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Instability of steady states for nonlinear wave and heat equations

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Abstract

We consider time-independent solutions of hyperbolic equations such as $\partial_{tt}u - \Delta u = f(x,u)$ where f is convex in u. We prove that linear instability with a positive eigenfunction implies nonlinear instability. In some cases the instability occurs as a blow up in finite time. We prove the same result for parabolic equations such as $\partial_t u - \Delta u = f(x,u)$. Then we treat several examples under very sharp conditions, including equations with potential terms and equations with supercritical nonlinearities.

Keywords: Nonlinear heat equation; Nonlinear wave equation; Instability; Steady states

1. Introduction

Given a linear second-order elliptic differential operator L whose coefficients are smooth and bounded, consider the parabolic equation

$$\partial_t u + Lu = f(x,u), \qquad x \in \mathbb{R}^n, \tag{1.1}$$

along with its hyperbolic analogue

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$$\partial_t^2 u + a\partial_t u + Lu = f(x,u), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where f is a nonlinear term and $a \in \mathbb{R}$ is arbitrary (possibly zero). A very important step in understanding the behavior of general solutions lies in understanding the qualitative properties of special types of solutions. In this paper, we focus on time-independent solutions, also known as steady states, and we address their stability properties in the context of both (1.1) and (1.2). Our main goal is to provide sufficient conditions under which linearized instability can be used to draw conclusions about nonlinear instability.

Before we turn to our main results, however, let us first introduce some assumptions on the time-independent solution φ and the nonlinear term f. We are going to assume that

(A1) the equation $L\varphi = f(x,\varphi)$ has a $\mathcal{C}^2$ solution φ;
(A2) the adjoint linearized operator $L^* - f_u(x,\varphi)$ has a negative eigenvalue $-\sigma^2$ and a corresponding eigenfunction $\chi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ that is nonnegative;
(A3) both $f(x,\varphi)$ and $f_u(x,\varphi)$ are bounded;
(A4) the nonlinear term $f(x,s)$ is convex in s and is $\mathcal{C}^1$.

Here, (A2) is mostly meant to ensure the presence of a negative eigenvalue; our assertions about the eigenfunction χ are already implied by this fact under pretty general conditions. Also, note that we do *not* require the steady state φ to be bounded.

A nontechnical description of our main result is that φ is a nonlinearly unstable solution of both (1.1) and (1.2) whenever (A1)–(A4) hold; namely, solutions which start out close to φ need not remain close to φ for all times. When it comes to the parabolic case (1.1), our result applies to a wide class of initial data, including all initial data for which

$$u(x,0) > \varphi(x). \tag{1.3}$$

To establish the instability of φ, we show that the norm

$$\left\| u(x,t) - \varphi(x) \right\|_{L^\infty(\mathbb{R}^n)} \tag{1.4}$$

must either grow exponentially at all times or else blow up in finite time. Our main result for the hyperbolic case (1.2) is almost identical. It too applies to a wide class of initial data, including all initial data for which

$$u(x,0) > \varphi(x), \quad \partial_t u(x,0) > 0, \tag{1.5}$$

and it shows that the energy norm[2]

$$\|u-\varphi\|_e \equiv \left\| u(x,t) - \varphi(x) \right\|_{H^1(\mathbb{R}^n)} + \left\| \partial_t u(x,t) \right\|_{L^2(\mathbb{R}^n)} \tag{1.6}$$

must either grow exponentially at all times or else blow up in finite time.

Note that the above results prove instability in a sense that is much stronger, and perhaps more natural, than the usual one. Namely, not only do they show that the solution exits any given

[2] In each case, our norm agrees with the one dictated by the local existence theory.

neighborhood of φ in finite time, but they also ensure that the solution does not reenter that neighborhood at any later time.

If one is willing to impose an additional positivity condition on the nonlinear term, then the above results can be further improved to show that instability occurs by blow up. This is the case, for instance, if one additionally assumes that

(A5) there exist $C_0 > 0$ and $p > 1$ such that $f(x,s) \geqslant C_0|s|^p$ for all $(x,s) \in \mathbb{R}^n \times \mathbb{R}$;
(A6) the product $\varphi\chi$ is integrable, where χ denotes the eigenfunction from (A2).

In our main results, we shall prove *instability* assuming (A1)–(A4), and also *instability by blow up* assuming (A1)–(A6). As the reader should note, (A3)–(A6) may be safely ignored in the special case $f(u) = |u|^p$ for some $p > 1$, provided that φ is bounded. For that special case, in particular, we establish instability by blow up assuming (A1), (A2) only.

To a large extent, our results are complementary to the abstract result of Grillakis, Shatah and Strauss [9]. These authors deal with arbitrary nonlinearities and the more general class of bound states, but their assumptions are more restrictive than ours. As we already mentioned, another advantage of our approach is that our notion of instability is much stronger than the one used in [9]. As we indicate later in the introduction, a final advantage is that our result applies to a very sharp class of perturbations, which is not the case with the result of [9].

Section 2 is devoted to the proof of our two main instability results, Theorems 2.1 and 2.2. Our proof is quite elementary and based on a variant of Kaplan's eigenfunction method [11] which allows us to reduce our analysis to the study of a certain functional. Although there are several other variants of this method, they are concerned with the associated Cauchy problem rather than the instability problem; see [7,8,13,16,17,29], for instance. In the last three sections of this paper, we apply our two main results to treat some examples that we briefly discuss below.

Our first and most general example appears in Theorem 3.1. Here, we shall only describe a very special case of the theorem that is itself of independent interest. Consider the nonlinear heat and wave equations with potential[3]

$$\partial_t^i u - \Delta u + V(x) \cdot u = |u|^p, \quad x \in \mathbb{R}^n,\ i = 1, 2. \tag{1.7}$$

We take $p > 1$ and assume that V is bounded, continuous and nonnegative, but we make no other assumptions. Then Theorem 3.1 implies the instability of all nonnegative, H^1 steady states that vanish at infinity. A sufficient condition on V which ensures the existence of such steady states (for some p) is provided by [24] and stated in Theorem 3.3. Together with Theorem 3.1, these results impose no restrictions on the linearized operator; we are not aware of any other general results with this feature.

In our second example, Theorem 4.2, we focus on the two-dimensional equations[4]

$$\partial_t^i u - \Delta u = e^u, \quad x \in \mathbb{R}^2,\ i = 1, 2. \tag{1.8}$$

The classification of all $\mathcal{C}^2$ steady states for which the nonlinear term is integrable is provided by [4] and stated in Lemma 4.1. They are *unbounded* and, as far as we know, their instability was not previously known. Theorem 4.2 shows they are all unstable.

[3] In the rest of the introduction, the possible damping term $a\partial_t u$ is suppressed merely for the sake of exposition.
[4] This is our only example for which the space dimension is restricted.

In our last example, Theorem 5.6, we focus on the equations

$$\partial_t^i u - \Delta u = |u|^p, \quad x \in \mathbb{R}^n,\ i = 1, 2, \tag{1.9}$$

where $n > 2$ and $p \geqslant \frac{n+2}{n-2}$. When it comes to the existence of positive, $\mathcal{C}^2$, radially symmetric steady states, the known results [3,4,6,18] are summarized in Theorem 5.2. Except for the special case $p = \frac{n+2}{n-2}$, these steady states are not in H^1, so our first example is no longer applicable. In Section 5, we analyze the spectrum of the linearized operator and we find the exact values of p for which a negative eigenvalue emerges. Using our main results, we then show that there is a critical value p_c, depending on the space dimension, such that the steady states are *nonlinearly unstable* if $p < p_c$ and *linearly stable* if $p \geqslant p_c$; see Theorem 5.6.

Our conclusions for Eq. (1.9) are new only in the hyperbolic case. As for the parabolic equation, the critical value p_c emerged in [10], where a quite different approach was used. The instability result of [10] is weaker than ours because it applies to a smaller class of initial data, but the stability result given there is stronger since it proves nonlinear stability instead. On the other hand, the approach in [10] relies on the maximum principle, so it cannot be applied to yield analogous results for the wave equation; see Remark 5.7 for more comments.

Finally, our result for the wave equation (1.9) is closely related to a recent paper of Krieger and Schlag [14]. These authors focus on the three-dimensional quintic case[5]

$$\partial_t^2 u - \Delta u = |u|^5, \quad x \in \mathbb{R}^3, \tag{1.10}$$

and a particular positive, $\mathcal{C}^2$ steady state φ. Theorem 5.6 implies that φ is unstable, while the result of [14] asserts the existence of a stable manifold associated with φ. Based on numerical evidence from [2,26], this stable manifold ought to separate the *small* perturbations for which solutions exist globally from those for which solutions blow up. In fact, Theorem 5.6 provides a partial proof of this conjecture, namely the blow up for all perturbations that lie strictly above the tangent plane to the stable manifold at the origin; see Remark 5.8 for more details.

2. The main instability results

In this section, we prove our two main instability results regarding the solutions of a general elliptic equation of the form

$$L\varphi = f(x, \varphi), \quad x \in \mathbb{R}^n.$$

Recall that L is a linear, second-order elliptic differential operator whose coefficients are smooth and bounded, while f is a nonlinear term. Our precise assumptions (A1)–(A6) were already mentioned in the introduction, so we shall not bother to repeat them here.

First, we deal with the hyperbolic case and thus focus on the equation

$$\partial_t^2 u + a\partial_t u + Lu = f(x, u), \quad u(x, 0) = \varphi(x) + \psi_0(x), \quad \partial_t u(x, 0) = \psi_1(x). \tag{2.1}$$

[5] Krieger and Schlag actually study this equation with u^5 instead of $|u|^5$, however the solutions they construct are nonnegative, so the distinction between the two nonlinearities is not crucial from their point of view.

Since the steady state φ is an exact solution when $\psi_0 \equiv \psi_1 \equiv 0$, we are mostly concerned with the case that the perturbation (ψ_0, ψ_1) is small in some sense. In our next result, we shall deal with finite-energy perturbations. Note, however, that φ itself need not be of finite energy.

Theorem 2.1 *(Hyperbolic equation). Let $a \in \mathbb{R}$. Assume* (A1)–(A4) *and let $(\psi_0, \psi_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ be such that*

$$\frac{a + \sqrt{a^2 + 4\sigma^2}}{2} \int_{\mathbb{R}^n} \chi(x)\psi_0(x)\,dx + \int_{\mathbb{R}^n} \chi(x)\psi_1(x)\,dx > 0. \tag{2.2}$$

Let $0 < T \leqslant \infty$ and let u be a solution of (2.1) *on $[0, T)$ such that $u - \varphi$ is continuous in t with values in the energy space and $f(x, u)$ is locally integrable.*

(a) *If $T = \infty$, then the energy norm*

$$\left\|u(t) - \varphi\right\|_e \equiv \left\|u(\cdot, t) - \varphi(\cdot)\right\|_{H^1(\mathbb{R}^n)} + \left\|\partial_t u(\cdot, t)\right\|_{L^2(\mathbb{R}^n)} \tag{2.3}$$

must grow exponentially.

(b) *Assume also* (A5), (A6)*. Then $T < \infty$.*

Proof. Consider the function

$$G(t) = \int_{\mathbb{R}^n} \chi(x) \cdot w(x, t)\,dx, \qquad w(x, t) = u(x, t) - \varphi(x). \tag{2.4}$$

By (A2), one certainly has

$$\left|G(t)\right| \leqslant \|\chi\|_{L^2(\mathbb{R}^n)} \cdot \left\|w(t)\right\|_{L^2(\mathbb{R}^n)} \leqslant C\left\|w(t)\right\|_e.$$

Thus $G(t)$ is well defined and bounded as long as the energy remains bounded; the energy grows exponentially, provided that $G(t)$ does; and the energy becomes infinite whenever $G(t)$ does. In view of these facts, our assertions about the energy (2.3) will follow once we have established analogous assertions for $G(t)$.

Let us first focus on part (a) and assume that $w = u - \varphi$ is continuous on $[0, \infty)$ with values in the energy space. Then w is a solution of

$$\partial_t^2 w + a\partial_t w + Lw = f(x, w + \varphi) - f(x, \varphi)$$

in the sense of distributions. In view of our convexity assumption (A4), we have

$$\partial_t^2 w + a\partial_t w + \left[L - f_u(x, \varphi)\right]w \geqslant 0.$$

Since $\chi(x) \geqslant 0$, we may multiply the inequality by $\chi(x)\theta(t)$, where $\theta(t)$ is an arbitrary nonnegative test function, and integrate to obtain

$$-\int_0^t \int_{\mathbb{R}^n} \chi(x) \cdot \partial_t w \cdot \theta'(\tau)\, dx\, d\tau + \int_0^t \int_{\mathbb{R}^n} \chi(x) \cdot a\partial_t w \cdot \theta(\tau)\, dx\, d\tau$$
$$+ \int_0^t \int_{\mathbb{R}^n} \big[L^* - f_u(x,\varphi)\big]\chi(x) \cdot w \cdot \theta(\tau)\, dx\, d\tau \geqslant 0. \tag{2.5}$$

To simplify the first two integrals, we note that

$$G'(t) = \int_{\mathbb{R}^n} \chi(x) \cdot \partial_t w(x,t)\, dx$$

is a continuous function of t by our definition (2.4) because $\partial_t w$ is continuous with values in $L^2(\mathbb{R}^n)$. To simplify the third integral in (2.5), we note that

$$\big[L^* - f_u(x,\varphi)\big]\chi = -\sigma^2 \chi$$

by our assumption (A2). Thus, Eq. (2.5) reduces to

$$-\int_0^t G'(\tau) \cdot \theta'(\tau)\ d\tau + a\int_0^t G'(\tau) \cdot \theta(\tau)\, d\tau - \sigma^2 \int_0^t G(\tau) \cdot \theta(\tau)\, d\tau \geqslant 0$$

for all nonnegative test functions $\theta(t)$, or equivalently,

$$G''(t) + aG'(t) - \sigma^2 G(t) \geqslant 0 \tag{2.6}$$

in the sense of distributions. Next, we note that our assumption (2.2) reads

$$\frac{a + \sqrt{a^2 + 4\sigma^2}}{2} \cdot G(0) + G'(0) > 0.$$

According to Lemma 2.3, both $G(t)$ and $G'(t)$ must then grow exponentially fast, so the proof of part (a) is complete.

Next, we turn to part (b). Suppose that $T = \infty$. As we have just shown, both $G(t)$ and $G'(t)$ must grow exponentially fast. Using our assumptions (A3) and (A5), we have

$$\begin{aligned} \partial_t^2 w + a\partial_t w + \big[L - f_u(x,\varphi)\big]w &= f(x, w+\varphi) - f(x,\varphi) - f_u(x,\varphi)w \\ &\geqslant C_0|w+\varphi|^p - C_1 - C_1|w|. \end{aligned}$$

Multiplying by the nonnegative eigenfunction χ and integrating over space, we then easily find that

$$G''(t) + aG'(t) - \sigma^2 G(t) \geqslant C_0 \int_{\mathbb{R}^n} \chi|w+\varphi|^p\, dx - C_1 \int_{\mathbb{R}^n} \chi\, dx - C_1 \int \chi|w|\, dx, \tag{2.7}$$

the second derivative being understood in the sense of distributions. This can be justified, as above, by introducing a nonnegative test function $\theta(t)$ to avoid the second-order derivative. Now, the last integral is at most

$$\int \chi|w|\,dx \leqslant \int \chi|w+\varphi|\,dx + \int \chi|\varphi|\,dx$$

by the triangle inequality. Since $\chi \in L^1$ by (A2) and $\chi\varphi \in L^1$ by (A6), we deduce that

$$G''(t) + aG'(t) \geqslant \sigma^2 G(t) + C_0 \int_{\mathbb{R}^n} \chi|w+\varphi|^p\,dx - C_1 \int \chi|w+\varphi|\,dx - C_2.$$

Since $G(t)$ grows exponentially fast, this implies

$$\begin{aligned} G''(t) + aG'(t) &\geqslant C_0 \int_{\mathbb{R}^n} \chi|w+\varphi|^p\,dx - C_1 \int_{\mathbb{R}^n} \chi|w+\varphi|\,dx \\ &\equiv C_0 A(t) - C_1 B(t) \end{aligned} \tag{2.8}$$

in the sense of distributions for all large enough t.

Now $B(t)$ itself grows exponentially fast, as the triangle inequality gives

$$G(t) \leqslant \int_{\mathbb{R}^n} \chi|w|\,dx \leqslant \int_{\mathbb{R}^n} \chi|w+\varphi|\,dx + \int_{\mathbb{R}^n} \chi|\varphi|\,dx = B(t) + C_3. \tag{2.9}$$

Similarly, $A(t)$ grows exponentially fast since Hölder's inequality gives

$$B(t) \leqslant \left(\int_{\mathbb{R}^n} \chi\,dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} \chi|w+\varphi|^p\,dx\right)^{\frac{1}{p}} = C_4 A(t)^{1/p}. \tag{2.10}$$

This actually forces $A(t)$ to grow faster than $B(t)$, namely

$$A(t) \geqslant C_4^{-p} B(t)^p, \tag{2.11}$$

hence $A(t)$ will eventually dominate $B(t)$. Combining (2.8), (2.9) and (2.11), we now get

$$G''(t) + aG'(t) \geqslant C_5 A(t) \geqslant C_6 B(t)^p \geqslant C_7 G(t)^p \tag{2.12}$$

in the sense of distributions for all large enough t. Moreover, both $G(t)$ and $G'(t)$ are eventually positive by above. Invoking Lemma 2.4, we reach the contradiction $T < \infty$. $\square$

Next, we modify our previous approach to obtain a simple parabolic analogue of Theorem 2.1.

Theorem 2.2 *(Parabolic equation). Assume* (A1)–(A4) *and let* $\psi_0 \in L^\infty(\mathbb{R}^n)$ *be continuous with*

$$\int_{\mathbb{R}^n} \chi(x)\psi_0(x)\,dx > 0. \tag{2.13}$$

Let $0 < T \leqslant \infty$ *and let* u *be a solution of*

$$\partial_t u + Lu = f(x,u), \quad u(x,0) = \varphi(x) + \psi_0(x) \tag{2.14}$$

on $[0,T)$ *such that* $u - \varphi$ *is continuous and bounded for each* t.

(a) *If* $T = \infty$, *then the norm* $\|u-\varphi\|_{L^\infty(\mathbb{R}^n)}$ *must grow exponentially.*
(b) *Assume also* (A5), (A6). *Then* $T < \infty$.

Proof. Once again, it suffices to establish analogous assertions for the function

$$G(t) = \int_{\mathbb{R}^n} \chi(x)\cdot w(x,t)\,dx, \quad w(x,t) = u(x,t) - \varphi(x).$$

Since

$$\bigl|G(t)\bigr| \leqslant \|\chi\|_{L^1(\mathbb{R}^n)} \cdot \|w\|_{L^\infty(\mathbb{R}^n)},$$

the L^∞-norm has to either grow exponentially or blow up whenever $G(t)$ does.

We now apply the same argument with minor changes. Note that $w = u - \varphi$ satisfies

$$\partial_t w + Lw = f(x, w+\varphi) - f(x,\varphi), \quad w(x,0) = \psi_0(x)$$

in the sense of distributions. Arguing as before, we then get the estimate

$$G'(t) - \sigma^2 G(t) \geqslant 0 \tag{2.15}$$

instead of (2.6). Since $G(0) > 0$ by assumption, the exponential growth of $G(t)$ follows directly.

Under the additional assumptions (A5), (A6), our previous approach yields the estimate

$$G'(t) - \sigma^2 G(t) \geqslant C_0 \int_{\mathbb{R}^n} \chi|w+\varphi|^p\,dx - C_1 \int_{\mathbb{R}^n} \chi\,dx - C_1 \int \chi|w|\,dx$$

instead of (2.7). Then the remaining part of our proof applies verbatim to give

$$G'(t) \geqslant C_7 G(t)^p$$

instead of (2.12), in the sense of distributions for all large enough t. Since $G(t)$ is positive, it is easy to deduce that $T < \infty$, as needed. $\square$

Lemma 2.3. *Let $a \in \mathbb{R}$ and $b > 0$. Suppose $y(t)$ is a $\mathcal{C}^1$ function such that*

$$y'' + ay' - by \geqslant 0$$

on some interval $[0, T)$ in the sense of distributions. If

$$\frac{a + \sqrt{a^2 + 4b}}{2} \cdot y(0) + y'(0) > 0, \tag{2.16}$$

then both $y(t)$ and $y'(t)$ must grow exponentially on $[0, T)$.

Proof. Let $\lambda_1 < 0 < \lambda_2$ be the roots of the characteristic equation $\lambda^2 + a\lambda - b = 0$ and set $z = y' - \lambda_1 y$. Then

$$z' - \lambda_2 z = y'' + ay' - by \geqslant 0.$$

Using the test function $\exp(-\lambda_2 t)\theta(t)$, where $\theta(t)$ is another test function, it follows that

$$-\int_0^t z(\tau)\exp(-\lambda_2\tau)\theta'(\tau)\,d\tau \geqslant 0.$$

Choosing $\theta(t)$ to be an approximation of the characteristic function of the interval $(0, t)$, we easily deduce that $z(t) \geqslant e^{\lambda_2 t} z(0)$. Thus $y' - \lambda_1 y \geqslant e^{\lambda_2 t} z(0)$, which implies that

$$y(t) \geqslant e^{\lambda_1 t} y(0) + \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \cdot z(0).$$

Since $\lambda_1 < 0 < \lambda_2$ by above, the exponential growth of y then follows, provided that

$$z(0) = y'(0) - \lambda_1 y(0) = y'(0) + \frac{a + \sqrt{a^2 + 4b}}{2} \cdot y(0)$$

is positive. In view of our assumption (2.16), the exponential growth of y thus follows.

Next, we use a similar argument to find that $w = y' - \lambda_2 y$ satisfies

$$w' - \lambda_1 w = y'' + ay' - by \geqslant 0,$$

hence also $w(t) \geqslant e^{\lambda_1 t} w(0)$. Since $y(t)$ grows exponentially by above, the equation

$$y'(t) \geqslant \lambda_2 y(t) + e^{\lambda_1 t} w(0)$$

then forces $y'(t)$ to grow exponentially fast as well because $\lambda_1 < 0 < \lambda_2$. $\square$

Lemma 2.4. *Let $a \in \mathbb{R}$, $b > 0$ and $p > 1$. Suppose $y(t)$ is a nonnegative $\mathcal{C}^1$ function such that*

$$y(T_1) > 0, \quad y'(T_1) > 0, \quad y''(t) + ay'(t) \geqslant by(t)^p$$

on some interval $[T_1, T_2)$ in the sense of distributions. Then $T_2 < \infty$.

Proof. Suppose first that $y \in \mathcal{C}^2$. Then the case $a = 1$ is treated for instance in [27, Proposition 3.1]; the case $a > 0$ is quite similar; and the case $a \leqslant 0$ is much easier. If y is merely $\mathcal{C}^1$, one can simply repeat the same argument using test functions as in the preceding lemma; we omit the details. □

3. Convex nonlinearity with potential term

In this section, we apply our main results to study nonnegative H^1 solutions of the equation

$$-\Delta\varphi + V(x) \cdot \varphi = f(\varphi), \quad x \in \mathbb{R}^n. \tag{3.1}$$

Although our approach applies verbatim to the more general case $f(x, \varphi)$, the assumptions of our next result would have to be suitably modified in that case; we shall only deal with (3.1) for the sake of simplicity.

Theorem 3.1. *Assume that the following conditions hold*:

(B1) *V is continuous and bounded on $\mathbb{R}^n$*;
(B2) *the essential spectrum of $-\Delta + V$ is contained in* $[0, \infty)$;
(B3) *Eq.* (3.1) *has a nonnegative $\mathcal{C}^2$ solution $\varphi \in H^1$ that vanishes at infinity*;
(B4) *$f \in \mathcal{C}^1(\mathbb{R})$ is convex with $f(0) = f'(0) = 0$ and $f(\varphi)$ is not identically zero.*

Then φ is an unstable solution of any of the equations

$$\partial_t u + (-\Delta + V)u = f(u), \quad \partial_t^2 u + a\partial_t u + (-\Delta + V)u = f(u),$$

where $a \in \mathbb{R}$. More precisely, the conclusions of Theorems 2.1(a) *and* 2.2(a) *remain valid.*

Remark 3.2. By Weyl's theorem, our spectral assumption (B2) essentially requires that $V(x)$ not be negative as $|x| \to \infty$. We do not know of any existence results regarding (3.1) for which this assumption is violated; there are several existence results [1,24,25] in case it holds.

Proof. We verify the assumptions (A1)–(A4) of Theorems 2.1 and 2.2. Note that the existence assumption (A1) holds by (B3), while the convexity assumption (A4) holds by (B4). Since φ is bounded by (B3), both $f(\varphi)$ and $f'(\varphi)$ are bounded as well, so the assumption (A3) also holds. To check the remaining assumption (A2), we consider the self-adjoint linearized operator

$$\mathscr{L} = -\Delta + V - f'(\varphi).$$

Noting that φ is a solution of (3.1), we use (B4) to find that

$$\mathscr{L}\varphi = f(\varphi) - \varphi \cdot f'(\varphi) = f(\varphi) - f(0) - \varphi \cdot f'(\varphi) \leqslant 0.$$

Let $\langle , \rangle$ denote the standard inner product on $L^2(\mathbb{R}^n)$. Then the expression

$$\langle \mathscr{L}\varphi, \varphi \rangle = \int_{\mathbb{R}^n} |\nabla\varphi|^2 \, dx + \int_{\mathbb{R}^n} \big[V - f'(\varphi) \big] \cdot \varphi^2 \, dx$$

is nonpositive because $\mathscr{L}\varphi \leqslant 0 \leqslant \varphi$ and is finite because $\varphi \in H^1$ and $V - f'(\varphi) \in L^\infty$. Thus, the first eigenvalue of $\mathscr{L}$ must also be nonpositive. It can only be zero if

$$\mathscr{L}\varphi = f(\varphi) - \varphi \cdot f'(\varphi)$$

is identically zero, in which case $f(y) = yf'(y)$ for all y in the image of φ. Solving this ordinary differential equation, we get $f(y) = Cy$ for some constant C. Then the fact that $f'(0) = 0$ implies $f(y) = 0$ for all y in the image of φ. Since this violates our assumption (B4), however, the first eigenvalue must actually be negative.

Now that the first eigenvalue is known to be negative, the remaining assertions of (A2) follow by standard facts. Using a variational argument, it is well known that the first eigenfunction can be chosen to be positive (see [19, Section 11], for instance). Finally, the first eigenfunction is in $L^1 \cap L^2$ because it decays exponentially by Agmon's estimate; see [23, Theorem C.3.5] or else a more general result of Nakamura [20]. In fact, Agmon's estimate applies to any eigenfunction for which the associated eigenvalue lies below the bottom of the essential spectrum. Since

$$\lim_{|x| \to \infty} f'\big(\varphi(x)\big) = f'(0) = 0$$

by (B3) and (B4), an application of Weyl's theorem (see [15, Theorem 1.1]) gives

$$\sigma_{\text{ess}}(\mathscr{L}) = \sigma_{\text{ess}}(-\Delta + V) \subset [0, \infty)$$

because of (B2). Thus, the first eigenfunction must decay exponentially, as needed. □

In our next result, we give a typical application of Theorem 3.1. Our precise assumptions on the potential $V(x)$ are taken from [24]; they are merely meant to ensure that Eq. (3.3) has a nonnegative $\mathcal{C}^2$ solution $\varphi \in H^1$ that vanishes at infinity. Needless to say, Theorem 3.1 implies the instability of such solutions for a much wider class of potentials. The only advantage of our next result is that all the assumptions are verified explicitly.

Theorem 3.3. *Let $n > 2$ and $1 < p < \frac{n+2}{n-2}$. Assume that $V(x)$ is radially symmetric, continuous and locally Hölder continuous with*

$$C_1\big(1 + |x|\big)^{-l} \leqslant V(x) \leqslant C_2 \tag{3.2}$$

for some constants $C_1, C_2 > 0$ and some $0 \leqslant l < \frac{2(n-1)(p-1)}{p+3}$. Then the equation

$$-\Delta\varphi + V(x) \cdot \varphi = \varphi^p, \quad x \in \mathbb{R}^n, \tag{3.3}$$

has a positive, $\mathcal{C}^2$, radially symmetric and exponentially decaying solution φ. Moreover, φ is an unstable solution of any of the equations

$$\partial_t u + (-\Delta + V)u = |u|^p, \quad \partial_t^2 u + a\partial_t u + (-\Delta + V)u = |u|^p,$$

where $a \in \mathbb{R}$. More precisely, the conclusions of Theorems 2.1 *and* 2.2 (*both* (a) *and* (b)) *remain valid. That is, the instability occurs by blow up in finite time.*

Proof. Under the given hypotheses, our assertions about the elliptic equation (3.3) follow from [24, Theorem 1.2]. These ensure that the assumption (B3) from the previous theorem holds with $f(u) = |u|^p$. It is clear that this nonlinear term satisfies (B4). Moreover, $V(x)$ is bounded and nonnegative by (3.2), so the remaining assumptions (B1), (B2) hold as well. According to the previous theorem then, the conclusions of Theorems 2.1 and 2.2 remain valid, indeed.

To show that instability occurs by blow up, it remains to check that the additional assumptions (A5), (A6) of these theorems remain valid as well. Since $f(u) = |u|^p$ satisfies (A5) and φ is bounded, these two assumptions hold trivially and the proof is complete. □

4. Exponential nonlinearity in 2D

In this section, we apply our main results to study solutions of the equation

$$-\Delta\varphi = e^{\varphi}, \quad e^{\varphi} \in L^1(\mathbb{R}^2). \tag{4.1}$$

The classification of all $\mathcal{C}^2$ solutions is due to Chen and Li [4]. It is perhaps worth noting that these solutions are *unbounded* and not necessarily of one sign.

Lemma 4.1. *Every $\mathcal{C}^2$ solution of* (4.1) *is of the form*

$$\varphi(x) = \log\left[32\lambda^2 \cdot \left(4 + \lambda^2|x-y|^2\right)^{-2}\right]$$

for some $\lambda > 0$ and some $y \in \mathbb{R}^2$.

Theorem 4.2. *Every $\mathcal{C}^2$ solution of* (4.1) *is an unstable solution of any of the equations*

$$\partial_t u - \Delta u = e^u, \quad \partial_t^2 u + a\partial_t u - \Delta u = e^u,$$

where $a \in \mathbb{R}$. More precisely, the conclusions of Theorems 2.1(a) *and* 2.2(a) *remain valid.*

Proof. We set $f(u) = e^u$ and verify the assumptions (A1)–(A4) of Theorems 2.1 and 2.2. The existence assumption (A1) and the convexity assumption (A4) obviously hold in this case. Assumption (A3) holds as well because

$$f(\varphi) = f'(\varphi) = 32\lambda^2 \cdot \left(4 + \lambda^2|x-y|^2\right)^{-2}$$

is bounded. To show that (A2) holds as well, we focus on the linearized operator

$$\mathscr{L} = -\Delta - \exp\varphi(x) = -\Delta - 32\lambda^2 \cdot \left(4 + \lambda^2|x-y|^2\right)^{-2}.$$

Its essential spectrum is

$$\sigma_{\text{ess}}(\mathscr{L}) = \sigma_{\text{ess}}(-\Delta) = [0, \infty)$$

by Weyl's theorem since $f'(\varphi)$ is bounded and vanishes at infinity. Exactly as in the proof of Theorem 3.1, the first eigenfunction will be positive and exponentially decaying, so long as the first eigenvalue is negative. In particular, it suffices to show that the associated energy

$$E(\zeta) = \int_{\mathbb{R}^2} \left|\nabla\zeta(x)\right|^2 dx - \int_{\mathbb{R}^2} \exp\varphi(x) \cdot \zeta(x)^2\, dx \tag{4.2}$$

is negative for some test function $\zeta \in H^1(\mathbb{R}^2)$. Choosing

$$\zeta(x) = \left(4 + \lambda^2|x - y|^2\right)^{-2},$$

the energy (4.2) is given by

$$\begin{aligned} E(\zeta) &= 16\lambda^4 \int_{\mathbb{R}^2} \frac{|x-y|^2\, dx}{(4+\lambda^2|x-y|^2)^6} - 32\lambda^2 \int_{\mathbb{R}^2} \frac{dx}{(4+\lambda^2|x-y|^2)^6} \\ &= 32\lambda^4\pi \int_0^\infty \frac{r^3\, dr}{(4+\lambda^2 r^2)^6} - 64\lambda^2\pi \int_0^\infty \frac{r\, dr}{(4+\lambda^2 r^2)^6} \end{aligned}$$

and then a short computation gives

$$E(\zeta) = 16\pi \int_4^\infty (s-4)s^{-6}\, ds - 32\pi \int_4^\infty s^{-6}\, ds = -\frac{\pi}{320}.$$

This implies the existence of a negative eigenvalue and completes the proof. □

5. Power nonlinearity with zero mass

In this last section, we focus on positive solutions of the equation

$$-\Delta\varphi(x) = \varphi(x)^p, \quad x \in \mathbb{R}^n. \tag{5.1}$$

Before we state our results, it is convenient to introduce the quadratic polynomial

$$Q(\alpha) = \alpha(n - 2 - \alpha). \tag{5.2}$$

This polynomial arises naturally through the computation

$$-\Delta|x|^{-\alpha} = \alpha(n-2-\alpha) \cdot |x|^{-\alpha-2}$$

and it is closely related to Hardy's inequality

$$\int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx \geqslant \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} |x|^{-2} u(x)^2 \, dx, \tag{5.3}$$

which is valid for all $u \in H^1(\mathbb{R}^n)$ and each $n > 2$. Namely, the coefficient on the right-hand side of (5.3) is the maximum value of $Q(\alpha)$ and it is known to be sharp in the following sense.

Lemma 5.1. *Let $n > 2$ and let V be a bounded function on $\mathbb{R}^n$ which vanishes at infinity. If there exists some $\varepsilon > 0$ such that*

$$V(x) \leqslant -(1+\varepsilon) \cdot \left(\frac{n-2}{2}\right)^2 \cdot |x|^{-2}$$

for all large enough $|x|$, then the operator $-\Delta + V$ has infinite negative spectrum.

For a proof of Hardy's inequality (5.3), see [22, p. 169]. For a proof of the lemma, see [5, the appendix], for instance.

We are now ready to state the known existence results for positive steady states. Parts (a) and (b) can be found in either [3] or [4]. For part (c), see [18, Theorem 1] and [6, Theorem 5.26].

Theorem 5.2. *Let $n \geqslant 1$ and $p > 1$. Denote by Q the quadratic in* (5.2)*.*

(a) *If either $n = 1, 2$ or $p < \frac{n+2}{n-2}$, then Eq.* (5.1) *has no positive, $\mathcal{C}^2$ solutions.*
(b) *If $n > 2$ and $p = \frac{n+2}{n-2}$, then any positive $\mathcal{C}^2$ solution of* (5.1) *is of the form*

$$\varphi_\lambda(x) = \left(\frac{\lambda\sqrt{n(n-2)}}{\lambda^2 + |x-y|^2}\right)^{\frac{2}{p-1}} \tag{5.4}$$

for some $\lambda > 0$ and some $y \in \mathbb{R}^n$.

(c) *If $n > 2$ and $p > \frac{n+2}{n-2}$, then the positive, $\mathcal{C}^2$, radially symmetric solutions of* (5.1) *form a one-parameter family $\{\varphi_\alpha\}_{\alpha>0}$, where each φ_α satisfies*

$$\varphi_\alpha(0) = \alpha, \qquad \lim_{|x|\to\infty} |x|^2 \varphi_\alpha(x)^{p-1} = Q\left(\frac{2}{p-1}\right) > 0. \tag{5.5}$$

In what follows, we use the previous two results to analyze the spectrum of the linearized operator associated with (5.1). In particular, we find the exact values of p for which a negative eigenvalue emerges. Although a preliminary characterization of these values is provided by the next two lemmas, a more concrete characterization will be given in Theorem 5.6.

Lemma 5.3. *Let $n > 2$ and $p \geqslant \frac{n+2}{n-2}$. Let Q be the quadratic in* (5.2) *and let φ denote any one of the steady states provided by Theorem* 5.2*. Then $-\Delta - p\varphi^{p-1}$ has a negative eigenvalue if*

$$\left(\frac{n-2}{2}\right)^2 < p \cdot Q\left(\frac{2}{p-1}\right). \tag{5.6}$$

In particular, it has a negative eigenvalue if $p = \frac{n+2}{n-2}$.

Proof. Suppose first that $p > \frac{n+2}{n-2}$. Then (5.5) and (5.6) allow us to find some $\varepsilon > 0$ such that

$$\lim_{|x|\to\infty} |x|^2\varphi(x)^{p-1} = Q\left(\frac{2}{p-1}\right) > p^{-1}(1+2\varepsilon)\cdot\left(\frac{n-2}{2}\right)^2.$$

Thus

$$V(x) = -p\varphi(x)^{p-1} < -(1+\varepsilon)\cdot\left(\frac{n-2}{2}\right)^2\cdot|x|^{-2}$$

for all large enough $|x|$, so the existence of a negative eigenvalue follows by Lemma 5.1.

Suppose now that $p = \frac{n+2}{n-2}$. Then inequality (5.6) automatically holds because

$$\left(\frac{n-2}{2}\right)^2 = Q\left(\frac{n-2}{2}\right) = Q\left(\frac{2}{p-1}\right) < p\cdot Q\left(\frac{2}{p-1}\right)$$

for this particular case. According to part (b) of Theorem 5.2, we also have

$$-\Delta - p\varphi(x)^{p-1} = -\Delta - \lambda^2 n(n+2)\cdot\left(\lambda^2 + |x-y|^2\right)^{-2}$$

for some $\lambda > 0$ and some $y \in \mathbb{R}^n$. Thus it suffices to check that the associated energy

$$E(\zeta) = \int_{\mathbb{R}^n} \left|\nabla\zeta(x)\right|^2 dx - \int_{\mathbb{R}^n} p\varphi(x)^{p-1}\zeta(x)^2\,dx \tag{5.7}$$

is negative for some test function $\zeta \in H^1(\mathbb{R}^n)$. Let us now consider the test function

$$\zeta(x) = \left(\lambda^2 + |x-y|^2\right)^{-n/2-1},$$

whose energy (5.7) is given by

$$E(\zeta) = (n+2)^2\int_{\mathbb{R}^n} \frac{|x-y|^2\,dx}{(\lambda^2+|x-y|^2)^{n+4}} - \lambda^2 n(n+2)\int_{\mathbb{R}^n} \frac{dx}{(\lambda^2+|x-y|^2)^{n+4}}.$$

Except for a positive factor, this expression is equal to

$$\widetilde{E}(\zeta) = (n+2)\int_0^\infty \frac{r^{n+1}\,dr}{(\lambda^2+r^2)^{n+4}} - \lambda^2 n\int_0^\infty \frac{r^{n-1}\,dr}{(\lambda^2+r^2)^{n+4}}. \tag{5.8}$$

Moreover, an integration by parts gives

$$\int_0^\infty \frac{r^{n+1}\,dr}{(\lambda^2+r^2)^{n+4}} = \frac{n}{2(n+3)} \int_0^\infty \frac{r^{n-1}\,dr}{(\lambda^2+r^2)^{n+3}}$$
$$= \frac{n}{2(n+3)} \left[\lambda^2 \int_0^\infty \frac{r^{n-1}\,dr}{(\lambda^2+r^2)^{n+4}} + \int_0^\infty \frac{r^{n+1}\,dr}{(\lambda^2+r^2)^{n+4}} \right];$$

hence also

$$\int_0^\infty \frac{r^{n+1}\,dr}{(\lambda^2+r^2)^{n+4}} = \frac{\lambda^2 n}{n+6} \int_0^\infty \frac{r^{n-1}\,dr}{(\lambda^2+r^2)^{n+4}}.$$

Inserting this equality in (5.8), we arrive at

$$\widetilde{E}(\zeta) = \lambda^2 n \left(\frac{n+2}{n+6} - 1 \right) \int_0^\infty \frac{r^{n-1}\,dr}{(\lambda^2+r^2)^{n+4}} = -\frac{4\lambda^2 n}{n+6} \int_0^\infty \frac{r^{n-1}\,dr}{(\lambda^2+r^2)^{n+4}}.$$

Since this expression is negative, the energy (5.7) must also be negative, as needed. □

Our next lemma is essentially due to Wang [28]; see also [10]. We have chosen to give a different proof here, as our approach avoids Sturm-type arguments and thus applies to higher-order analogues of (5.1) as well; see [12].

Lemma 5.4. *Let $n > 2$ and $p > \frac{n+2}{n-2}$. Let Q denote the quadratic in* (5.2) *and assume that*

$$\left(\frac{n-2}{2} \right)^2 \geqslant p \cdot Q\left(\frac{2}{p-1} \right). \tag{5.9}$$

Then the steady states provided by Theorem 5.2 satisfy

$$|x|^2 \varphi(x)^{p-1} \leqslant Q\left(\frac{2}{p-1} \right) \tag{5.10}$$

for each $x \in \mathbb{R}^n$, and the operator $-\Delta - p\varphi^{p-1}$ has no negative spectrum.

Proof. First suppose the estimate (5.10) does hold. Using our assumption (5.9), we then get

$$-p\varphi(x)^{p-1} \geqslant -p \cdot Q\left(\frac{2}{p-1} \right) \cdot |x|^{-2} \geqslant -\left(\frac{n-2}{2} \right)^2 \cdot |x|^{-2}$$

for each $x \in \mathbb{R}^n$. According to Hardy's inequality (5.3), this already implies that

$$\int_{\mathbb{R}^n} \left|\nabla u(x)\right|^2 dx - \int_{\mathbb{R}^n} p\varphi(x)^{p-1} \cdot u(x)^2\, dx \geqslant 0$$

for all $u \in H^1(\mathbb{R}^n)$, so that the operator $-\Delta - p\varphi^{p-1}$ has no negative spectrum, indeed.

Let us now prove (5.10). Set $k = \frac{2}{p-1}$ and consider the function

$$W(s) = e^{ks}\varphi(e^s) = r^k\varphi(r), \quad s = \log r = \log|x|.$$

Then $W(s)$ is positive and bounded because of (5.5). It also satisfies the equation

$$Q(k - \partial_s)W(s) = W(s)^p \tag{5.11}$$

by Lemma 5.5. We note that s ranges over $(-\infty, \infty)$ as r ranges from 0 to ∞, while

$$\lim_{s\to-\infty} W(s) = \lim_{r\to 0^+} r^k\varphi(r) = 0.$$

The derivatives of $W(s)$ must also vanish at $s = -\infty$ because

$$\lim_{s\to-\infty} W'(s) = \lim_{r\to 0^+} r \cdot \partial_r\big[r^k\varphi(r)\big] = 0,$$

and so on. Now the fact that $f(x) = x^p$ is convex on $(0, \infty)$ ensures that

$$f\big(W(s)\big) - f\big(Q(k)^{\frac{1}{p-1}}\big) \geqslant f'\big(Q(k)^{\frac{1}{p-1}}\big) \cdot \big(W(s) - Q(k)^{\frac{1}{p-1}}\big),$$

that is,

$$W(s)^p - Q(k)^{\frac{p}{p-1}} \geqslant pQ(k) \cdot \big(W(s) - Q(k)^{\frac{1}{p-1}}\big).$$

Inserting this inequality into (5.11), we thus find

$$Q(k - \partial_s)W(s) - Q(k)^{\frac{p}{p-1}} \geqslant pQ(k) \cdot \big(W(s) - Q(k)^{\frac{1}{p-1}}\big). \tag{5.12}$$

To eliminate the constant term on the left-hand side, we change variables by

$$Y(s) = W(s) - Q(k)^{\frac{1}{p-1}}.$$

Then Eq. (5.12) can also be written in the equivalent form

$$\big[pQ(k) - Q(k - \partial_s)\big]Y(s) \leqslant 0. \tag{5.13}$$

We will prove below that the characteristic polynomial

$$\mathscr{P}(\lambda) \equiv pQ(k) - Q(k - \lambda) \tag{5.14}$$

has two negative roots λ_1, λ_2. Temporarily assuming this and noting that $\mathscr{P}(\lambda)$ has its highest-order coefficient equal to 1 by (5.2), we can then factor the ordinary differential inequality (5.13) as

$$Z(s) \equiv Y'(s) - \lambda_1 Y(s), \quad Z'(s) - \lambda_2 Z(s) \leqslant 0.$$

The last inequality makes $e^{-\lambda_2 s} Z(s)$ decreasing, so we must actually have

$$e^{-\lambda_2 s} Z(s) \leqslant \lim_{s \to -\infty} e^{-\lambda_2 s} Z(s) = 0$$

because $\lambda_2 < 0$ by above. This gives $Y'(s) - \lambda_1 Y(s) = Z(s) \leqslant 0$, and we similarly find

$$e^{-\lambda_1 s} Y(s) \leqslant \lim_{s \to -\infty} e^{-\lambda_1 s} Y(s) = 0$$

because $\lambda_1 < 0$ as well. Now that $Y(s) \leqslant 0$, we have

$$W(s) \leqslant Q(k)^{\frac{1}{p-1}}. \tag{5.15}$$

This is precisely the desired inequality (5.10) since $W(s) = r^{\frac{2}{p-1}} \varphi(r)$ by definition.

Thus it remains to show that the quadratic in (5.14) has two negative roots in case of assumption (5.9). Since the product of the two roots is

$$\mathscr{P}(0) = (p-1) \cdot Q(k) > 0 \tag{5.16}$$

by (5.14) and (5.5), it suffices to show that $\mathscr{P}(\lambda_*) \leqslant 0$ for some $\lambda_* < 0$. We choose

$$\lambda_* = k - \frac{n-2}{2} = \frac{2}{p-1} - \frac{n-2}{2}$$

and note that $\lambda_* < 0$ because $p > \frac{n+2}{n-2}$ by assumption. Combining (5.14), (5.2) and (5.9), we find

$$\mathscr{P}(\lambda_*) = pQ(k) - Q\left(\frac{n-2}{2}\right) = pQ(k) - \left(\frac{n-2}{2}\right)^2 \leqslant 0.$$

Thus $\mathscr{P}(\lambda)$ has two negative roots. $\square$

Lemma 5.5. *Let $n > 2$. Let $p \geqslant \frac{n+2}{n-2}$ and $k = \frac{2}{p-1}$. Given any one of the steady states φ provided by Theorem* 5.2*, the function*

$$W(s) = e^{ks} \varphi(e^s) = r^k \varphi(r), \quad s = \log r = \log |x|$$

must then satisfy the ordinary differential equation

$$(\partial_s - k)(\partial_s - k + n - 2) W(s) = -W(s)^p. \tag{5.17}$$

Moreover, this ordinary differential equation can be written in the form

$$Q(k - \partial_s) W(s) = W(s)^p,$$

where Q denotes the quadratic in (5.2).

Proof. Since $\partial_r = e^{-s}\partial_s$, a short computation allows us to write the radial Laplacian as

$$\Delta = \partial_r^2 + (n-1)r^{-1}\partial_r = e^{-2s}(n-2+\partial_s)\partial_s.$$

Since $-\Delta\varphi = \varphi^p$ and $kp = k+2$ by assumption, this easily leads us to

$$-W^p = -e^{kps}\varphi^p = e^{ks+2s}\Delta\varphi = e^{ks}(n-2+\partial_s)\partial_s\big(e^{-ks}W\big).$$

Using the operator identity $\partial_s e^{-ks} = e^{-ks}(\partial_s - k)$ twice, we then get

$$-W^p = e^{ks}(n-2+\partial_s)e^{-ks}(\partial_s - k)W = (n-2+\partial_s-k)(\partial_s-k)W.$$

This proves our first assertion (5.17), from which our second assertion follows trivially. $\square$

Theorem 5.6. *Let $n > 2$ and $p \geqslant \frac{n+2}{n-2}$. Let φ denote any one of the steady states provided by Theorem 5.2 and set*

$$p_{\mathrm{c}} = \begin{cases} \infty & \text{if } n \leqslant 10, \\ \frac{n^2-8n+4+8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n > 10, \end{cases} \tag{5.18}$$

for convenience. Then $p_{\mathrm{c}} > \frac{n+2}{n-2}$ and the following dichotomy holds.

If $p < p_{\mathrm{c}}$, then φ is a nonlinearly unstable solution of any of the equations

$$\partial_t u - \Delta u = |u|^p, \quad \partial_t^2 u + a\partial_t u - \Delta u = |u|^p, \tag{5.19}$$

where $a \in \mathbb{R}$. More precisely, the conclusions of Theorems 2.1 and 2.2 (both (a) and (b)) remain valid. That is, the instability occurs by blow up in finite time.

If $p \geqslant p_{\mathrm{c}}$, on the other hand, then φ is a linearly stable solution in the sense that the linearized operator has no negative spectrum.

Proof. Let Q denote the quadratic in (5.2) and consider the expression

$$\mathcal{Q}(p) \equiv 4(p-1)^2 \cdot \left[\left(\frac{n-2}{2}\right)^2 - p \cdot Q\left(\frac{2}{p-1}\right)\right]. \tag{5.20}$$

When this is nonnegative, Lemma 5.4 implies that φ is linearly stable. When it is negative, on the other hand, Lemma 5.3 implies that the linearized operator has a negative eigenvalue. Since φ vanishes at infinity, each of the functions

$$\varphi, \quad f(\varphi) = \varphi^p, \quad f'(\varphi) = p\varphi^{p-1}$$

is bounded, whence (A3)–(A6) hold and the linearized operator has essential spectrum

$$\sigma_{\mathrm{ess}}\big(-\Delta - p\varphi^{p-1}\big) = \sigma_{\mathrm{ess}}(-\Delta) = [0, \infty),$$

whence (A1), (A2) follow by standard arguments, as in the proof of Theorem 3.1.

In view of these observations, it thus suffices to check that

$$\mathcal{Q}(p) < 0 \quad \Leftrightarrow \quad \frac{n+2}{n-2} \leqslant p < p_{\mathrm{c}}. \tag{5.21}$$

Combining our definitions (5.2) and (5.20), we have

$$\mathcal{Q}(p) = (n-2)(n-10) \cdot p^2 - 2\bigl(n^2 - 8n + 4\bigr) \cdot p + (n-2)^2. \tag{5.22}$$

The last equation easily leads to

$$\mathcal{Q}\left(\frac{n+2}{n-2}\right) = -\frac{64}{n-2} < 0. \tag{5.23}$$

Case 1. Suppose first that $n > 10$. Then $\mathcal{Q}(p)$ has two roots, the largest one of which is p_{c}, given by (5.18). Since $\mathcal{Q}(p)$ is negative only between its two roots, (5.23) implies that $\mathcal{Q}(p)$ is negative if $\frac{n+2}{n-2} \leqslant p < p_{\mathrm{c}}$ and nonnegative if $p \geqslant p_{\mathrm{c}}$.

Case 2. Suppose now that $2 < n \leqslant 10$. Then the computation

$$\mathcal{Q}'(p) = 2(n-2)(n-10)\left(p - \frac{n+2}{n-2}\right) - 48$$

forces $\mathcal{Q}(p)$ to be decreasing for each $p \geqslant \frac{n+2}{n-2}$, and hence negative by (5.23). This shows that the desired condition (5.21) holds with $p_{\mathrm{c}} = \infty$. $\square$

Remark 5.7. When it comes to the nonlinear heat equation (5.19), the above result of Theorem 5.6 is originally due to Gui, Ni and Wang [10]. Their proof is based on an earlier result of Wang [28], according to which the graphs of two steady states do not intersect if $p \geqslant p_{\mathrm{c}}$, while the graphs of any two steady states do intersect if $p < p_{\mathrm{c}}$. Using this characterization of the critical value p_{c}, they are then able to employ comparison arguments which rely on the strong maximum principle. As far as the unstable case $p < p_{\mathrm{c}}$ is concerned, they show that blow up occurs provided the perturbation ψ_0 in (2.14) satisfies $0 \leqslant \psi_0 \not\equiv 0$, while solutions exist globally provided $0 \geqslant \psi_0 \not\equiv 0$. Theorem 5.6 yields a refinement of their blow up result for the nonlinear heat equation since our exact assumption (2.13) on ψ_0 reads

$$\int_{\mathbb{R}^n} \chi(x)\psi_0(x)\,dx > 0$$

for a certain positive function $\chi(x)$. As for the stable case $p \geqslant p_{\mathrm{c}}$, the result in [10] is much stronger than Theorem 5.6 because it proves nonlinear stability, not merely linear stability. For more recent stability results in the parabolic case, we refer the reader to [21] and the references cited therein. A proof of nonlinear stability in the hyperbolic case remains difficult to imagine.

Remark 5.8. When it comes to the wave equation (5.19) in the critical case $p = \frac{n+2}{n-2}$, our conclusions are closely related to a result of Krieger and Schlag [14] in three dimensions. According to Theorem 5.2, the equation

$$\partial_t^2 u - \Delta u = |u|^5, \quad x \in \mathbb{R}^3, \tag{5.24}$$

admits a family of steady states

$$\varphi_\lambda(x) = \frac{(3\lambda^2)^{1/4}}{\sqrt{\lambda^2 + |x|^2}}, \quad \lambda > 0, \tag{5.25}$$

all of which are unstable by Theorem 5.6. For the equation with u^5 rather than $|u|^5$, Krieger and Schlag [14] construct a stable manifold Σ associated with φ_1. That is, if the initial data

$$u(x,0) = \varphi_1(x) + \psi_0(x), \quad \partial_t u(x,0) = \psi_1(x)$$

correspond to a small perturbation $(\psi_0, \psi_1) \in \Sigma$, then the solution to (5.24) exists globally and remains near the curve $\{\varphi_\lambda\}_{\lambda>0}$ for all times. Numerical computations [2,26] suggest that the stable manifold Σ ought to represent the borderline case between global existence and blow up of solutions. To see that Theorem 5.6 provides a partial proof of this conjecture, we note that the tangent plane to Σ at the origin is given by the equation

$$\sigma \int_{\mathbb{R}^n} \chi(x)\psi_0(x)\,dx + \int_{\mathbb{R}^n} \chi(x)\psi_1(x)\,dx = 0, \tag{5.26}$$

where $\sigma > 0$ and $-\sigma^2$ is the first eigenvalue of the linearized operator; see [14]. Thus the set of perturbations (2.2) to which our instability theorem applies is precisely the set of perturbations that lie strictly above the tangent plane to Σ at the origin.

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