras.

This is true because \( \tilde{\varphi} \) is the composition of \( \mathcal{F}/2^m \), which is an isomorphism between a Hadamard algebra and a group algebra, with \( \varphi \), which is a monomorphism of group algebras, and with \( \mathcal{F} \), which is an isomorphism between a group algebra and a Hadamard algebra.

(b) There exist \( Z_i \subset \mathbb{F}^{tm} \) such that \( \tilde{\varphi}(Y_i) = Z_i \) for all \( i \in \{1, \ldots, r\} \).

Let \( \lambda_{ix} \in \mathbb{C} \) be the coefficients for which \( \sum_{x \in \mathbb{F}^m} \lambda_{ix} x = \tilde{\varphi}(Y_i) \) for all \( i \in \{1, \ldots, r\} \). Then,

\[
\sum_{x \in \mathbb{F}^m} \lambda_{ix} x = \tilde{\varphi}(Y_i) = \tilde{\varphi}(Y_i \circ Y_i) = \tilde{\varphi}(Y_i) \circ \tilde{\varphi}(Y_i)
\]

\[
= \left( \sum_{x \in \mathbb{F}^m} \lambda_{ix} x \right) \circ \left( \sum_{x \in \mathbb{F}^m} \lambda_{ix} x \right) = \sum_{x \in \mathbb{F}^m} (\lambda_{ix})^2 x.
\]

So for all \( x \in \mathbb{F}^m \), we have \( (\lambda_{ix})^2 = \lambda_{ix} \), therefore \( \lambda_{ix} = 0 \) or \( \lambda_{ix} = 1 \).

(c) \( \{Z_1, \ldots, Z_r\} \) is a partition of \( \mathbb{F}^{tm} \).

Since \( \tilde{\varphi} \) is a homomorphism of Hadamard algebras, we have

\[
\sum_{i=1}^{r} Z_i = \sum_{i=1}^{r} \tilde{\varphi}(Y_i) = \tilde{\varphi}\left( \sum_{i=1}^{r} Y_i \right) = \tilde{\varphi}\left( \sum_{x \in \mathbb{F}^m} x \right) = \sum_{x \in \mathbb{F}^m} x.
\]
(d) \((Z_1, \ldots, Z_r) \boxplus S_1 \subset (Z_1, \ldots, Z_r)\).

First, we observe that
\[
\langle Z_1, \ldots, Z_r \rangle = \langle \bar{\phi}(Y_1), \ldots, \bar{\phi}(Y_r) \rangle = \bar{\phi}((Y_1, \ldots, Y_r))
\]
\[
= \bar{\phi}((U_1, \ldots, U_r)) = \langle \bar{\phi}(U_1), \ldots, \bar{\phi}(U_r) \rangle.
\]

Finally, using (8) and (18) and \(S_i^* \circ S_i^* = \delta_i^j S_i^*\), we deduce that
\[
S_1 \boxplus \bar{\phi}(U_i) = \frac{\mathcal{F}}{2^m} (\mathcal{F}(S_1 \boxplus \bar{\phi}(U_i))) = \frac{\mathcal{F}}{2^m} (\mathcal{F}(S_1) \circ \mathcal{F}(\bar{\phi}(U_i)))
\]
\[
= \mathcal{F} \left( \mathcal{F}(S_1) \circ \mathcal{F} \left( \frac{\mathcal{F}}{2^{2m}} \left( \mathcal{F} \left( \frac{\mathcal{F}}{2^m}(U_i) \right) \right) \right) \right)
\]
\[
= \mathcal{F} \left( \mathcal{F}(S_1) \circ \phi \left( \frac{\mathcal{F}}{2^m}(U_i) \right) \right)
\]
\[
= \mathcal{F} \left( \sum_{j=0}^{tm} (tm - 2j)S_j^* \circ \phi \left( \frac{\mathcal{F}}{2^m}(U_i) \right) \right)
\]
\[
= \mathcal{F} \left( \sum_{j=0}^{tm} (tm - 2j)S_j^* \circ \frac{S_{tm-j}}{2} \circ \phi \left( \frac{\mathcal{F}}{2^m}(U_i) \right) \right)
\]
\[
= \mathcal{F} \left( \sum_{j=0}^{tm} (tm - 2j)S_j^* \circ \frac{S_{tm-j}}{2} \circ \phi \left( \frac{\mathcal{F}}{2^m}(U_i) \right) \right)
\]
\[
= t\theta_i \mathcal{F} \left( \phi \left( \frac{\mathcal{F}}{2^m}(U_i) \right) \right) = t\theta_i \bar{\phi}(U_i)
\]
for all \(i \in \{1, \ldots, r\}\).

(e) \(\{Z_1, \ldots, Z_r\}\) is a partition design.

By Proposition 2.2 and step (d) we obtain that \(\{Z_1, \ldots, Z_r\}\) is a partition design. Let \(M'\) be the adjacency matrix of \(\{Z_1, \ldots, Z_r\}\).

(f) \(u_i\) is an eigenvector of \(M'\) with eigenvalue \(t\theta_i\) for all \(i \in \{1, \ldots, r\}\).

This claim is proved by Proposition 3.2. Using (20), we show that the hypothesis of this proposition holds:
\[
S_1 \boxplus \left( \sum_{j=1}^r u_{ij} Z_j \right) = S_1 \boxplus \left( \sum_{j=1}^r u_{ij} \bar{\phi}(Y_j) \right) = S_1 \boxplus \bar{\phi} \left( \sum_{j=1}^r u_{ij} Y_j \right)
\]
\[
= S_1 \boxplus \bar{\phi}(U_i) = t\theta_i \bar{\phi}(U_i) = \theta_i \left( \sum_{j=1}^r u_{ij} Z_j \right).
\]

Step (f) implies that \(M' = tM\). \qed
The Theorem 4.1 describes a new construction called twist.

**Definition 4.1.** Let \( \{Z_1, \ldots, Z_r\} \) be the partition design obtained from Theorem 4.1 by using the partition design \( \{Y_1, \ldots, Y_r\} \) and the monomorphism \( \varphi \), i.e., \( \mathcal{F} \left( \varphi \left( \mathcal{F}^2(Y_1) \right) \right) = Z_i \) for all \( i \in \{1, \ldots, r\} \). We say that \( \{Z_1, \ldots, Z_r\} \) is obtained by \( \varphi \)-twisting \( \{Y_1, \ldots, Y_r\} \).

Now we are ready to introduce the \( \tau^t \)-twisting. We will show that \( \tau^t \)-twisting and raising a partition design to the power \( t \) are equivalent constructions. We have expressed raising a partition design to a power in another way because \( \tau^t \)-twisting is better suited for our needs. What we need is a way to compare the \( \tau^t \)-twisting with the \( \tau^t \)-twisting construction.

**Definition 4.2.** For \( t \in \mathbb{N} \) we define \( \tau^t : \mathbb{C}(\mathbb{F}^m)^* \to \mathbb{C}(\mathbb{F}^m)^* \) to be the monomorphism such that

\[
\tau^t((x_1, \ldots, x_m)^*) = \left(\frac{t}{2}\right) \sum_{y_1 \in \mathbb{F}} (-1)^{\langle y_1 \rangle(x_1)} (y_1)^*
\]

for all \( (x_1, \ldots, x_m) \in \mathbb{F}^m \).

In a few places we will need a monomorphism like \( \tau^t \) but without stars, e.g., \( (1, 0) \to (1, 1, 0, 0, 0) \). For convenience, we will also use \( \tau^t \) in that context.

Note that \( \tau^t((x_1, \ldots, x_m)^*) = (\tau^t((x_1)^*); \ldots; \tau^t((x_m)^*)) \). In fact, concatenation and \( \tau^t \) commute: \( \tau^t((U^*; V^*)) = (\tau^t(U^*); \tau^t(V^*)) \) for all \( U, V \in \mathbb{C}\mathbb{F}^m \). This is analogous to property (vi) of Proposition 3.1, namely, concatenation and \( \mathcal{F} \) commute.

**Lemma 4.1.** For all \( Y \subset \mathbb{F}^m \) and for all \( t \in \mathbb{N} \), we have \( \mathcal{F} \left( \tau^t \left( \frac{\mathcal{F}^2(Y)}{2} \right) \right) = \sum_{x \in \mathcal{Y}} x \).

**Proof.** Let us prove the lemma for the special case where \( m = 1 \) and \( Y \) has only one element. Then, \( Y = \{x\} \) with \( x = (x_1) \) a vector with only one coordinate.

\[
\mathcal{F} \left( \tau^t \left( \frac{\mathcal{F}^2((x_1))}{2} \right) \right) = \mathcal{F} \left( \tau^t \left( \frac{1}{2} \sum_{y_1 \in \mathbb{F}} (-1)^{\langle y_1 \rangle \cdot \langle x_1 \rangle} (y_1)^* \right) \right)
\]

\[
= \frac{1}{2} \mathcal{F} \left( \sum_{y_1 \in \mathbb{F}} (-1)^{y_1x_1} (y_1)^* \right)
\]

\[
= \sum_{y_1 \in \mathbb{F}} (-1)^{y_1x_1} (\mathcal{F}(y_1))^*
\]

\[
= \sum_{y_1 \in \mathbb{F}} (-1)^{y_1x_1} \left( \sum_{z_1 \in \mathbb{F}} (-1)^{y_1z_1} (z_1)^*; \ldots; \sum_{z_t \in \mathbb{F}} (-1)^{y_1z_t} (z_t)^* \right)
\]

\[
= \sum_{z_1, \ldots, z_t \in \mathbb{F}} \left( \sum_{y_1 \in \mathbb{F}} (-1)^{y_1(x_1 \oplus z_1 \oplus \cdots \oplus z_t)} (z_1, \ldots, z_t) \right)
\]

\[
= \sum_{z_1 \oplus \cdots \oplus z_t = x_1} (z_1, \ldots, z_t) = \sum_{z \in \{x\}^t} z.
\]
Using this simple case we can easily prove it in general. The trick is that both $F$ and $\tau^t$ commute with the concatenation.

$$F \left( \tau^t \left( \frac{F}{2m} (Y) \right) \right) = \sum_{(x_1, \ldots, x_m) \in Y} F \left( \tau^t \left( \frac{F}{2m} ((x_1, \ldots, x_m)) \right) \right)$$

$$= \sum_{(x_1, \ldots, x_m) \in Y} \left( F \left( \tau^t \left( \frac{F}{2m} ((x_1)) \right) \right) ; \ldots ; F \left( \tau^t \left( \frac{F}{2m} ((x_1)) \right) \right) \right)$$

$$= \sum_{(x_1, \ldots, x_m) \in Y} \left( \sum_{z_1 \in \{x_1\}} ; \ldots ; \sum_{z_m \in \{x_m\}} z \right)$$

$$= \sum_{(x_1, \ldots, x_m) \in Y} \sum_{z \in \{x_1, \ldots, x_m\}^t} z = \sum_{z \in \{y_1, \ldots, y_r\}^t}$$

for all $Y \subset \mathbb{P}^m$. □

A consequence of Lemma 4.1 is that the construction of raising a partition design to some power, given in Proposition 1.1, is a particular case of Theorem 4.1.

We will often define a homomorphism of group algebras $\varphi : \mathbb{C}(\mathbb{P}^m)^* \rightarrow \mathbb{C}(\mathbb{P}^m)^*$ and, for convenience, make no difference between $\varphi$ and its restriction to $\langle F(U_1), \ldots, F(U_r) \rangle$.

We want to avoid the more correct but awful notation $\varphi|_{\langle F(U_1), \ldots, F(U_r) \rangle}$.

**Corollary 4.1.** For every partition design $\mathcal{Y}$, we have that $\mathcal{Y}^t$ is equal to the partition design obtained by $\tau^t$-twisting $\mathcal{Y}$.

**Proof.** To be able to $\tau^t$-twist $\mathcal{Y}$ it must be proved that $\tau^t$ is a monomorphism of group algebras, and that (18) is verified (for $\varphi = \tau^t$). The first part is clear considering $\tau^t$ to be a homomorphism from $\mathbb{C}(\mathbb{P}^m)^*$ to $\mathbb{C}(\mathbb{P}^m)^*$. Any restriction of $\tau^t$ will have the same properties. That (18) is verified follows from that fact that $x^* \in S_j$ implies $\tau^t(x^*) \in S_{(j)}$.

Notice that $F(U_i)$ is a linear combination of vectors of $S_{(m-i)}^t$; therefore $\tau^t(F(U_i))$ is a linear combination of vectors of $S_{(m-i)}^t$.

Let $\mathcal{Y} = \{Y_1, \ldots, Y_r\}$. Finally, applying Lemma 4.1 to $Y_1, \ldots, Y_r$ we deduce that by $\tau^t$-twisting $\mathcal{Y}$ we obtain $\mathcal{Y}^t$. □

**5. The homomorphism $\tau^t_{pq}$**

The other type of twisting that we will use is the $\tau^t_{pq}$-twisting. It is a variation of the $\tau^t$-twisting, but it is sufficiently different so that both constructions applied to the same partition design normally give non-equivalent partition designs (but with the same adjacency matrix). In fact, if they give equivalent partition designs then there are no more partition designs with the same adjacency matrix. There is only one exception: the $\tau^t$-twisting of $\mathcal{H}_4$ is equivalent to the $\tau^t_{pq}$-twisting of $\mathcal{H}_4$ (for all $p, q$), but there are two other partition designs with the same adjacency matrix.
Definition 5.1. For $t \geq 2$ and $m \geq 2$ we define $\tau_{12}^t : \mathbb{C}(\mathbb{F}^m)^* \rightarrow \mathbb{C}(\mathbb{F}^m)^*$ to be the monomorphism such that

$$
\tau_{12}^t((x_1, x_2, x_3, \ldots)^*) = \frac{1}{t} \left( \begin{array}{c}
\tau_{t-2}^{-2}((x_1, x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2, x_3, \ldots, x_3, \ldots)^*) \\
\tau_{t-2}^{-2}((x_2, x_2, x_1, \ldots, x_1, x_1, x_2, \ldots, x_2, x_3, \ldots, x_3, \ldots)^*) \\
\tau_{t-2}^{-2}((x_1, x_2, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2, x_3, \ldots, x_3, \ldots)^*) \\
\tau_{t-2}^{-2}((x_2, x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2, x_3, \ldots, x_3, \ldots)^*)
\end{array} \right)^* \equiv (x_1, x_2, x_3, \ldots, x_m)^*
$$

for all $(x_1, x_2, x_3, \ldots) \in \mathbb{F}^m$. Analogously, the definition of $\tau_{pq}^t$, for $1 \leq p < q \leq m$, is like $\tau_{12}^t$ but for coordinates $x_p$ and $x_q$.

The homomorphisms $\tau_{12}^t$ and $\tau^t$ act the same for the coordinates $x_3, \ldots, x_m$, and quite differently for coordinates $x_1, x_2$. Using concatenation we may express this in the following way:

$$
\tau_{12}^t((x_1, x_2, x_3, \ldots, x_m)^*) = (\tau_{12}^t((x_1, x_2)^*); \tau^t((x_3, \ldots, x_m)^*)).
$$

(21)

We can $\tau^t$-twist any partition design. We want to prove that $\tau_{pq}^t$ has the same property. First, we prove that $\tau_{pq}^t$ is a homomorphism of group algebras. After that we will prove that (18) is verified.

Proposition 5.1. The homomorphism $\tau_{pq}^t : \mathbb{C}(\mathbb{F}^m)^* \rightarrow \mathbb{C}(\mathbb{F}^m)^*$ is a homomorphism of group algebras.

Proof. For convenience we assume that $p = 1$ and $q = 2$. Furthermore, if we prove the proposition for $m = 2$ the general case follows from (21) and from the fact that $\tau^t$ is a homomorphism of group algebras.

So we must show that

$$
\tau_{12}^t((x_1, x_2)^* \oplus (y_1, y_2)^*) = \tau_{12}^t((x_1, x_2)^*) \oplus \tau_{12}^t((y_1, y_2)^*)
$$

(22)

for all $(x_1, x_2), (y_1, y_2) \in \mathbb{C}(\mathbb{F}^2)^*$. To clarify computations, an overlined symbol will be a shorthand for repeating that symbol $t - 2$ times:

$$
\overline{x_i} = x_i, \ldots, x_i.
$$

(23)

We split the proof of (22) in three cases:
(i) \((x_1, x_2)^* = (y_1, y_2)^*\). On one hand \(\tau_{12}'((x_1, x_2)^* \boxplus (y_1, y_2)^*) = \tau_{12}'((0, 0)^*) = (0, 0, 0, 0, 0)^*\). On the other hand

\[
\tau_{12}'((x_1, x_2)^*) \boxplus \tau_{12}'((x_1, x_2)^*) = \frac{1}{4}((x_1, x_1, x_1, x_2, x_2)^* - (x_2, x_2, x_1, x_1, x_2)^* + (x_1, x_2, x_1, x_1, x_2)^* - (x_2, x_2, x_1, x_1, x_2)^*) + \frac{1}{4}[(x_1, x_1, x_2, x_2, x_2)^* + (x_1, x_2, x_1, x_1, x_2)^* + (x_2, x_1, x_1, x_1, x_2)^*] = (0, 0, 0, 0, 0)^* - \frac{1}{2}(z, z, 0, z, 0)^* + \frac{1}{2}(0, 0, 0, 0, 0)^* + \frac{1}{4}(0, z, 0, 0, z)^* + \frac{1}{4}(z, z, 0, 0, z)^* + \frac{1}{2}(0, 0, 0, 0, 0)^*
\]

where \(z = x_1 \boxplus x_2\).

(ii) \(y_1 = y_2\). We have that

\[
\tau_{12}'((x_1, x_2)^*) \boxplus \tau_{12}'((y_1, y_1)^*) = \frac{1}{4}((x_1, x_1, x_1, x_2, x_2)^* - (x_2, x_2, x_1, x_1, x_2)^* + (x_1, x_2, x_1, x_1, x_2)^* - (x_2, x_2, x_1, x_1, x_2)^*) + \frac{1}{4}[(x_1, x_1, x_2, x_2, x_2)^* + (x_1, x_2, x_1, x_1, x_2)^* + (x_2, x_1, x_1, x_1, x_2)^*] = \tau_{12}'((x_1, x_2)^* \boxplus (y_1, y_1)^*) = \tau_{12}'((x_1, x_2)^* \boxplus (y_1, y_1)^*) = \tau_{12}'((0, 0)^*) \boxplus \tau_{12}'((0, 1)^*) = \tau_{12}'((1, 0)^*) \boxplus \tau_{12}'((1, 1)^*) = \tau_{12}'((1, 0)^*) \boxplus \tau_{12}'((1, 1)^*)
\]

(iii) \((x_1, x_2)^* = (1, 0)^*\) and \((y_1, y_2)^* = (0, 1)^*\). To prove this case we use (i) and (ii):

\[
\tau_{12}'((1, 0)^*) \boxplus \tau_{12}'((1, 1)^*) = \tau_{12}'((1, 0)^*) \boxplus \tau_{12}'((1, 0)^*) \boxplus \tau_{12}'((0, 1)^*) = \tau_{12}'((1, 0)^*) \boxplus \tau_{12}'((0, 0)^*) \boxplus \tau_{12}'((1, 1)^*) = \tau_{12}'((1, 1)^*) = \tau_{12}'((1, 0)^*) \boxplus \tau_{12}'((0, 1)^*)
\]

Considering that \(\boxplus\) is commutative, these three cases cover all possibilities of (22). \(\Box\)

It only remains to prove (18) (for \(\varphi = \tau_{pq}'\)) to be able to use \(\tau_{pq}'\) in Theorem 4.1. From the definition of \(\tau_{pq}'\) we deduce that

\[
x^* \in S_{pq}' \implies \tau_{pq}'(x^*) \text{ is a linear combination of vectors of } S_{pq}^n.
\]

Since \(\mathcal{F}(U_i)\) is a linear combination of vectors of \(S_{(m-\theta_{ij})/2}^n\), then \(\tau_{pq}'(\mathcal{F}(U_i))\) is a linear combination of vectors of \(S_{(m-\theta_{ij})/2}^n\). This implies (18).

Corollary 4.1 gives us a way to understand how \(\tau'^i\)-twisting acts on the sets of a partition design. The \(\tau'_{pq}\)-twisting also has a nice interpretation.

By looking at the structure of \(\tau_{12}'\) displayed in Definition 5.1 we observe that \(\tau_{12}'\) is equal to \(\tau'\) composed with a linear combination of permutations. More precisely,

\[
\tau_{pq}' = \frac{1}{4}(\sigma_0 - \sigma_1 + \sigma_2 + \sigma_3) \circ \tau'
\]
where

\[ \sigma_0 = \text{id}, \quad \sigma_1 = \tau (p-1) + 1, t(q-1) + 2, \quad \sigma_3 = \tau (p-1) + 2, t(q-1) + 1 \]

and \( \sigma_1 = \sigma_2 \sigma_3 \).

Let \( \mathcal{Y} = \{ Y_1, \ldots, Y_r \} \) be a partition design and let \( \{ Z_1, \ldots, Z_r \} \) be the partition design obtained by \( \tau_{pq} \)-twisting \( \mathcal{Y} \). Our aim is to express \( Z_i \) in terms of \( Y_i \) without using \( F \). Remember that \( F \) commutes with all permutations, so \( F \) also commutes with a linear combination of permutations. Consequently

\[
Z_i = F \left( \tau_{pq} \left( \frac{F}{2^m} (Y_i) \right) \right) = F \left( \left( \frac{\sigma_0}{2} - \frac{\sigma_1}{2} + \frac{\sigma_2}{2} + \frac{\sigma_3}{2} \right) \circ \tau_{pq} \left( \frac{F}{2^m} (Y_i) \right) \right)
\]

\[
= \left( \frac{\sigma_0}{2} - \frac{\sigma_1}{2} + \frac{\sigma_2}{2} + \frac{\sigma_3}{2} \right) \sum_{x \in Y_i} x
\]

\[
= \frac{1}{2} Z_{i0} - \frac{1}{2} Z_{i1} + \frac{1}{2} Z_{i2} + \frac{1}{2} Z_{i3}
\]

where

\[ Z_{i0} = Y_i^t \quad \text{and} \quad Z_{ij} = \{ \sigma_j(x) \}_{x \in Y_i^t} \quad \text{for} \ j = 1, 2, 3. \]

So \( Z_i \) is a linear combination of 4 permutations of \( Y_i^t \). In a more informal way we may say that to obtain \( Z_i \) we have “twisted” \( Y_i^t \) using 4 permutations. In fact, the term twist of Definition 4.1 was chosen because of this informal interpretation. Finally, note that if instead of using 4 permutations we had only used one, then \( \{ Z_1, \ldots, Z_r \} \) would be equivalent to \( \{ Y_1, \ldots, Y_r \} \).

6. The proof of Theorem 1.1

Before we start the proof of the main theorem we need the following technical lemma.

**Lemma 6.1.** Let \( \mathcal{Y} = \{ Y_1, \ldots, Y_r \} \) be a partition design of \( H(m) \). We have seen that \( \mathcal{Y}^t \) is the partition design obtained by \( \tau^t \)-twisting \( \mathcal{Y} \). For all \( 1 \leq p < q \leq m \), let \( \mathcal{Y}_{pq}^t \) be the partition design obtained by \( \tau_{pq}^t \)-twisting \( \mathcal{Y} \). Then \( \mathcal{Y}^t \) and \( \mathcal{Y}_{pq}^t \) have the same adjacency matrix, but are almost always non-equivalent. There are only three exceptions for which \( \mathcal{Y}^t \) and \( \mathcal{Y}_{pq}^t \) are equivalent:

1. for \( t \geq 2 \), when for all \( (\sum_{y \in \mathbb{F}^n} \lambda_y y^*) \in F(\langle Y_1, \ldots, Y_r \rangle) \), and all \( x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_{q-1}, x_{q+1}, \ldots, x_m \in \mathbb{F} \) we have

\[
\lambda(x_1, \ldots, x_p, 0, \ldots, x_m) = \lambda(x_1, \ldots, 1, \ldots, x_m) = 0;
\]
2. for \( t = 2 \), when for all \( (\sum_{y \in \mathbb{F}^m} \lambda y^* y) \in \mathcal{F}(\langle Y_1, \ldots, Y_r \rangle) \), and all \( x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_{q-1}, x_{q+1}, \ldots, x_m \in \mathbb{F} \) we have
\[
\lambda_{\frac{p}{q}}(x_1, \ldots, 0, \ldots, x_m) = \lambda_{\frac{p}{q}}(x_1, \ldots, 1, \ldots, x_m);
\]

3. for \( t = 2 \), when for all \( (\sum_{y \in \mathbb{F}^m} \lambda y^* y) \in \mathcal{F}(\langle Y_1, \ldots, Y_r \rangle) \), and all \( x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_{q-1}, x_{q+1}, \ldots, x_m \in \mathbb{F} \) we have
\[
\lambda_{\frac{p}{q}}(x_1, \ldots, 0, \ldots, x_m) = -\lambda_{\frac{p}{q}}(x_1, \ldots, 1, \ldots, x_m);
\]

**Proof.** For convenience, we assume that \( p = 1 \) and \( q = 2 \).

Let \( M \) be the adjacency matrix of \( Y \). By Theorem 4.1, the adjacency matrix of \( Y^t \) and \( Y^t_{12} \) is \( tM \). In the rest of the proof we will find the partition designs \( Y \) such that \( Y^t \) is equivalent to \( Y^t_{12} \).

The proof goes as follows. We suppose that \( Y^t \) and \( Y^t_{12} \) are equivalent partition designs. Then the cardinality of the support of \( Y^t \) is equal to the cardinality of the support of \( Y^t_{12} \). Since this is one of the invariants introduced in Section 3.2, from this equality we deduce, only for \( t \geq 3 \), necessary and sufficient conditions on \( Y \) so that \( Y^t \) and \( Y^t_{12} \) are equivalent partition designs.

For the case \( t = 2 \) we use \( \gamma \), which is a thinner invariant than the cardinality of the support. So we assume that the distribution vector of the support of \( Y^2 \) is equal to the distribution vector of the support \( Y^t_{12} \). From careful analysis of the difference between both distribution vectors, we deduce necessary and sufficient conditions on \( Y \) so that \( Y^2 \) and \( Y^t_{12} \) are equivalent partition designs.

Let us begin. We define the following sets for all \( v \in \mathbb{F}^{m-2t} \):
\[
A_v := \{ u \in \mathbb{F}^{2t} \mid (u; v) \in \text{SUPP}(Y^t) \}
\quad \text{and}
\]
\[
B_v := \{ u \in \mathbb{F}^{2t} \mid (u; v) \in \text{SUPP}(Y^t_{12}) \}.
\]

Observe that
\[
\sum_{v \in \mathbb{F}^{m-2t}} |A_v| = |\text{SUPP}(Y^t)| \quad \text{and} \quad \sum_{v \in \mathbb{F}^{m-2t}} |B_v| = |\text{SUPP}(Y^t_{12})|.
\]

Since we have assumed that \( Y^t \) and \( Y^t_{12} \) are equivalent, then
\[
\sum_{v \in \mathbb{F}^{m-2t}} |A_v| = \sum_{v \in \mathbb{F}^{m-2t}} |B_v|.
\]

(24)

Now we want to prove that
\[
|A_v| \leq |B_v| \quad \text{for all } v \in \mathbb{F}^{m-2t}.
\]

(25)

Before doing this, we must rewrite \( A_v \) and \( B_v \) without using the support. To do this we use the definition of the support.
\[
A_v = \{ u \in \mathbb{F}^{2t} \mid \pi(u;v)(t^t(\mathcal{F}(\langle Y_1, \ldots, Y_r \rangle))) \neq 0 \}
\]

(26)
\[
B_v = \{ u \in \mathbb{F}^{2t} \mid \pi(u;v)(t^t_{12}(\mathcal{F}(\langle Y_1, \ldots, Y_r \rangle))) \neq 0 \}.
\]

(27)
We split the proof of (25) into two cases:

(a) \( t = 2 \). Let \( W = \sum_{z \in \mathbb{F}^m} \lambda_z z^\ast \in \mathcal{F}(\{Y_1, \ldots, Y_t\}) \). Then

\[
\tau_{12}^2(W) = \tau_{12}^2 \left( \sum_{y \in \mathbb{F}^{m-2}} \sum_{x \in \mathbb{F}^2} \lambda_{(x,y)}(x; y)^\ast \right)
\]

\[
= \sum_{y \in \mathbb{F}^{m-2}} \sum_{x \in \mathbb{F}^2} \lambda_{(x,y)} \left( \tau_{12}^2(x^\ast); \tau^2(y^\ast) \right)
\]

\[
= \sum_{y \in \mathbb{F}^{m-2}} \left[ \lambda_{((0,0);y)}((0,0,0,0)^\ast; \tau^2(y^\ast)) + \frac{\lambda_{((0,1);y)}}{2}((0,1,1,0)^\ast + (0,1,0,1)^\ast + (1,0,1,0)^\ast - (1,1,0,0)^\ast; \tau^2(y^\ast)) + \frac{\lambda_{((1,0);y)}}{2}((0,0,1,1)^\ast + (0,1,0,1)^\ast + (1,0,1,0)^\ast; \tau^2(y^\ast)) + \lambda_{((1,1);y)}((1,1,1,1)^\ast; \tau^2(y^\ast)) \right].
\]

So, for every \( y \in \mathbb{F}^{m-2} \), the only possibly non-zero coefficients of \( \tau_{12}^2(W) \) are

\[
\left\{
\begin{array}{l}
\pi_{((0,0,0,0); \tau^2(y))}(\tau_{12}^2(W)) = \lambda_{((0,0);y)}, \\
\pi_{((0,0,1,1); \tau^2(y))}(\tau_{12}^2(W)) = 1/2(\lambda_{((0,1);y)} - \lambda_{((1,0);y)}), \\
\pi_{((0,1,0,1); \tau^2(y))}(\tau_{12}^2(W)) = 1/2(\lambda_{((0,1);y)} + \lambda_{((1,0);y)}), \\
\pi_{((1,0,1,0); \tau^2(y))}(\tau_{12}^2(W)) = 1/2(\lambda_{((0,1);y)} - \lambda_{((1,0);y)}), \\
\pi_{((1,1,0,0); \tau^2(y))}(\tau_{12}^2(W)) = 1/2(\lambda_{((0,1);y)} + \lambda_{((1,0);y)}), \\
\pi_{((1,1,1,1); \tau^2(y))}(\tau_{12}^2(W)) = \lambda_{((1,1);y)}. \\
\end{array}
\right\}
\]

(28)

Note that, in this case, \( \tau^2 \) is used for vectors without stars.

Since \( \tau^2(W) = \sum_{y \in \mathbb{F}^{m-2}} \sum_{x \in \mathbb{F}^2} \lambda_{(x,y)}(\tau^2(x^\ast); \tau^2(y^\ast)) \), then the only possibly non-zero coefficients of \( \tau^2(W) \) are

\[
\left\{
\begin{array}{l}
\pi_{((0,0,0,0); \tau^2(y))}(\tau^2(W)) = \lambda_{((0,0);y)}, \\
\pi_{((0,0,1,1); \tau^2(y))}(\tau^2(W)) = \lambda_{((0,1);y)}, \\
\pi_{((0,1,0,1); \tau^2(y))}(\tau^2(W)) = \lambda_{((1,0);y)}, \\
\pi_{((1,0,1,0); \tau^2(y))}(\tau^2(W)) = \lambda_{((1,1);y)}. \\
\end{array}
\right\}
\]

(29)

The above happens for all \( W \in \mathcal{F}(\{Y_1, \ldots, Y_t\}) \). We deduce that if \( v \in \mathbb{F}^{m-2t} \) cannot be written as \( v = \tau^2(y) \) for some \( y \in \mathbb{F}^{m-2} \), then \( A_v = B_v = \emptyset \), and so \( |A_v| = |B_v| = 0 \). Therefore, we are only interested in the sets \( A_{\tau^2(y)} \) and \( B_{\tau^2(y)} \) for \( y \in \mathbb{F}^{m-2} \).
By using the definitions (26) and (27), and formulas (28) and (29), we obtain that

\[(0, 0, 0, 0) \in A_{z(y)} \iff (0, 0, 0, 0) \in B_{z(y)}, \quad \text{and} \]

\[(1, 1, 1) \in A_{z(y)} \iff (1, 1, 1, 1) \in B_{z(y)}.\]

The set \(A_{z(y)}\) contains at most two more elements: \((0, 0, 1, 1)\) and \((1, 1, 0, 0)\); meanwhile \(B_{z(y)}\) contains at most four more elements: \((0, 0, 1, 1), (0, 1, 0, 1), (1, 0, 1, 0)\) and \((1, 1, 0, 0)\). Let us analyze each of the possibilities of \(A_{z(y)}\) for \(y \in \mathbb{F}^{m-2}\):

(i) \((0, 0, 1, 1), (1, 1, 0, 0) \notin A_{z(y)}\). Then for all \(W = \sum_{z \in \mathbb{F}^n} \lambda(z) z^* \in \mathcal{F}(Y_1, \ldots, Y_r)\) it is verified that

\[\pi((0, 0, 1, 1); z(y))(r_2^2(W)) = 0\]

and

\[\pi((1, 1, 0, 0); z(y))(r_2^2(W)) = 0.\]

By (29), \(\lambda((0, 1); y) = 0\) and \(\lambda((1, 0); y) = 0\). And, by (28), we have

\[\pi((0, 0, 1, 1); z(y))(r_2^2(W)) = \pi((1, 1, 0, 0); z(y))(r_2^2(W)) = \pi((1, 1, 0, 0); z(y))(r_2^2(W)) = 0.\]

So \((0, 0, 1, 1), (0, 0, 0, 0), (0, 1, 0, 1), (1, 0, 1, 0) \notin B_{z(y)}\). Finally \(|A_{z(y)}| = |B_{z(y)}| + 3\).

(ii) \((0, 0, 1, 1) \in A_{z(y)}, (1, 1, 0, 0) \notin A_{z(y)}\). Then, there exists \(W = \sum_{z \in \mathbb{F}^n} \lambda(z) z^* \in \mathcal{F}(Y_1, \ldots, Y_r)\) such that \(\lambda((0,1); y) \neq 0\) and \(\lambda((1,0); y) = 0\). Consequently,

\[\pi((0, 0, 1, 1); z(y))(r_2^2(W)) \neq 0, \quad \pi((1, 1, 0, 0); z(y))(r_2^2(W)) = 0, \]

\[\pi((1, 1, 0, 0); z(y))(r_2^2(W)) \neq 0.\]

So \((0, 0, 1, 1), (0, 1, 0, 0), (1, 0, 1, 0), (1, 1, 0, 0) \in B_{z(y)}\). That is, \(|A_{z(y)}| + 3 = |B_{z(y)}|\).

(iii) \((0, 0, 1, 1) \notin A_{z(y)}, (1, 1, 0, 0) \in A_{z(y)}\). Analogously we deduce that \((0, 0, 1, 1), (0, 0, 0, 1), (0, 1, 0, 1), (1, 0, 1, 0) \in B_{z(y)}\). And again, \(|A_{z(y)}| + 3 = |B_{z(y)}|\).

(iv) \((0, 0, 1, 1) \in A_{z(y)}, (1, 1, 0, 0) \in A_{z(y)}\). Then, there exists \(W = \sum_{z \in \mathbb{F}^n} \lambda(z) z^* \in \mathcal{F}(Y_1, \ldots, Y_r)\) such that \(\lambda((0,1); y) \neq 0\). Depending on the value of \(\lambda((1,0); y)\), we come to different conclusions:

1. \(\lambda((0,1); y) = \lambda((1,0); y)\).

   In this case,

   \[\pi((0, 0, 1, 1); z(y))(r_2^2(W)) = 0, \quad \pi((1, 1, 0, 0); z(y))(r_2^2(W)) = 0,\]

   \[\pi((1, 1, 0, 0); z(y))(r_2^2(W)) \neq 0, \quad \pi((0, 1, 0, 1); z(y))(r_2^2(W)) = 0.\]

   So \((0, 1, 0, 1), (1, 0, 1, 0) \in B_{z(y)}\). Thus \(|A_{z(y)}| \leq |B_{z(y)}|\).

2. \(\lambda((0,1); y) = -\lambda((1,0); y)\).

   In this case,

   \[\pi((0, 0, 1, 1); z(y))(r_2^2(W)) = 0, \quad \pi((0, 1, 0, 1); z(y))(r_2^2(W)) \neq 0,\]

   \[\pi((0, 1, 0, 1); z(y))(r_2^2(W)) = 0, \quad \pi((1, 1, 0, 0); z(y))(r_2^2(W)) = 0.\]

   So \((0, 0, 1, 1), (1, 1, 0, 0) \in B_{z(y)}\). Thus \(|A_{z(y)}| \leq |B_{z(y)}|\).
(3) $|\lambda_{(0,1):y}| \neq |\lambda_{(1,0):y}|$. And finally, in this case
\[
\pi_{((0,0,1,1):\tau^2(y))}(\tau^{12}_W) \neq 0, \quad \pi_{((0,1,0,1):\tau^2(y))}(\tau^{12}_W) \neq 0, \\
\pi_{((1,0,1,0):\tau^2(y))}(\tau^{12}_W) \neq 0, \quad \pi_{((1,1,0,0):\tau^2(y))}(\tau^{12}_W) \neq 0.
\]

So $(0, 0, 1, 1), (0, 1, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0) \in \mathcal{B}_{\tau^2(y)}$. Thus $|\mathcal{A}_{\tau^2(y)}| + 2 = |\mathcal{B}_{\tau^2(y)}|$. 

Note that if $|\mathcal{A}_{\tau^2(y)}| = |\mathcal{B}_{\tau^2(y)}|$, then necessarily case (a)(iv)(1) occurs for all $W \in \mathcal{F}(\mathcal{Y}_1, \ldots, \mathcal{Y}_r)$, or case (a)(iv)(2) occurs for all $W \in \mathcal{F}(\mathcal{Y}_1, \ldots, \mathcal{Y}_r)$.

(b) $t \geq 3$. As in (23), an overlined symbol is a shorthand for repeating that symbol $t$ times. In this case we proceed like we did for $t = 2$, but this time the analysis becomes easier. Let $W = \sum_{z \in \mathbb{Z}^{m}} \lambda_{z} z^* \in \mathcal{F}(\mathcal{Y}_1, \ldots, \mathcal{Y}_r)$. Then
\[
\tau^{12}_W = \sum_{y \in \mathbb{Z}^{m-2}} \left[ \lambda_{(0,0,0,0,0,\bar{0},0,0,0,\bar{0})^*; \tau'(y^*)} \\
+ \frac{\lambda_{(0,1,0,1,0,\bar{1},0,0,0,\bar{1})^*}}{2} ((0, 0, \bar{0}, 1, 0, 0, 1, 0, \bar{0}, 0) + (0, 1, \bar{0}, 0, 0, 0, 0, 0, 0, \bar{0})^*; \tau'(y^*)) \\
+ (1, 0, 0, 0, 0, 0, 0, 0, 0, 0)^*; \tau'(y^*)) \\
+ \frac{\lambda_{(1,1,0,0,0,0,0,0,0,0,0)^*}}{2} ((0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0) + (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)^*; \tau'(y^*)) \\
+ (1, 0, 0, 0, 0, 0, 0, 0, 0, 0)^*; \tau'(y^*)) \\
+ \lambda_{(1,1,0,1,0,0,0,0,0,0,0)^*; \tau'(y^*)) \right].
\]
So, for every $y$, the only possibly non-zero coefficients of $\tau^{12}_W$ are
\[
\pi_{((0,0,0,0,0,0,0,0,0,0)^*; \tau'(y^*))}(\tau^{12}_W) = \lambda_{((0,0):y)}, \\
\pi_{((0,0,0,0,0,0,0,0,0,0)^*; \tau'(y^*))}(\tau^{12}_W) = \lambda_{((1,0):y)}/2, \\
\pi_{((1,0,0,0,0,0,0,0,0,0)^*; \tau'(y^*))}(\tau^{12}_W) = \lambda_{((0,1):y)}/2, \\
\pi_{((1,1,0,0,0,0,0,0,0,0)^*; \tau'(y^*))}(\tau^{12}_W) = \lambda_{((0,1):y)}/2, \\
\pi_{((0,0,0,0,0,0,0,0,0,0)^*; \tau'(y^*))}(\tau^{12}_W) = \lambda_{((0,1):y)}/2, \\
\pi_{((0,0,0,0,0,0,0,0,0,0)^*; \tau'(y^*))}(\tau^{12}_W) = \lambda_{((1,0):y)}/2, \\
\pi_{((0,1,0,0,0,0,0,0,0,0)^*; \tau'(y^*))}(\tau^{12}_W) = \lambda_{((0,1):y)}/2, \\
\pi_{((1,0,0,0,0,0,0,0,0,0)^*; \tau'(y^*))}(\tau^{12}_W) = \lambda_{((1,0):y)}/2, \\
\pi_{((1,1,0,0,0,0,0,0,0,0)^*; \tau'(y^*))}(\tau^{12}_W) = \lambda_{((1,0):y)}/2, \\
\pi_{((0,1,0,0,0,0,0,0,0,0)^*; \tau'(y^*))}(\tau^{12}_W) = \lambda_{((1,0):y)}/2.
\]
And for $\tau'(W)$ we get
\[
\pi_{((0,0,0,0,0,0,0,0,0,0)^*; \tau'(y^*))}(\tau'(W)) = \lambda_{((0,0):y)}, \\
\pi_{((0,0,0,0,0,0,0,0,0,0)^*; \tau'(y^*))}(\tau'(W)) = \lambda_{((0,1):y)}. 
\]
Then let us use the following notation

\[
\pi_{(1,1,0,0,0)}(\pi'_{(1)}(W)) = \lambda_{(1,0)}(y),
\]

\[
\pi_{(1,1,1,1,0)}(\pi'_{(1)}(W)) = \lambda_{(1,1)}(y).
\]

With the same arguments used for \( t = 2 \), we obtain

\[
(0, 0, \tilde{0}, 0, 0, \tilde{0}) \in \mathcal{A}_{t'(1)} \iff (0, 0, \tilde{0}, 0, 0, \tilde{0}) \in \mathcal{B}_{t'(1)},
\]

\[
(0, 0, \tilde{0}, 1, 1, \tilde{1}) \in \mathcal{A}_{t'(1)} \iff (0, 0, \tilde{0}, 1, 1, \tilde{1}), (0, 1, \tilde{0}, 0, 1, \tilde{1}),
\]

\[
(1, 0, \tilde{0}, 1, 0, \tilde{1}), (1, 1, \tilde{0}, 0, 0, \tilde{1}) \in \mathcal{B}_{t'(1)},
\]

\[
(1, 1, \tilde{1}, 0, 0, \tilde{0}) \in \mathcal{A}_{t'(1)} \iff (0, 0, \tilde{1}, 1, 0, \tilde{0}), (0, 1, \tilde{1}, 0, 1, \tilde{0}),
\]

\[
(1, 0, \tilde{1}, 1, 0, \tilde{0}), (1, 1, \tilde{1}, 0, 0, \tilde{0}) \in \mathcal{B}_{t'(1)},
\]

\[
(1, 1, \tilde{1}, 1, 1, \tilde{1}) \in \mathcal{A}_{t'(1)} \iff (1, 1, \tilde{1}, 1, 1, \tilde{1}) \in \mathcal{B}_{t'(1)}.
\]

So \( |\mathcal{A}_{t'(1)}| \leq |\mathcal{B}_{t'(1)}| \). Note that, \( |\mathcal{A}_{t'(1)}| = |\mathcal{B}_{t'(1)}| \) if and only if

\[
\text{for all } \sum_{y \in \mathbb{R}^m} \lambda_{z} z^* \in \mathcal{F}(\{Y_1, \ldots, Y_t\}) \text{ we have } \lambda_{((0,1);y)} = \lambda_{((1,0);y)} = 0.
\]

From (24) and (25) we obtain that

\[
|\mathcal{A}_{t'(1)}| = |\mathcal{B}_{t'(1)}| \quad \text{for all } y \in \mathbb{R}^{m-2}.
\]

For \( t \geq 3 \), condition (30) implies that for all \( \sum_{y \in \mathbb{R}^m} \lambda_{z} z^* \in \mathcal{F}(\{Y_1, \ldots, Y_t\}) \) and for all \( y \in \mathbb{R}^{m-2} \), we have \( \lambda_{((0,1);y)} = \lambda_{((1,0);y)} = 0 \). This corresponds to exception 1. Note that in this case \( \pi'_{(W)} = \pi'_{(2)}(W) \) for all \( W \in \mathcal{F}(\{Y_1, \ldots, Y_t\}) \). So for this exception, indeed, \( \gamma^y \) and \( \gamma_{12}^y \) are equivalent; in fact, they are equal.

In the rest of the proof we assume that \( t = 2 \). Condition (30) implies that for each \( y \in \mathbb{R}^{m-2} \) we have case (a)(i), (a)(iv)(1) or (a)(iv)(2). Equivalently, each \( y \in \mathbb{R}^{m-2} \) belongs to one of the following disjoint sets:

\[
Q := \left\{ y \in \mathbb{R}^{m-2} \mid (0, 0, 1, 1), (1, 1, 0, 0) \notin \mathcal{A}_{t'(1)} \text{ and } (0, 0, 1, 1), (1, 1, 0, 0) \notin \mathcal{B}_{t'(1)} \right\}
\]

\[
P_+ := \left\{ y \in \mathbb{R}^{m-2} \mid (0, 0, 1, 1), (1, 1, 0, 0) \in \mathcal{A}_{t'(1)} \text{ and } (0, 0, 1, 1), (1, 1, 0, 0) \notin \mathcal{B}_{t'(1)} \right\}
\]

\[
P_- := \left\{ y \in \mathbb{R}^{m-2} \mid (0, 0, 1, 1), (1, 1, 0, 0) \in \mathcal{A}_{t'(1)} \text{ and } (0, 0, 1, 1), (1, 1, 0, 0) \in \mathcal{B}_{t'(1)} \right\}
\]

Now we will prove that at least one of the sets \( P_+ \), \( P_- \) is empty. We need two more sets:

\[
P_0 := \{ y \in \mathbb{R}^{m-2} \mid (0, 0, 0, 0) \notin \mathcal{A}_{t'(1)} \text{ and } (0, 0, 0, 0) \in \mathcal{B}_{t'(1)} \}
\]

\[
P_1 := \{ y \in \mathbb{R}^{m-2} \mid (1, 1, 1, 1) \notin \mathcal{A}_{t'(1)} \text{ and } (1, 1, 1, 1) \in \mathcal{B}_{t'(1)} \}
\]

Let us use the following notation

\[
\mathcal{P}^u_{s}(y) = \{ (u; \pi^2(y)) \} \in \mathcal{P}_s \text{ where } s \in \{ +, - , 1, 0 \} \text{ and } u \in \mathbb{F}^4.
\]

Then

\[
\text{Supp}(\gamma^y) = \mathcal{P}^{(0,0,1,1)}_+ \cup \mathcal{P}^{(1,1,0,0)}_+ \cup \mathcal{P}^{(0,0,1,1)}_- \cup \mathcal{P}^{(1,1,0,0)}_- \cup \mathcal{P}^{(0,0,0,0)}_0 \cup \mathcal{P}^{(1,1,1,1)}_1.
\]
and

$$\text{SUPP}(\gamma_{12}) = \mathcal{P}_+^{(1,0,1,0)} \cup \mathcal{P}_+^{(1,0,1,0)} \cup \mathcal{P}_-^{(0,0,0,0)} \cup \mathcal{P}_0^{(0,1,1,0)} \cup \mathcal{P}_1^{(1,1,1,1)}.$$  

Since $\gamma$ is an invariant for equivalent partition designs, using the definition of the distribution vector and simplifying opposite terms we obtain

$$(0, \ldots, 0) = \gamma(\gamma^2) - \gamma(\gamma_{12}^2) = A_{\text{SUPP}(\gamma^2)} - A_{\text{SUPP}(\gamma_{12}^2)}$$

$$= \frac{1}{|\text{SUPP}(\gamma^2)|} \left( \sum_{\mathcal{R}, \mathcal{S} \in \{\mathcal{P}_+^{(1,1,0,0)}, \mathcal{P}_+^{(1,1,0,0)}, \mathcal{P}_-^{(0,0,0,0)}, \mathcal{P}_0^{(0,0,0,0)}, \mathcal{P}_1^{(1,1,1,1)}\}} |\{(z_1, z_2) \in \mathcal{R} \times \mathcal{S} \mid w(z_1 \boxplus z_2) = i\}| \right)_{0 \leq i \leq 2m}$$

$$- \sum_{\mathcal{R}, \mathcal{S} \in \{\mathcal{P}_+^{(1,1,0,0)}, \mathcal{P}_+^{(1,1,0,0)}, \mathcal{P}_-^{(0,0,0,0)}, \mathcal{P}_0^{(0,0,0,0)}, \mathcal{P}_1^{(1,1,1,1)}\}} |\{(z_1, z_2) \in \mathcal{R} \times \mathcal{S} \mid w(z_1 \boxplus z_2) = i\}|$$

$$= \frac{2}{|\text{SUPP}(\gamma^2)|} \left( \sum_{\mathcal{R} \in \{\mathcal{P}_+^{(1,1,0,0)}, \mathcal{P}_+^{(1,1,0,0)}\}} |\{(z_1, z_2) \in \mathcal{R} \times \mathcal{S} \mid w(z_1 \boxplus z_2) = i\}| \right)_{0 \leq i \leq 2m}$$

$$- \sum_{\mathcal{R} \in \{\mathcal{P}_+^{(1,1,0,0)}, \mathcal{P}_+^{(1,1,0,0)}\}} |\{(z_1, z_2) \in \mathcal{R} \times \mathcal{S} \mid w(z_1 \boxplus z_2) = i\}|$$

$$= \frac{2}{|\text{SUPP}(\gamma^2)|} [2(B; (0, 0, 0, 0)) + 2((0, 0, 0, 0); B) - 4((0, 0); B; (0, 0))] \quad (31)$$

where $B = (\{(x, y) \in \mathcal{P}_+ \times \mathcal{P}_- \mid w(x \boxplus y) = i\})_{0 \leq i \leq 2m-4} \in \mathbb{Z}^{2m-3}$, and where we have used concatenation for vectors with coefficients in $\mathbb{Z}$.

From (31) we deduce that $B = (0, \ldots, 0)$, and this implies that at least one of the sets $\mathcal{P}_+, \mathcal{P}_-$ is empty.

(A) If $\mathcal{P}_+ = \emptyset$ and $\mathcal{P}_- = \emptyset$ then for each $y \in \mathbb{F}^{m-2}$ we have case (a)(i). In other words, for all $\sum_{i=1}^{m} \lambda_i z_i^* \in \mathcal{F}(Y_1, \ldots, Y_t)$ we have that $\lambda_{(0,1); y} = \lambda_{(1,0); y} = 0$. This corresponds to exception 1, as stated in the lemma. If we observe formulas
(28) and (29), we conclude that $\tau_{12}^2(W) = \tau_1(W)$ for all $W \in \mathcal{F}(\mathbf{Y}_1, \ldots, \mathbf{Y}_r)$. Therefore $\mathcal{Y}_2 = \mathcal{Y}_{12}^2$.

(B) If $\mathcal{P}_+ = \emptyset$ then for each $y \in \mathbb{F}^m$ we have case (a)(i) or case (a)(iv)(2). In other words, for all $\sum_{z \in \mathbb{F}^m} \lambda_z z^y \in \mathcal{F}(\mathbf{Y}_1, \ldots, \mathbf{Y}_r)$ we have that $\lambda_{((0,1);y)} = -\lambda_{((1,0);y)}$.

This corresponds to exception 3. If we observe formulas (28) and (29), taking into account that $\lambda_{((0,1);y)} + \lambda_{((1,0);y)} = 0$, we conclude that $\tau_{12}^2(W) = \tau_1(W)$ for all $W \in \mathcal{F}(\mathbf{Y}_1, \ldots, \mathbf{Y}_r)$. Therefore $\mathcal{Y}_2 = \mathcal{Y}_{12}^2$.

(C) If $\mathcal{P}_- = \emptyset$ then for each $y \in \mathbb{F}^m$ we have case (a)(i) or case (a)(iv)(1). In other words, for all $\sum_{z \in \mathbb{F}^m} \lambda_z z^y \in \mathcal{F}(\mathbf{Y}_1, \ldots, \mathbf{Y}_r)$ we have that $\lambda_{((0,1);y)} = \lambda_{((1,0);y)}$.

This corresponds to exception 2. If we observe formulas (28) and (29), we conclude that $\tau_{12}^2(W) = \sigma_2 \circ \tau_1(W)$ for all $W \in \mathcal{F}(\mathbf{Y}_1, \ldots, \mathbf{Y}_r)$, where $\sigma_2$ is the permutation $(1, 4)$. Therefore $\mathcal{Y}_2$ and $\mathcal{Y}_{12}^2$ are equivalent partition designs. □

Note that for $t = 2$, exception 1 is a particular case of exceptions 2 and 3.

Proof of Theorem 1.1. In the first part of the proof we will find the set $\mathcal{V}$ of all pairs $(\mathcal{Y}, t)$, where $\mathcal{Y}$ is a partition design of $H(m)$ and $t \geq 2$, such that $\mathcal{Y}^d$ is equivalent to $\mathcal{Y}_{pq}$ for all $1 \leq p < q \leq m$ (remember that $\mathcal{Y}^d$ and $\mathcal{Y}_{pq}$ have the same adjacency matrix). This is done using Lemma 6.1.

The aim of the theorem is to find the set of all imprimitive partition designs that are characterized by their adjacency matrix. This set is equivalent to the set $\mathcal{U}$ of all pairs $(\mathcal{Y}, t)$, where $\mathcal{Y}$ is a partition design of $H(m)$ and $t \geq 2$, such that $\mathcal{Y}^d$ is characterized by its adjacency matrix. Clearly $\mathcal{U} \subset \mathcal{V}$.

The second part of the proof consists of determining which elements of $\mathcal{V}$ are in $\mathcal{U}$. Surprisingly there is only one element of $\mathcal{V}$ that is not contained in $\mathcal{U}$, namely $(\mathcal{Y}_4, 2)$.

Let $\mathcal{Y} = \{Y_1, \ldots, Y_r\}$ be a partition design of $H(m)$ such that $\mathcal{Y}^d$ is equivalent to $\mathcal{Y}_{pq}$ for all $1 \leq p < q \leq m$. By Lemma 6.1, for each $(p, q)$ the partition design $\mathcal{Y}$ verifies exceptions 1, 2 or 3.

We remember some notation that we will need. Let $M$ be the adjacency matrix of $\mathcal{Y}$. In (13), we saw that the eigenvalues of $M$ verify $m = \theta_1 \geq \cdots \geq \theta_r \geq -m$. The elements $U_1, U_2, \ldots, U_r$ are as in Definition 3.4, and by the paragraph after Definition 3.4 we may assume that $U_1 = \sum_{i=1}^r Y_i$. In Proposition 3.3 we saw that there exists $\alpha_{i,x} \in \mathbb{R}$ such that

$$\mathcal{F}(U_i) = \sum_{x \in S_{m-\theta_i}} \alpha_{i,x} x^y. \tag{32}$$

Now we prove an important consequence of Lemma 6.1:

$$|\alpha_{i,x}| = |\alpha_{i,y}| \quad \text{for all } 1 \leq i \leq r \text{ and for all } x, y \in S_{m-\theta_i}. \tag{33}$$

Let $x, y \in S_{m-\theta_i}$, where $1 \leq i \leq r$. Choose a permutation $\rho$ of the coordinates that transforms $x$ into $y$: $\rho(x) = y$. The permutation $\rho$ can be decomposed into transpositions: $\rho = \rho_1 \circ \cdots \circ \rho_s$. From the set $\{1, \ldots, s\}$ we delete the elements $i$ such that $\rho_i$ interchanges two 0’s or two 1’s of $\rho_{i+1}(\rho_{i+2}(\cdots \rho_{i}(x) \cdots))$. We get the subset $\{i_1, \ldots, i_t\} \subseteq \{1, \ldots, s\}$. It is still true that $\rho_1(\rho_2(\cdots \rho_k(x) \cdots)) = y$. The transposition $\rho_i$ switches a 1 and a 0.
of $x$, let us say that the 1 is in position $p$ and that the 0 is in position $q$ and that $p < q$.
By hypothesis, $\gamma$ and $\gamma^{\rho_{pq}}$ are equivalent. Then, one of the exceptions described in Lemma 6.1 is verified. In either case, it is true that $|\alpha| = |\alpha_{\rho_{pq}(x)}|$. Using the same argument for $\rho_{l_1}, \ldots, \rho_{l_2}, \rho_1$ we obtain that

$$|\alpha_{l_1,x}| = |\alpha_{l_1,\rho_{l_1}(x)}| = \cdots = |\alpha_{l_1,\rho_{l_1}(\cdots \rho_{l_1}(x)\cdots))| = |\alpha_{l_1,y}|.$$ 

Let us continue. By $(32)$ and since $\theta_1 = m$, we have $F(U_1) = \alpha_{1,(0,\ldots,0)}(0, \ldots, 0)^*$. This element verifies the three exceptions.

It is convenient to distinguish two cases:

(A) **Exception 1 happens for some pair $1 \leq p < q \leq m$.** Now we distinguish two more cases:

(a) $r = 1$. Then, $Y_1 = F^m$. So $\gamma$ is the trivial partition $T_m$. Finally, $(T_m, t) \in \mathcal{V}$ for

$m \geq 1$ and $t \geq 2$, where $\mathcal{V}$ is the set of all pairs $(\gamma, t)$ such that $\gamma$ is equivalent to $\gamma^{\rho_{pq}}$ for all $1 \leq p < q \leq m$. Note that $T_m^t = T_{lm}$. This case corresponds to (i) of Theorem 1.1.

(b) $r \geq 2$. Suppose that $(m) = (\theta_1 = \theta_2)$, then $F(U_2) = \alpha_{2,(0,\ldots,0)}(0, \ldots, 0)^*$ because of $(32)$. Then, $F(U_1)$ and $F(U_2)$ would be linearly dependent, but this is impossible.

Now, suppose that $m > \theta_2 > -m$. Then, $0 < \frac{m-\theta_2}{2} < m$. So there exists $x \in S_{m, -\theta_2}$ such that $x_p = 1$ and $x_q = 0$. Because exception 1 is verified for the pair $(p, q)$, we have $\alpha_{2,x} = 0$; and by $(33)$, $\alpha_{2,x} = 0$ for all $x \in S_{m, \theta_2}$. This implies that $F(U_2) = 0$, which is impossible.

We conclude that $\theta_2 = -m$. That is, $F(U_2) = \alpha_{2,(1,\ldots,1)}(1, \ldots, 1)^*$. Necessarily $r = 2$. Otherwise, $\theta_3 = \theta_2 = -m$ and this would imply that $F(U_3)$ and $F(U_2)$ would be linear dependent, which is impossible.

Observe that, from Proposition 3.1 we obtain that

$$F^{-1}((1, \ldots, 1)^*) = \frac{1}{2m} (F(1, \ldots, 1))^*$$

$$= \frac{1}{2m} \sum_{x \in \mathbb{Z}^m} (-1)^{(1,\ldots,1)\cdot x} = \sum_{i \text{ even}} S_i - \sum_{i \text{ odd}} S_i.$$

Therefore, $\gamma$ is the even–odd partition $\mathcal{E}_m$. Finally, $(\mathcal{E}_m, t) \in \mathcal{V}$ for $m \geq 1$ and $t \geq 2$. Note that $\mathcal{E}_m^t = \mathcal{E}_m$. This case corresponds to (ii) of Theorem 1.1.

(B) **Exception 1 does not happen for any pair.** Then $t = 2$. We analyze this case by proving a sequence of claims:

**Claim 1.** All eigenvalues of $M$ are different, i.e., $\theta_1 > \theta_2 > \cdots > \theta_r$.

Suppose that $\theta_i = \theta_j$ for $i \neq j$, then consider for an arbitrary $x \in S_{m, -\theta}$, the following element of $F((Y_1, \ldots, Y_r))$:

$$\frac{F(U_i)}{\alpha_{i,x}} + \frac{F(U_j)}{\alpha_{j,x}} = \sum_{y \in S_{m, -\theta}} \left( \frac{\alpha_{i,y}}{\alpha_{i,x}} + \frac{\alpha_{j,y}}{\alpha_{j,x}} \right) y^*.$$ 

(34)
Note that \( \alpha_{i,x} \neq 0 \); otherwise, \( \alpha_{i,y} = 0 \) for all \( y \in S_{m-\theta_2} \), by (33), and so \( \mathcal{F}(U_i) = 0 \), which is impossible. For the same reason \( \alpha_{j,x} \neq 0 \). The fractions \( \frac{\alpha_{i,x}}{\alpha_{j,x}} \) are equal to \( \pm 1 \) by (33). Thus the coefficients of the element (34) are \( \pm 2 \) or 0. If \( \frac{\alpha_{i,x}}{\alpha_{j,x}} + \frac{\alpha_{j,x}}{\alpha_{i,x}} = 2 \) for all \( y \in S_{m-\theta_2} \), then \( \mathcal{F}(U_i) \) and \( \mathcal{F}(U_j) \) would be linear dependent, which is impossible. The same would happen if all coefficients are 0. So there is a \( z_1 \in S_{m-\theta_2} \) such that \( \frac{\alpha_{i,z_1}}{\alpha_{j,z_1}} = 0 \), and a \( z_2 \in S_{m-\theta_2} \) such that \( \frac{\alpha_{i,z_2}}{\alpha_{j,z_2}} = 2 \). Now we use a similar argument to the one used to prove (33). Observe that all the coefficients of \( \frac{F(U_i)}{\alpha_{i,x}} \) are real. Let \( \sigma = \sigma_1 \circ \cdots \circ \sigma_s \) be a permutation such that \( \sigma(z_1) = z_2 \) and such that \( \sigma_i \) is a transposition that interchanges a 1 and 0 of \( \sigma_{i+1}(\cdots \sigma(z_1) \cdots) \) for all \( i = 1, \ldots, s \). Then,

\[
0 = \left| \frac{\alpha_{i,z_1}}{\alpha_{i,x}} + \frac{\alpha_{i,z_2}}{\alpha_{i,x}} \right| = \left| \frac{\alpha_{i,\sigma(z_1)}}{\alpha_{i,x}} + \frac{\alpha_{i,\sigma(z_2)}}{\alpha_{i,x}} \right| = \cdots = \left| \frac{\alpha_{i,z_2}}{\alpha_{i,x}} + \frac{\alpha_{i,z_2}}{\alpha_{j,x}} \right| = 2.
\]

This is a contradiction.

**Claim 2.** \( m - 4 \) is an eigenvalue of \( M \).

Suppose that \( \theta_1 = m \) is the only eigenvalue of \( M \), then \( r = 1 \), so \( \mathcal{V} = \{ Y_1 \} = \{ \mathbb{F}^m \} \) and \( \mathcal{F}(Y_1) = \langle (0, \ldots, 0)^* \rangle \). Thus, exception 1 would be verified for every pair \((p, q)\). In (B) we have assumed that exception 1 does not happen, so there is at least a second eigenvalue \( \theta_2 \) of \( M \) different from \( \theta_1 \).

Suppose that \( \theta_2 = -m \), then \( r = 2 \), and

\[
\mathcal{F}(Y_1, Y_2) = \mathcal{F}(U_1, U_2) = \langle \mathcal{F}(U_1), \mathcal{F}(U_2) \rangle = \langle (0, \ldots, 0)^*, (1, \ldots, 1)^* \rangle,
\]

and so exception 1 would also be verified for every pair \((p, q)\). In (B) we have assumed that exception 1 does not happen, so \( m > \theta_2 = -m \) or equivalently \( 0 < \frac{m-\theta_2}{2} < m \). We deduce that there are \( x_1, \ldots, x_m \in \mathbb{F} \) such that \( (1, 0, x_3, \ldots, x_m) \in S_{m-\theta_2} \).

If the pair \((1, 2)\) verifies exception 2, then \( \alpha_{2,1,0,x_3,\ldots,x_m} = \alpha_{2,0,1,x_3,\ldots,x_m} \) for all \( x_3, \ldots, x_m \in \mathbb{F} \) such that \( (1, 0, x_3, \ldots, x_m) \in S_{m-\theta_2} \). Note that \( \alpha_{2,1,0,x_3,\ldots,x_m} \neq 0 \); otherwise, \( \alpha_{i,x} = 0 \) for all \( x \in S_{m-\theta_2} \), by (33), and so \( \mathcal{F}(U_i) = 0 \), which is impossible. We also know that \( \alpha_{2,x} = 0 \) for all \( x \notin S_{m-\theta_2} \). Therefore, \( \alpha_{2,1,1,x_3,\ldots,x_m} \alpha_{2,0,0,x_3,\ldots,x_m} = 0 \) for all \( x_3, \ldots, x_m \in \mathbb{F} \) (note that \( w((1, 1, x_3, \ldots, x_m)) = w((0, 0, x_3, \ldots, x_m)) + 2 \)).

Let us prove that the coefficient of \((1, 1, 0, \ldots, 0)^*\) for the vector \( \mathcal{F}(U_2) \oplus \mathcal{F}(U_2) \in \mathcal{F}(Y_1, \ldots, Y_r) \) is non-zero:

\[
\pi_{(1,1,0,\ldots,0)}(\mathcal{F}(U_2) \oplus \mathcal{F}(U_2))
= \pi_{(1,1,0,\ldots,0)} \left( \sum_{y \in S_{m-\theta_2}} \alpha_{2,y} y^* \right) \oplus \left( \sum_{y \in S_{m-\theta_2}} \alpha_{2,y} y^* \right)
\]
Now we have got two possibilities:

Let

\[
\alpha \left( 1, 0, x_3, \ldots, x_m \right) \alpha \left( 0, 1, x_3, \ldots, x_m \right)
\]

Thus, (1, 1, 0, \ldots, 0) \in \text{Supp}(\mathcal{Y}). So \exists \ell \text{ such that } \frac{m-\theta_1}{2} = 2, \text{ i.e. } \theta_1 = m - 4.

If the pair (1, 2) verifies exception 3, then the coefficient of (1, 1, 0, \ldots, 0)^* for the vector \(\mathcal{F}(U_2) \oplus \mathcal{F}(U_2) \in \mathcal{F}(Y_1, \ldots, Y_r)\) is also non-zero:

\[
\pi_{1,1,0,0}(\mathcal{F}(U_2) \oplus \mathcal{F}(U_2)) = 2 \sum_{x_3, \ldots, x_m \in \mathcal{Y}, x \in S_{m-2}} -\left( \alpha \left( 1, 0, x_3, \ldots, x_m \right) \right)^2 \neq 0.
\]

Claim 3. There is a partition design equivalent to \(\{Y_1, \ldots, Y_r\}\) where only exception 2 occurs.

Remember that \(\theta_1 = m - 4\) and so \(\mathcal{F}(U_1) = \sum_{y \in S_2} \alpha_{i,y} y^x\), where \(|\alpha_{i,x}| = |\alpha_{i,y}|\) for all \(x, y \in S_2\).

For all \(T \subset \{1, \ldots, m\}\), let \(x_T \in S_{|T|}\) be the vector:

\[
(x_T)_i := \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{otherwise.} \end{cases}
\]

Let us define the following set:

\[
N := \{ k \in \{2, \ldots, m\} | \alpha_{i,x_{1,k}} < 0 \}.
\]

Now we define a partition design equivalent to the original one:

\[
\{Y'_1, \ldots, Y'_r\} := \{x_N \oplus Y_1, \ldots, x_N \oplus Y_r\}.
\]

Let \(U'_j = \sum_{j=1}^r u_{ij} Y'_j\). By Section 3.1, and in particular by (17), we have that

\[
\mathcal{F}(U'_j) = \sum_{x \in S_2} a'_{i,j,x} x^x \quad \text{where } a'_{i,j,x_{(p,q)}}, x^x = (-1)^{x_{(p,q)}, x_{N}} a_{i,j,x_{(p,q)}} \cdot
\]

Since \(\langle x_{(1,q)}, x_N \rangle = 1\) if \(\alpha_{i,x_{(1,q)}} < 0\) and \(\langle x_{(1,q)}, x_N \rangle = 0\) if \(\alpha_{i,x_{(1,q)}} > 0\) then \(a'_{i,j,x_{(1,q)}} > 0\) for all \(2 \leq q \leq m\). So \(N\) is empty for \(\{Y'_1, \ldots, Y'_r\}\). This means that for \(\{Y'_1, \ldots, Y'_r\}\) exception 2 occurs for all pairs \((p, q)\) such that \(2 \leq p < q \leq m\). So for all \(2 \leq p_1 < q_1 \leq m\) and all \(2 \leq p_2 < q_2 \leq m\), we have \(a'_{i,j,x_{(p_1,q_1)}} = a'_{i,j,x_{(p_2,q_2)}} = a'_{i,j,x_{(p_2,q_2)}}\).

Now we have got two possibilities:

(i) \(a'_{i,j,x_{(p,q)}} > 0\) for all \(2 \leq p < q \leq m\). Then all pairs \((1, q)\) for \(2 \leq q \leq m\) verify exception 2 and we are finished.
(ii) $\alpha'_{x[q-p]} < 0$ for all $2 \leq p < q \leq m$. Then all the pairs $(1, q)$ verify exception 3. In this case we re-define

$$\{Y'_1, \ldots, Y'_r\} \equiv \{x_{1}\cup N \oplus Y_1, \ldots, x_{1}\cup N \oplus Y_r\}.$$  

Finally we obtain that $\alpha'_{t,y} < 0$ for all $y \in S_2$. Therefore, we are finished.

From now on we assume that for $\{Y_1, \ldots, Y_r\}$ only exception 2 occurs.

**Claim 4.**

$$F((Y_1, \ldots, Y_r)) = \begin{cases} 
(S_0^*, S_2^*, S_4^*, \ldots), & \text{or} \\
(S_0^*, S_1^*, S_2^*, \ldots, S_m^*)
\end{cases}$$

By ((B) Claim 2) and ((B) Claim 3), we have $F(U_i) = \beta S_2^*$ for some $\beta \in \mathbb{R}$. Thus $S_2^* \in F((Y_1, \ldots, Y_r))$. The convolution product closure of $S_2^*$ is $(S_0^*, S_2^*, S_4^*, \ldots)$. This subalgebra verifies that exception 2 happens for all pairs and that all eigenvalues are different: $\theta_1 = m, \theta_2 = m - 4, \theta_3 = m - 8, \ldots$ If we make this subalgebra larger, we must adjoin $S_j^*$ for an odd $j$. Then the closure is $(S_0^*, S_1^*, S_2^*, \ldots, S_m^*)$. We cannot make this subalgebra even bigger since all the possible eigenvalues are used.

**Claim 5.** The case $F((Y_1, \ldots, Y_r)) = (S_0^*, S_1^*, S_2^*, \ldots, S_m^*)$ corresponds to $\mathcal{V} = \mathcal{D}_m$.

By (8), we have $F(S_1^*) = \sum_{i=0}^{m}(m - 2i)S_i$. So

$$\{Y_1, \ldots, Y_r\} = \{S_0, S_1, S_2, \ldots, S_m\}.$$  

So $(\mathcal{D}_m, 2) \in \mathcal{V}$ for all $m \geq 2$. Note that $\mathcal{D}_1 \equiv \mathcal{E}_1$. This case corresponds to (iii) of Theorem 1.1.

**Claim 6.** The case $F((Y_1, \ldots, Y_r)) = (S_0^*, S_2^*, S_4^*, \ldots)$ corresponds to $\mathcal{V} = \mathcal{H}_m$.

By making a similar calculation as in (8), we obtain:

$$F(S_2^*) = \sum_{i=0}^{m} \binom{m}{2} - 2(m - i)i S_i.$$  

Let us see for which $i \neq j$ we have

$$\binom{m}{2} - 2(m - i)i = \binom{m}{2} - 2(m - j)j.$$  

By a straightforward process we deduce that $m = i + j$. We conclude that

$$\{Y_1, \ldots, Y_r\} = \{S_0 \cup S_m, S_1 \cup S_{m-1}, S_2 \cup S_{m-2}, \ldots\}.$$  

So $(\mathcal{H}_m, 2) \in \mathcal{V}$ for all $m \geq 3$. Note that $\mathcal{H}_i \equiv \mathcal{E}_i$ for $i = 1, 2$. This case corresponds to (iv) of Theorem 1.1.

We have seen that

$$\mathcal{V} = \{(\mathcal{T}_m, t)\}_{m \geq 1, t \geq 2} \cup \{(\mathcal{E}_m, t)\}_{m \geq 1, t \geq 2} \cup \{(\mathcal{D}_m, 2)\}_{m \geq 2} \cup \{(\mathcal{H}_m, 2)\}_{m \geq 3}.$$
Remember that \( \mathcal{U} \) is the set of all pairs \((\mathcal{V}, t)\), where \( \mathcal{V} \) is a partition design of \( H(m) \) and \( t \geq 2 \), such that \( \mathcal{V}' \) is characterized by its adjacency matrix. The proof will be finished if we determine \( \mathcal{U} \). We know that \( \mathcal{U} \subseteq \mathcal{V} \). In the second part of the proof we will study all the pairs of \( \mathcal{V} \) and see if they belong to \( \mathcal{U} \).

(a) All partition designs \( \mathcal{T}_m \equiv \mathcal{T}_{m,t} \), where \( m \geq 1 \) and \( t \geq 2 \), are obviously characterized by their adjacency matrix, which is \( (tm, 0) \).

(b) All partition designs \( \mathcal{E}_m \equiv \mathcal{E}_{m,t} \), where \( m \geq 1 \) and \( t \geq 2 \), are clearly characterized by their adjacency matrix, which is \( \begin{pmatrix} 0 & tm \\ tm & 0 \end{pmatrix} \).

(c) Let us prove that all partition designs \( \mathcal{D}_m^2 \) and \( \mathcal{H}_m^2 \) (except \( \mathcal{H}_2^2 \)), where \( m \geq 2 \), are characterized by their adjacency matrix.

We begin by describing, in a unified way, the adjacency matrix \( M \) of \( \mathcal{D}_m^2 \) and of \( \mathcal{H}_m^2 \). Let \( r \) be the number of subsets of a partition design, so \( r \) is the number of rows (and columns) of \( M \). For \( \mathcal{D}_m^2 \) we have \( r = m + 1 \), and for \( \mathcal{H}_m^2 \) we have \( r = \lceil \frac{m}{2} \rceil + 1 \). Let us define a surjective function \( f: \{0, 1, \ldots, m\} \rightarrow \{0, 1, \ldots, r - 1\} \). For the case of \( \mathcal{D}_m^2 \) the function \( f \) is the identity. And for the case of \( \mathcal{H}_m^2 \) we define \( f(i) = \min\{i, m - i\} \) for all \( 0 \leq i \leq m \). Let \( M = (m_{ij})_{1 \leq i, j \leq r} \), then

\[
m_{f(i)+1, f(i-1)+1} = 2i \quad \text{for all } i = 1, \ldots, m
\]

and

\[
m_{f(i)+1, f(i)+1+1} = 2m - 2i \quad \text{for all } i = 0, \ldots, m - 1;
\]

the rest of the \( m_{ij} \) are 0. Note that we obtain two times the adjacency matrices of \( \mathcal{D}_m \) and \( \mathcal{H}_m \), which are described in (1) and (3) and (4).

Let \( \mathcal{V} = \{Y_1, \ldots, Y_r\} \) be \( \mathcal{D}_m^2 \) or \( \mathcal{H}_m^2 \) for \( m \geq 2 \). We observe again that \( r, f \) and \( M \) depend on the case we are considering. On this way we develop a proof that is valid for both cases.

From \( M \) we obtain that for all \( i \in \{0, \ldots, m\} \) we have that all \( x \in Y_{f(i)+1} \) has 2i adjacent vertices in \( Y_{f(i-1)+1} \) (if \( i \neq 0 \)), and \( 2m - 2i \) vertices in \( Y_{f(i)+1+1} \) (if \( i \neq m \)).

Without loss of generality we may suppose that \( (0, \ldots, 0) \in Y_1 \) (for a class of equivalent partition design that has adjacency matrix \( M \), there is a subset of partition designs such that \( (0, \ldots, 0) \in Y_1 \)). Then \( S_1 \subseteq Y_2 \), since the first row of \( M \) is of the form \( (0, 2m, 0, \ldots, 0) \). For every subset \( T \subseteq \{1, \ldots, 2m\} \) we define \( x_T \in \mathbb{F}^{2m} \) as in (35). Note that \( m_{2,1} = 2 \) and, therefore, every \( x_{\{j\}} \in S_1 \subseteq Y_2 \) has two adjacent vertices in \( Y_1 \): one of them will be \( x_{\emptyset} = (0, \ldots, 0) \) and the other vertex \( w \) has to be an element of \( S_2 \). That is, there exists a function \( \sigma: \{1, \ldots, 2m\} \rightarrow \{1, \ldots, 2m\} \) such that \( w = x_{\{j, \sigma(j)\}} \). Let us prove three claims about \( \sigma \):

(i) **The function \( \sigma \) is a permutation.** Suppose that there are \( j \neq k \) such that \( \sigma(j) = \sigma(k) \), then we have a contradiction because \( x_{\{j, k\}} \), which is an element of \( Y_2 \), would be adjacent to more than two elements of \( Y_1 : x_{\emptyset}, x_{\{j, \sigma(j)\}}, x_{\{k, \sigma(k)\}} \).

(ii) **The permutation \( \sigma \) has no fixed points.** We deduce that from the definition of \( \sigma \).

(iii) **The permutation \( \sigma \) is an involution.** Otherwise there would be a \( j \) such that \( \sigma(\sigma(j)) = j \), and then \( x_{\{j\}} \), which is an element of \( Y_2 \), would be adjacent to more than two elements of \( Y_1 : x_{\emptyset}, x_{\{j, \sigma(j)\}}, x_{\{\sigma(j), \sigma(\sigma(j))\}} \).
Then $\sigma$ decomposes into $m$ disjoint transpositions. We permute the coordinates in such a way that $\sigma = (1, 2)(3, 4) \cdots (2m - 1, 2m)$. We conclude that

$$\{x_{1, \sigma(1)}, x_{3, \sigma(3)}, \ldots, x_{2(m-1), \sigma(2m-1)}\}$$

$$= \{x_{1,2}, x_{3,4}, \ldots, x_{2(m-1),2m}\} \subset Y_1.$$  \hspace{1cm} (36)

Before we continue we introduce a new family of sets, a figure that will guide our proof and a useful result.

Let us define the following sets:

$$W_i := \{(x_1, \ldots, x_{2m}) \in \mathbb{F}^{2m} | w((x_1 \oplus x_2, x_3 \oplus x_4, \ldots, x_{2m-1} \oplus x_{2m})) = i\}$$

for all $0 \leq i \leq m$. In the following figure the circle of row $W_i$ and column $S_j$ represents the set $W_i \cap S_j$ where $W_i, S_j \subset \mathbb{F}^{2m}$. A segment will join the circle of row $W_i$ and column $S_j$ with the circle of row $W_{i-1}$ and column $S_k$ if and only if there are edges in $H(2m)$ that join elements of the set $W_i \cap S_j$ with elements of the set $W_{i-1} \cap S_k$. Moreover, the number $i$ placed in the top of the segment indicates that every element of $W_i \cap S_j$ has exactly $i$ adjacent elements in $W_{i-1} \cap S_k$. Let us prove this last claim. If $(x_1, \ldots, x_{2m}) \in W_i \cap S_j$ then the distance between $(x_1, x_3, \ldots, x_{2m-1})$ and $(x_2, x_4, \ldots, x_{2m})$ is $i$; that is, there exists $T \subset \{1, \ldots, m\}$ where $|T| = i$ such that $x_{2k-1} \neq x_{2k} \Leftrightarrow k \in T$. If we want to change a coordinate of $(x_1, \ldots, x_{2m})$ such that the new vertex belongs to $W_{i-1}$ then it can be the $(2k-1)$th coordinate or the $(2k)th$ coordinate for all $k \in T$; that is, we have $2i$ possibilities to obtain an element of $W_{i-1}$ by changing only one coordinate. These possibilities can be divided into two groups, the first group consists of changing a 1 into a 0 and we obtain $i$ elements of $W_{i-1} \cap S_{j-1}$, and the second group consists of changing a 0 into a 1 and we obtain $i$ elements of $W_{i-1} \cap S_{j+1}$.

Analogously, we can prove that if there is a segment that joins the circle of row $W_{i-1}$ and column $S_k$ with the circle of row $W_i$ and column $S_j$, then every element of $W_{i-1} \cap S_k$ is joined to a fixed number, different from zero, of elements of $W_i \cap S_j$. We are only interested in the fact that these numbers are fixed and non-zero, for this reason they are not included in the figure.
Let us prove that \( W_i \subset Y_{f(i)+1} \) for all \( 0 \leq i \leq m \). We begin by proving the following: for \( i, j \geq 0 \) such that \( i + j \leq 2m - 2 \),

\[
\begin{align*}
\text{if } W_i \cap S_j & \subset Y_{f(i)+1}, W_{i-1} \cap S_{j-1} \subset Y_{f(i-1)+1}, W_{i-1} \cap S_{j+1} \subset Y_{f(i-1)+1}, \\
\text{then } W_{i+1} \cap S_{j+1} & \subset Y_{f(i+1)+1}.
\end{align*}
\]

(37)

Let \( x \in W_{i+1} \cap S_{j+1} \), and let \( y \in W_i \cap S_j \subset Y_{f(i)+1} \) be a vertex adjacent to \( x \). The vertex \( y \in Y_{f(i)+1} \) has \( 2i \) adjacent vertices in \( Y_{f(i-1)+1} \) and \( 2m - 2i \) adjacent vertices in \( Y_{f(i+1)+1} \). Since \( y \) is an element of \( W_i \cap S_j \), it has \( i \) adjacent vertices in \( W_{i-1} \cap S_{j-1} \subset Y_{f(i-1)+1} \) and other \( i \) adjacent vertices in \( W_{i-1} \cap S_{j+1} \subset Y_{f(i-1)+1} \); therefore \( x \in Y_{f(i+1)+1} \). Thus, we obtain that \( W_{i+1} \cap S_{j+1} \subset Y_{f(i+1)+1} \).

Note that if \( i = 0 \), then the sets \( W_{i-1} \cap S_{j-1} \) and \( W_{i-1} \cap S_{j+1} \) do not exist. In this case we only need to verify that \( W_0 \cap S_j \subset Y_{f(0)+1} = Y_1 \) to deduce that \( W_1 \cap S_j \subset Y_{f(1)+1} = Y_2 \).

The proof that \( W_i \subset Y_{f(i)+1} \) for all \( 0 \leq i \leq m \), is divided into different steps:

- We have supposed that \( \{(0, \ldots, 0)\} = W_0 \cap S_0 \subset Y_1 \).
- Furthermore we have seen that \( W_1 \cap S_1 = S_1 \subset Y_2 = Y_{f(1)+1} \).
- From (36) we have \( W_0 \cap S_2 = \{x_{1, 2}, x_{3, 4}, \ldots, x_{2m-1, 2m}\} \subset Y_1 = Y_{f(0)+1} \).
- By (37) with \( i = 1 \) and \( j = 1 \), we have \( W_2 \cap S_2 \subset Y_{f(2)+1} \).
- Let us continue by induction. We have proved that \( W_i \cap S_j \subset Y_{f(i)+1} \) for \( 0 \leq i \leq 3 \) and \( 0 \leq j \leq 2 \). Suppose that we have proved \( W_i \cap S_j \subset Y_{f(i)+1} \) for \( 0 \leq i \leq 3 \) and \( 0 \leq j \leq 2k \). The induction will be done over the variable \( k \).

(1) By (37) with \( i = 0 \) and \( j = 2k \), we have \( W_1 \cap S_{2k+1} \subset Y_{f(1)+1} \).

(2) By (37) with \( i = 2 \) and \( j = 2k \), we conclude that \( W_3 \cap S_{2k+1} \subset Y_{f(3)+1} \).

(3) Let \( x \in W_0 \cap S_{2k+2} \), and let \( y \in W_1 \cap S_{2k+1} \subset Y_{f(1)+1} \) be a vertex adjacent to \( x \). Since \( y \) is an element of \( Y_{f(1)+1} \), it has exactly 2 adjacent elements in \( Y_{f(0)+1} \). One of them is the only adjacent vertex of \( y \) that belongs to \( W_0 \cap S_{2k} \), and the other, \( x' \), cannot belong to \( W_2 \cap S_{2k} \) because this set is contained in \( Y_{f(2)+1} \). Thus, \( x' \in S_{2k+2} \). By the diagram, \( x'' \in W_0 \cap S_{2k+2} \) or \( x'' \in W_2 \cap S_{2k+2} \). Suppose that the second case occurs, then it is possible to find a vertex \( x'' \) adjacent to \( x' \), such that \( x'' \in W_3 \cap S_{2k+1} \subset Y_{f(3)+1} \). Note that \( f(3) + 1 \neq f(1) + 1 \), except for \( H_4^2 \) (at the end of the proof we will say something more about this exception). We obtain a contradiction because \( x'' \in Y_{f(0)+1} \) would be adjacent to \( x'' \in Y_{f(3)+1} \neq Y_{f(1)+1} \). Then, necessarily the first case occurs: \( x'' \in W_0 \cap S_{2k+2} \). Since \( y \) has only one adjacent vertex in \( W_0 \cap S_{2k+2} \), then \( x = x'' \in Y_{f(0)+1} \). We deduce that \( W_0 \cap S_{2k+2} \subset Y_{f(0)+1} \).

(4) By (37) with \( i = 1 \) and \( j = 2k + 1 \), we have \( W_2 \cap S_{2k+2} \subset Y_{f(2)+1} \).

With this induction we have proved that \( W_i \cap S_j \subset Y_{f(i)+1} \) for all \( 0 \leq i \leq 3 \) and all \( 0 \leq j \leq 2m \). Observe that when \( k = m - 1 \), steps (2) and (4) are not applied because \( W_3 \cap S_{2m-1} \) and \( W_2 \cap S_{2m} \) are empty. Moreover, step (3) is simplified because \( W_2 \cap S_{2m} \) is empty.

- To prove that \( W_i \cap S_j \subset Y_{f(i)+1} \) for all \( 4 \leq j \leq 2m - 4 \), we apply (37) with \( i = 3 \) and \( 3 \leq j \leq 2m - 5 \). Analogously, we apply (37) successively until we obtain the desired result: \( W_i \subset Y_{f(i)+1} \) for all \( 0 \leq i \leq m \).
Finally, $Y_i = \bigcup_{j \in f^{-1}(i-1)} W_j$ for all $1 \leq i \leq r$. So, up to equivalence, there is only one partition design, $\mathcal{Y} = \{Y_1, \ldots, Y_r\}$, with adjacency matrix $M$.

For completeness we give the three non-equivalent partition designs that have adjacency matrix $\begin{pmatrix} 0 & 8 & 0 \\ 2 & 0 & 6 \\ 0 & 8 & 0 \end{pmatrix}$. They are $\mathcal{H}_2^3$, $\{Y'_1, Y'_2, Y'_3\}$ and $\{Y''_1, Y''_2, Y''_3\}$, where

$$Y'_i = \{(0,0,0,0,0,0,0,0,1,1,1,1), (0,0,0,0,0,0,1,1,0,0,1,1), (0,0,0,0,1,1,0,0,0,0,1,1), (0,0,0,1,1,0,0,0,0,1,1,0), (0,1,0,1,1,0,1,0,1,1,0,0), (0,1,0,0,1,1,0,1,0,1,1,0), (0,1,1,0,0,0,0,1,1,0,0,1), (0,1,1,1,0,0,0,0,1,1,0,0), (0,1,1,1,0,0,1,0,1,1,1,0), (1,0,0,1,1,0,0,1,1,0,1,1), (1,0,1,0,1,1,0,0,1,1,0,0), (1,0,1,1,0,0,1,0,1,1,0,0), (1,1,0,0,0,0,0,0,1,1,0,0), (1,1,0,0,0,0,1,0,1,1,0,0), (1,1,0,0,0,1,0,0,1,1,0,0), (1,1,1,1,0,0,0,0,1,1,0,0), (1,1,1,1,1,0,0,0,0,1,1,0)\},$$

and

$$Y''_i = \{(0,0,0,0,0,0,0,0,0,1,1,1,1), (0,0,0,0,0,0,0,1,1,0,0,1,1), (0,0,0,0,0,1,1,0,0,0,1,1), (0,0,0,0,1,1,0,0,0,1,1,0), (0,0,0,1,1,0,1,0,1,1,0,0), (0,0,0,1,1,1,0,0,1,1,0,0), (0,0,0,1,1,1,1,0,0,1,1,0), (0,0,0,1,1,1,1,1,0,0,1,1), (0,0,0,1,1,1,1,1,1,0,0,1), (0,0,1,1,1,0,0,0,1,1,0,0), (0,0,1,1,1,0,0,1,1,0,0,1,1), (0,0,1,1,1,0,1,1,0,1,1,0), (0,0,1,1,1,1,0,0,1,1,0,0,1), (0,0,1,1,1,1,0,1,1,0,1,1,0), (0,0,1,1,1,1,1,0,1,1,0,1,1), (0,0,1,1,1,1,1,1,0,1,1,0,0), (0,0,1,1,1,1,1,1,1,0,1,1,0), (0,0,1,1,1,1,1,1,1,1,0,1,1,0), (0,0,1,1,1,1,1,1,1,1,1,0,1,1,0), (0,0,1,1,1,1,1,1,1,1,1,1,0,1,1,0), (0,0,1,1,1,1,1,1,1,1,1,1,1,0,1,1,0), (0,0,1,1,1,1,1,1,1,1,1,1,1,1,0,1,1), (0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,0,1), (0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0)\}.$$

The sets $Y'_2, Y'_3$ are easily obtained from $Y'_1$, and $Y''_2, Y''_3$ are easily obtained from $Y''_1$. □

**Note.** In the paper [3] it is shown that, up to equivalence, there is a unique partition design, which is $\mathcal{H}_2^3$, with adjacency matrix $\begin{pmatrix} 0 & 10 & 0 \\ 2 & 0 & 8 \\ 0 & 4 & 6 \end{pmatrix}$. In [3] relations between $\mathcal{H}_2^3$ and the Best code are also studied.
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