Higher twist corrections to the sum rule for semileptonic $B$ decay

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Abstract

The sum rule for charmless inclusive semileptonic $B$-meson decays allows a theoretically clean and experimentally efficient determination of $|V_{ub}|$. The leading twist contribution to the sum rule is known in QCD. We compute higher twist corrections to the sum rule using the heavy-quark effective theory.

A new method to determine the Cabibbo–Kobayashi–Maskawa (CKM) matrix element $|V_{ub}|$ has been proposed [1] that takes advantage of the sum rule for charmless inclusive semileptonic $B$-meson decays $\bar{B} \to X_u \ell \bar{\nu}_\ell$ ($\ell = e$ or $\mu$). The sum rule establishes a clean relationship between $|V_{ub}|$ and the observable

$$S \equiv \int_0^1 d\xi_u \frac{1}{\xi_u^2} \frac{d\Gamma}{d\xi_u} (\bar{B} \to X_u \ell \bar{\nu}_\ell)$$

with the kinematic variable $\xi_u = (q^0 + |\mathbf{q}|)/M_B$ in the $B$-meson rest frame, where $q$ is the momentum transfer to the lepton pair and $M_B$ denotes the $B$ meson mass. Moreover, this method of determining $|V_{ub}|$ has experimental virtue too. The kinematic variable $\xi_u$ is the most efficient discriminator between $\bar{B} \to X_u \ell \bar{\nu}_\ell$ signal and $\bar{B} \to X_c \ell \bar{\nu}_\ell$ background. A majority of $\bar{B} \to X_u \ell \bar{\nu}_\ell$ events have a value of $\xi_u$ beyond the limit allowed for $\bar{B} \to X_c \ell \bar{\nu}_\ell$ decays with charm in the final state, $\xi_u > 1 - M_D/M_B = 0.65$ with $M_D$ being the $D$-meson mass. Therefore, only a small extrapolation is needed to obtain $S$.

The charmless inclusive semileptonic decay of the $B$ meson is a light-cone dominated process. The light-cone expansion allows a rigorous and systematic ordering of nonperturbative QCD effects, providing an effective technique for a separation and classification of higher twist (HT) effects [2]. The leading term in this expansion gives the leading twist contribution. HT contributions are contained in the light-cone expansion beyond the leading order. The sum rule at the leading twist order measures the bottomness carried by a $B$ meson. There are no perturbative QCD corrections to the sum rule. Thus the primary hadronic uncertainty and the potential uncertainty of perturbative QCD are eliminated, dramatically reducing the theoretical error on $|V_{ub}|$. This inclusive method is to be contrasted with the determination of $|V_{ub}|$ from the charmless inclusive semileptonic branching fraction of $B$ mesons where the calculation of the total semileptonic decay rate is model dependent or assumes quark–hadron duality, there are uncertainties due to perturbative QCD corrections and, in addition, a larger extrapolation is necessary to extract the total rate if the kinematic cut on a certain observable, such as the charged-lepton energy or the invariant mass of the lepton pair, is applied for the suppression of $b \to c$ background.

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Only uncertainties due to HT effects remain in the sum rule. Including the HT contribution $\Delta HT$, the sum rule reads

$$S = \int \frac{1}{d\xi_u} \frac{1}{d\xi_v} \frac{d\Gamma}{dE} (\overline{B} \rightarrow X_u \ell \bar{\nu}_\ell)$$

$$= |V_{ub}|^2 \frac{G_F^2 M_B^5}{192\pi^3} (1 + \Delta HT).$$

(2)

Although HT contributions are expected to be suppressed by powers of $\Lambda_{QCD}/M_B$ ($\Lambda_{QCD}$ being the QCD scale), a quantitative estimate of them is indispensable for a complete understanding of remaining theoretical uncertainties in this determination of $|V_{ub}|$. In this Letter, we investigate HT effects on the sum rule for charmless inclusive semileptonic $B$ decays using the heavy-quark effective theory (HQET) [3–5].

Charmless inclusive semileptonic decays of the $B$ meson are induced by the weak interactions. The differential decay rate to lowest order in the weak interactions is

$$d\Gamma = \frac{G_F^2 |V_{ub}|^2}{(2\pi)^3 E} L^{\mu\nu} W_{\mu\nu} d^3 k_\mu d^3 k_\nu.$$

(3)

Here $E$ ($P$), $E_\ell$ ($k_\ell$), and $E_\nu$ ($k_\nu$) denote the energies (four-momenta) of the $B$ meson, the charged lepton, and the antineutrino, respectively. The leptonic tensor for the lepton pair is completely determined by the standard electroweak theory since leptons do not have strong interactions:

$$L^{\mu\nu} = 2 \left( k_\mu k_\nu + g^{\mu\nu}_v k_\ell \cdot k_\nu - g^{\mu\nu}_v k_\ell \cdot k_\nu + i e^{\mu\nu}_v g_{\alpha\beta} k_\ell^\alpha k_\nu^\beta \right).$$

(4)

The hadronic tensor incorporates all nonperturbative QCD physics for the inclusive semileptonic $B$ decay. It is summed over all hadronic final states and can be expressed in terms of a current commutator taken between the $B$ meson states:

$$W_{\mu\nu} = \frac{1}{2\pi} \int d^4 y e^{i q \cdot y} \langle B | [j_\mu(y), j_\nu^0(0)] | B \rangle,$$

(5)

where $j_\mu(y) = \bar{u}(y)\gamma_\mu(1 - \gamma_5)b(y)$ is the charged weak current for the $b \rightarrow u$ transition. We adopt a covariant normalization for one-particle states, i.e.,

$$\langle B | B(P) \rangle = (2\pi)^3 2 P^{0\beta}(\mathbf{P} - P').$$

The most general hadronic tensor form that can be constructed is a linear combination of $P_\mu P_\nu$, $P_\mu q_\nu$, $q_\mu P_\nu$, $q_\mu q_\nu$, $e_{\mu\nu\alpha\beta} P^{\alpha\beta}$ and $g_{\mu\nu}$, with coefficients being scalar functions $W_\alpha(v, q^2)$ of the two independent Lorentz invariants, $v \equiv q \cdot P/M_B$ and $q^2$. However, the combination $P_\mu q_\nu - q_\mu P_\nu$ does not contribute since $L^{\mu\nu}(P_\mu q_\nu - q_\mu P_\nu) = 0$. Thus the hadronic tensor must take the form

$$W_{\mu\nu} = -g_{\mu\nu} W_1 + \frac{P_\mu P_\nu}{M_B^2} W_2 - i e_{\mu\nu\alpha\beta} \frac{P^{\alpha\beta} q^2}{M_B^2} W_3$$

$$+ \frac{q_\mu P_\nu}{M_B^2} W_4 + \frac{q_\mu P_\nu + q_\mu P_\nu}{M_B^2} W_5.$$

(6)

Eq. (5) shows that $W_\alpha = W_\alpha$, so $W_\alpha, \alpha = 1, \ldots, 5$ are real. The interesting physics describing the hadron structure and the strong interactions is wrapped up in the five dimensionless real structure functions $W_\alpha(v, q^2), \alpha = 1, \ldots, 5$ for the unpolarized processes.

In the following we will neglect the masses of the charged lepton and the $\nu$-quark. From Eqs. (3) and (6), we obtain the double differential decay rate for $\overline{B} \rightarrow X_u \ell \bar{\nu}_\ell$ in the rest frame of the $B$ meson

$$\frac{d^2 \Gamma}{d\xi_a \, dq^2} = \frac{G_F^2 |V_{ub}|^2 |q|^2}{48\pi^3 M_B} (W_1 3q^2 + W_2 |q|^2),$$

where

$$|q| = \frac{1}{2} M_B \xi_a \left(1 - \frac{q^2}{M_B^2 \xi_a^2}\right).$$

(7)

(8)

By integrating Eq. (7) over $q^2$, one gets the decay distribution of the kinematic variable $\xi_a$

$$\frac{d\Gamma}{d\xi_a} = \int_0^{M_B^2 \xi_a^2} dq^2 \frac{d^2 \Gamma}{d\xi_a \, dq^2}.$$  

(9)

Computing the current commutator one obtains from Eq. (5)

$$W_{\mu\nu} = -\frac{1}{\pi} \left( S_{\mu\nu\beta} - i e_{\mu\nu\alpha\beta} \right) \times$$

$$\times \int d^4 y e^{i q \cdot y} \left[ \partial^\beta \Delta_\mu(y) \right] \times \langle B | \overline{b}(0) y^\beta U(0, y) b(y) | B \rangle,$$

(10)

where $S_{\mu\nu\beta} = g_{\mu\nu} S_{\beta\bar{d}} + g_{\nu\beta} S_{\alpha\bar{d}} - g_{\mu\beta} S_{\alpha\bar{d}}$. In the above we have used

$$\{u(x), \overline{u}(y)\} = i (y \cdot \partial) \Delta_\mu(x - y) U(x, y).$$

(11)
with the Wilson link
\[ U(x, y) = \mathcal{P} \exp \left[ i g_s \int_y^x d\zeta A_\mu(\zeta) \right] , \] (12)

\[ \Delta_\alpha(y) = -\frac{i}{(2\pi)^4} \int d^4 k e^{-ik\cdot y} \delta(\epsilon(0)\delta(k^2)), \] (13)

where \( A_\mu \) is the background gluon field and \( \epsilon(x) \) satisfies \( \epsilon(|x|) = 1 \) and \( \epsilon(-|x|) = -1 \).

The matrix element \( \langle B|\hat{b}(0)\gamma^\beta U(0, y)\hat{b}(y)|B \rangle \) is the basic building block of the description of inclusive \( B \) decays in QCD. In general one can decompose it in the following form:

\[ \langle B|\hat{b}(0)\gamma^\beta U(0, y)\hat{b}(y)|B \rangle = 2 \left[ p^\beta F(y^2, y \cdot P) + y^\beta G(y^2, y \cdot P) \right], \] (14)

where \( F(y^2, y \cdot P) \) and \( G(y^2, y \cdot P) \) are functions of the two independent Lorentz scalars, \( y^2 \) and \( y \cdot P \). The dominant part of the integrand in the hadronic matrix elements below is suppressed by \( (\Lambda_{QCD}/M_B)^2 \) and \( (\Lambda_{QCD}/M_B)^2 \) is the heavy quark effective theory to compute the twist-4 matrix elements below.

The matrix element \( \langle B|\hat{b}(0)\gamma^\beta U(0, y)\hat{b}(y)|B \rangle \) arises from the two relevant structure functions \( y^2 \) and \( y \cdot P \). The dominant part of the integrand in the hadronic matrix elements below is suppressed by \( (\Lambda_{QCD}/M_B)^2 \) and \( (\Lambda_{QCD}/M_B)^2 \) is the heavy quark effective theory to compute the twist-4 matrix elements below.

The twist decomposition for the decay rate thus takes the form

\[ d\Gamma = \sum_{n=0}^\infty d\Gamma^{(2n+2)}, \] (16)

where

\[ d\Gamma^{(2n+2)} = \frac{G_4^2 |V_{ub}|^2}{(2\pi)^3} \frac{L_{\mu\nu}W_{\mu\nu}^{(2n+2)}}{2E_i} \] (17)

is the twist-(2n+2) contribution to the decay rate with \( W_{\mu\nu}^{(2n+2)} \)

\[ = -g_{\mu\nu}W_1^{(2n+2)} + \frac{P_\mu P_\nu}{M_B^2} W_2^{(2n+2)} \]

\[ -i\epsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta M_B^2 W_3^{(2n+2)} + \frac{q_\mu q_\nu}{M_B^2} W_4^{(2n+2)} + \frac{P_\mu q_\nu + q_\mu P_\nu}{M_B^2} W_5^{(2n+2)} \]

\[ = \frac{2}{\pi} (S_{\mu\nu\alpha\beta} - i\epsilon_{\mu\nu\alpha\beta}) \int d^4 y e^{i(q\cdot y)} [\partial^\sigma \Delta_\alpha(y)] \]

\[ \times \left[ p^\beta (y^2)^n F^{(2n+2)}(y \cdot P) + y^\beta (y^2)^{n-1} G^{(2n+2)}(y \cdot P) \right]. \] (19)

The leading twist contribution to the sum rule (2) results from \( F^{(2)}(y \cdot P) \) of twist 2 and is known in QCD to be [1]

\[ \int \limits_0^1 d\xi_u \frac{1}{\xi_u} \frac{d\Gamma^{(2)}}{d\xi_u} (B \to X_u \ell\bar{\nu}_\ell) = |V_{ub}|^2 \frac{G_4^2 M_B^5}{192\pi^3} \] (20)

which is a consequence of the conservation of the \( b \)-quark vector current by the strong interactions. The next-to-leading twist contribution to the sum rule arises from \( F^{(4)}(y \cdot P) \) and \( G^{(4)}(y \cdot P) \) of twist 4. It can be obtained by integrating Eq. (9) over \( \xi_u \) with the two relevant structure functions \( W_1^{(4)}(\mu, q^2) \) and \( W_2^{(4)}(\mu, q^2) \) of twist 4.

We use the operator product expansion and the heavy quark effective theory to compute the twist-4 structure functions. The Wilson link is a gauge dependent operator. It is convenient to use the Fock–Schwinger gauge such that \( U(0, y) = \) unity. Since
the $b$ quark inside the $B$ meson behaves as almost free due to its large mass, relative to which its binding to the light constituents is weak, one can extract the large space–time dependence

$$b(y) = e^{-i m_b y} p_b(y),$$  \hfill (21)

where $m_b$ is the $b$-quark mass and $v = P/M_B$ is the four-velocity of the $B$ meson. This factorization makes clear why the large scale in matrix elements does not affect the relative size of terms in the light-cone expansion (15). The large scale hidden in matrix elements of $b$-quark operators is contained in an overall factor $e^{-i m_b y}$, so reduced matrix elements of the operators containing the rescaled operator $h(y)$ involve only momenta of order $\Lambda_{\text{QCD}}$, which determine the relative size of terms in the light-cone expansion (15), i.e., schematically

$$\langle B|\hat{b}(0)\gamma^\mu h(y)|B\rangle = e^{-i m_b y} \langle B|\hat{b}(0)\gamma^\mu h(y)|B\rangle$$

$$\sim e^{-i m_b y} \sum_{n=0}^{\infty} \left( \frac{\Lambda_{\text{QCD}}^2}{M_B^2} \right)^n. \hfill (22)$$

The rescaled operator for a free $b$-quark no longer depends on the space–time, so $b(y) = e^{-i m_b y} b(0)$. In this case all the coefficients $f^{2n+2}(y \cdot P)$ and $g^{2n+4}(y \cdot P)$ in the light-cone expansion (15) vanish except for $f^{(2)}(y \cdot P) = e^{-i m_b y}$, because the conservation of the $b$-quark vector current implies that $\langle B|\hat{b}(0)\gamma^\mu h(0)|B\rangle = 0$. The leading-twist sum rule (20) is consistently reproduced in the free quark decay $b \to u\ell \nu \ell$. The conserved vector current $\hat{h} = \gamma^\mu \bar{b} \gamma^\nu \gamma^\rho b$ is not renormalized by the strong interactions. This explains why there are no perturbative QCD corrections to the sum rule.

A Taylor expansion of the field in a gauge-covariant form relates the bilocal and local operators. This leads to an operator product expansion

$$\hat{b}(0)\gamma^\mu b(y) = e^{-i m_b y} \hat{b}(0)\gamma^\mu h(y)$$

$$= e^{-i m_b y} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \gamma_{\mu_1} \cdots \gamma_{\mu_n}$$

$$\times \hat{b}(0)\gamma^\mu k^{[\mu_1 \cdots k^{\mu_n}] h(0)}. \hfill (23)$$

where $k_{\mu} = iD_{\mu} = i(\partial_{\mu} - ig_{\mu A} A_{\mu})$ and the symbol $\cdots$ means symmetrization with respect to the en-
closed indices. Because of the weak dependence of the rescaled operator $b(y)$ on $y$, we attempt to estimate the matrix element of the bilocal operator sandwiched between the $B$ meson states with the truncated $y$-expansion in Eq. (23). To obtain a twist-4 accuracy it suffices to keep only the first three terms

$$\langle B|\hat{b}(0)\gamma^\mu b(y)|B\rangle = e^{-i m_b y} \left[ \langle B|\hat{b}(0)\gamma^\mu h(0)|B\rangle \right.$$  

$$+ (-i) y_{\mu} \langle B|\hat{b}(0) h(y) \gamma^\rho i D_{\nu} b(0)|B\rangle$$  

$$+ \frac{(-i)^2}{2} y_{\mu} y_{\nu} \langle B|\hat{b}(0) h(y) \gamma^\rho \gamma^\sigma b(0)|B\rangle$$  

$$\times i D^{[\mu} D^{\nu]} b(0)|B\rangle \right]. \hfill (24)$$

In the heavy quark effective theory the QCD $b$-quark field $b(y)$ is related to its HQET counterpart $h(y)$ by means of an expansion in powers of $1/m_b$:

$$b(y) = e^{-i m_b y} \left[ 1 + \frac{i \hat{b}}{2 m_b} + O \left( \frac{\Lambda_{\text{QCD}}^2}{m_b^2} \right) \right] h(y). \hfill (25)$$

The effective Lagrangian takes the form

$$\mathcal{L}_{\text{HQET}} = \bar{h} i \gamma \cdot D h + \bar{h} \frac{(i D)^2}{2 m_b} h + \bar{h} g_5 G_{\mu\nu} \epsilon^{\mu\nu} - h + \mathcal{O} \left( \frac{1}{m_b^2} \right). \hfill (26)$$

where $g_5 G^{\mu\nu} = i [D^{\mu}, D^{\nu}]$ is the gluon field-strength tensor. At the level of accuracy of the present discussion we take into account only the leading, 1/m_b correction to the heavy quark limit $m_b \to \infty$. Relating the matrix elements of the local operators in full QCD in Eq. (24) to those in HQET, it follows that

$$\langle B|\hat{b}(0)\gamma^\mu h(y)\rangle = 2 e^{-i m_b y} \left\{ p^\mu \left[ -y \cdot P i \frac{5}{3 M_B} E_b \right.$$

$$- \left. (y \cdot P)^2 \frac{1}{3} \frac{m_b^2}{M_B^2} k_b + y^2 \frac{1}{3} \frac{m_b^2}{M_B^2} K_b \right] \right.$$  

$$\left. + \frac{y \cdot P}{3} \frac{m_b}{M_B} E_b \right\}. \hfill (27)$$

where $E_b = K_b + G_b$ and $K_b$ and $G_b$ are the dimensionless HQET parameters of order $(\Lambda_{\text{QCD}}/m_b)^2$, which are often referred to by the alternate names

$$\{ \cdots \}.$$
\begin{align*}
\lambda_1 &= -2m_b^2K_b \quad \text{and} \quad \lambda_2 = -2m_b^2G_b/3, \quad \text{defined as} \\
\lambda_1 &= \frac{1}{2M_B}(B\bar{h}(iD)^2h|B), \\
\lambda_2 &= \frac{1}{12M_B}(B|\bar{g}_G G_{\mu\nu}\sigma_{\mu\nu}h|B).
\end{align*}

Comparing Eq. (27) with Eq. (15) yields

\begin{align*}
F^{(4)}(y \cdot P) &= \frac{1}{3}m_b^2K_b\epsilon e^{-i\mu\nu\gamma}, \\
G^{(4)}(y \cdot P) &= \frac{2}{3}m_bM_bE_B\epsilon e^{-i\mu\nu\gamma}.
\end{align*}

We see that the coefficients \(F^{(4)}(y \cdot P)\) and \(G^{(4)}(y \cdot P)\) of the light-cone expansion (15) are indeed of order \(\Lambda_{QCD}^2\) as expected. Substituting Eqs. (30) and (31) in Eq. (19) and integrating by parts, we arrive at

\begin{align*}
W^{(4)}_{\mu\nu} &= \frac{16m_b}{3M_B}\left[-\frac{1}{4}m_b(m_b - v)K_bX \\
&\quad + E_b\epsilon(q^0 - m_bv^0)\delta(q^2 - 2m_bv + m_b^2) \\
&\quad + \frac{1}{3}m_b(m_b - v)K_b\epsilon\delta(q^0 - m_bv^0) \\
&\quad \times \delta'(q^2 - 2m_bv + m_b^2) \\
&\quad + (q^2 - 2m_bv + m_b^2)E_b \\
&\quad \times \epsilon(q^0 - m_bv^0)\delta(q^2 - 2m_bv + m_b^2)\right] \\
&\quad + P_{\mu}P_{\nu}m_b^2\left[\frac{1}{4}K_bX + 2(K_b + E_b) \\
&\quad + \epsilon(q^0 - m_bv^0)\delta(q^2 - 2m_bv + m_b^2)\right] \\
&\quad + \frac{i\epsilon_{\mu\nu\alpha\beta}}{2}P_{\alpha}q_{\beta}^2m_bM_bK_b \\
&\quad \times \left[\frac{1}{4}X + \epsilon(q^0 - m_bv^0) \\
&\quad \times \delta(q^2 - 2m_bv + m_b^2)\right] \\
&\quad + q_{\mu}q_{\nu}M_B^2E_b\epsilon(q^0 - m_bv^0) \\
&\quad \times \delta(q^2 - 2m_bv + m_b^2) \\
&\quad + (P_{\mu}q_{\nu} + q_{\mu}P_{\nu})m_bM_B\left[-\frac{1}{4}K_bX \\
&\quad - \frac{1}{9}K_b\epsilon(q^0 - m_bv^0)\delta(q^2 - 2m_bv + m_b^2) \\
&\quad - 2E_b\epsilon(q^0 - m_bv^0)\delta(q^2 - 2m_bv + m_b^2)\right]\right].
\end{align*}

Comparing Eq. (32) with Eq. (18), we find

\begin{align*}
W_1^{(4)}(v, q^2) &= \frac{15}{2}m_bM_B\left[\frac{1}{4}m_b(m_b - v)K_bX \\
&\quad + E_b\epsilon(q^0 - m_bv^0)\delta(q^2 - 2m_bv + m_b^2) \\
&\quad + \frac{1}{3}m_b(m_b - v)K_b\epsilon\delta(q^0 - m_bv^0) \\
&\quad \times \delta'(q^2 - 2m_bv + m_b^2)\right].
\end{align*}

\begin{align*}
W_2^{(4)}(v, q^2) &= \frac{16}{3}m_b^3M_B\left[\frac{1}{4}K_bX + 2(K_b + E_b) \\
&\quad + \epsilon(q^0 - m_bv^0)\delta(q^2 - 2m_bv + m_b^2)\right],
\end{align*}

\begin{align*}
W_3^{(4)}(v, q^2) &= \frac{16}{3}m_b^3M_B^2K_b\left[\frac{1}{4}X + \epsilon(q^0 - m_bv^0) \\
&\quad \times \delta'(q^2 - 2m_bv + m_b^2)\right],
\end{align*}

\begin{align*}
W_4^{(4)}(v, q^2) &= \frac{32}{3}m_bM_bE_b\epsilon(q^0 - m_bv^0)\delta(q^2 - 2m_bv + m_b^2),
\end{align*}

\begin{align*}
W_5^{(4)}(v, q^2) &= \frac{16}{3}m_b^3M_B^2\left[\frac{1}{4}K_bX + (K_b + 2E_b) \\
&\quad + \epsilon(q^0 - m_bv^0)\delta(q^2 - 2m_bv + m_b^2)\right].
\end{align*}

The twist-4 contribution to the sum rule can be obtained from Eqs. (9), (7), (34) and (35). The result is

\begin{align*}
\int_0^1 d\xi_u \frac{1}{\xi_u} d\Gamma^{(4)}(\bar{B} \to X_u \bar{c}c)
&= |V_{ub}|^2 \frac{G_F^2M_B^5}{192\pi^3} \left[\frac{304}{45}K_b + \frac{76}{45}E_b + \frac{2m_b^2}{3M_B}K_b \\
&\quad - \frac{68}{9}m_b^3\frac{1}{M_B}K_b - \frac{80}{9}m_b^2\frac{1}{M_B}E_b + \frac{26}{3}m_b\frac{1}{M_B}K_b \\
&\quad + \frac{28}{3}\frac{m_b^4}{M_B}E_b + \frac{112}{15}\frac{m_b^5}{M_B}K_b - \frac{32}{15}\frac{m_b^3}{M_B}E_b\right]
\end{align*}

This can serve as an estimate of HT contributions to the sum rule (2).
For the numerical analysis, we need to know the values for the parameters involved. The HQET parameter \( \lambda_2 \) can be extracted from the \( B^* \to B \) mass splitting: \( \lambda_2 = (M_{B^*}^2 - M_B^2)/4 \approx 0.12 \text{ GeV}^2 \), while \( \lambda_1 \) and \( m_b \) are less determined. For the purpose of estimation, we take \( \lambda_1 = -0.5 \text{ GeV}^2 \) [7,8] and \( m_b = 4.9 \text{ GeV} \) [9]. From Eq. (39), the HT correction to the sum rule (2) is then estimated to be \( \Delta HT = 0.012 \). This quantitative study shows that HT corrections are at the expected level of \( \sim \Lambda_{\text{QCD}}/M_B^2 \).

The sum rule (2) requires measuring the \( \xi_u \) spectrum dividing by \( \xi_u^5 \), which emphasizes the contribution from very small \( \xi_u \). In the free quark decay \( b \to u\ell\bar{v}_\ell \), the weighted \( \xi_u \) spectrum at tree level is

\[
\frac{1}{\xi_u^5} \frac{d\Gamma}{d\xi_u} (b \to u\ell\bar{v}_\ell) = |V_{ub}|^2 \frac{G_F^2 M_B^5}{192\pi^3} f(\xi_u - \frac{m_b}{M_B}),
\]

(40)

because \( \xi_u \) is fixed by kinematics to be \( \xi_u = m_b/M_B \). We see that in the free quark decay the \( \xi_u \) spectrum weighted with \( \xi_u^{-5} \) is just a discrete line at \( \xi_u = m_b/M_B \), which is well above the charm threshold, and there are no contributions from small \( \xi_u \). This is the reason why the kinematic variable \( \xi_u \) is the most efficient discriminator between \( B \to X_u\ell\bar{v}_\ell \) signal and \( B \to X_c\ell\bar{v}_\ell \) background. The inclusion of strong interactions amounts to a smearing of the spectrum. However, since in reality the \( b \) quark in the \( B \) meson is nearly free, we would expect the actual weighted \( \xi_u \) spectrum to remain small in the small \( \xi_u \) region from a general point of view independent of light-cone dominance.

Let us examine how strong interactions affect the weighted \( \xi_u \) spectrum at small values of \( \xi_u \). The above calculation in the framework of HQET suggests that HT contributions to \( S \) in this region may be neglected. The weighted \( \xi_u \) spectrum to leading twist is given by [1]

\[
\frac{1}{\xi_u^5} \frac{d\Gamma^{(2)}}{d\xi_u} (B \to X_u\ell\bar{v}_\ell) = |V_{ub}|^2 \frac{G_F^2 M_B^5}{192\pi^3} f(\xi_u),
\]

(41)

where the distribution function \( f(\xi) \) is the Fourier transformation of \( \mathcal{F}^{(2)}(y \cdot P) \) [2]. The distribution function \( f(\xi) \) is known [2] in QCD to be sharply peaked around \( \xi = m_b/M_B \) and vanish at \( \xi = 0 \). This result is consistent with the physical picture that the \( b \) quark in the \( B \) meson is nearly free. Eq. (41) shows that the weighted \( \xi_u \) spectrum has the same sharp shape as the distribution function, in particular, vanishing at \( \xi_u = 0 \). We also note that perturbative QCD corrections to the weighted spectrum are fairly small in the small \( \xi_u \) region and tend to vanish as \( \xi_u \to 0 \). The small correction is from the bremsstrahlung process of hard gluons. Together, these imply a small contribution to the sum rule value \( S \) from small \( \xi_u \), despite the presence of the \( \xi_u^{-5} \) weighting. A detailed numerical study of the weighted spectrum has been presented in the second reference in [1], including both perturbative and nonperturbative QCD effects. It turns out that about 80% of the weighted spectrum, \( \xi_u^{-5} d\Gamma/d\xi_u \), would lie above the charm threshold, \( \xi_u > 1 - M_D/M_B \), as opposed to about 10% of the charged-lepton energy spectrum above the charm threshold, \( E_\ell > (M^2_B - M^2_D)/(2M_B) \). Thus a smaller extrapolation into an unmeasured \( \xi_u \) region is needed to obtain the integral \( S \). Since the distribution function is universal, one can use the distribution function \( f(\xi) \) measured from other observables, such as the photon energy spectrum in \( B \to X_u \gamma \) (see the last reference in [2]), to extrapolate the weighted \( \xi_u \) spectrum, without making any model dependent assumptions. In practice, the sum rule (2) requires precision measurements over a wide range of \( \xi_u \). It is important to have data with good accuracy down to \( \xi_u \) as small as possible.

In summary, we have elaborated a quantitative way of estimating HT contributions in inclusive \( B \) decays, which is based on the heavy quark effective theory. As an application we have calculated the twist-4 correction to the sum rule (2) for charmless inclusive semileptonic \( B \) decays. Using the sum rule, \( |V_{ub}| \) can be determined from a measurement of the weighted integral \( S \). The error on \( |V_{ub}| \) due to HT corrections to the sum rule is estimated to be 1%. Combining theoretical cleanliness and experimental efficiency, together with the better understanding of remaining theoretical uncertainties, the sum rule holds high promise of a precise and model-independent determination of \( |V_{ub}| \).

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