Leader type contractions, periodic and fixed points and new completivity in quasi-gauge spaces with generalized quasi-pseudodistances

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1. Introduction

The necessity of defining the various concepts of completivity in quasi-gauge spaces became apparent with the investigation of asymmetric structures in these spaces. General results of this sort were progressively shown in a series of papers and important ideas are to be found in [1,7,15,26,25,27,28], which also contains many examples. Also note that the studies of asymmetric structures and their applications in theoretical computer science are important.

We recall the definition of quasi-gauge spaces.

Definition 1.1. Let $X$ be a nonempty set.

(i) A quasi-pseudometric on $X$ is a map $p : X \times X \to [0, \infty)$ such that:

(P1) $\forall x \in X \{ p(x, x) = 0 \}$; and

(P2) $\forall x, y, z \in X \{ p(x, z) \leq p(x, y) + p(y, z) \}$.

For given quasi-pseudometric $p$ on $X$ a pair $(X, p)$ is called quasi-pseudometric space. A quasi-pseudometric space $(X, p)$ is called Hausdorff if $\forall x, y \in X \{ x \neq y \Rightarrow p(x, y) > 0 \lor p(y, x) > 0 \}$.

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(ii) Each family $\mathcal{P} = \{p_\alpha: \alpha \in A\}$ of quasi-pseudometrics $p_\alpha: X \times X \to [0, \infty)$, $\alpha \in A$, is called a quasi-gauge on $X$.

(iii) Let the family $\mathcal{P} = \{p_\alpha: \alpha \in A\}$ be a quasi-gauge on $X$. The topology $\mathcal{T}(\mathcal{P})$, having as a subbase the family $\mathcal{B}(\mathcal{P}) = \{B(x, \varepsilon_\alpha): x \in X, \varepsilon_\alpha > 0, \alpha \in A\}$ of all balls $B(x, \varepsilon_\alpha) = \{y \in X: d_\alpha(x, y) < \varepsilon_\alpha\}$, $x \in X, \varepsilon_\alpha > 0, \alpha \in A$, is called the topology induced by $\mathcal{P}$ on $X$.

(iv) (Dugundji [7], Reilly [25, 26]) A topological space $(X, \mathcal{T})$ such that there is a quasi-gauge $\mathcal{P}$ on $X$ with $\mathcal{T} = \mathcal{T}(\mathcal{P})$ is called a quasi-gauge space and is denoted by $(X, \mathcal{P})$.

(v) A quasi-gauge space $(X, \mathcal{P})$ is called Hausdorff if quasi-gauge $\mathcal{P}$ has the property: $\forall x, y \in X, x \neq y \Rightarrow \exists \alpha \in A, \{p_\alpha(x, y) > 0 \vee p_\alpha(y, x) > 0\}$.

Remark 1.1. Each quasi-uniform space and each topological space is a quasi-gauge space (Reilly [25, Theorems 4.2 and 2.6]). The quasi-gauge spaces are the greatest general spaces with asymmetric structures.

By $\text{Fix}(T)$ we denote the set of all fixed points of $T: X \to X$, i.e., $\text{Fix}(T) = \{w \in X: w = T(w)\}$. By a contractive fixed point of $T: X \to X$ we mean a point $w \in \text{Fix}(T)$ such that, for each $w^0 \in X$, $\lim_{m \to \infty} T^m(w^0) = w$.

In 1983 Leader [17, Theorem 3] discovered the following interesting phenomenon in the metric fixed point theory.

Theorem 1.1. Let $(X, d)$ be a metric space and let $T: X \to X$ be a map with a complete graph (i.e. closed in $Y^2$ where $Y$ is the completion of $X$). The following hold:

(a) $T$ has a contractive fixed point if and only if (L1) $\forall x, y \in X \exists \varepsilon > 0 \exists \alpha \in \mathbb{N} \exists \varepsilon_\alpha \in \mathbb{N} \forall i, j \in \mathbb{N}[d(T^i(x), T^j(y)) < \varepsilon + \varepsilon_\alpha \Rightarrow d(T^{i+r}(x), T^{j+r}(y)) < \varepsilon]$.

(b) $T$ has a fixed point if and only if (L2) $\forall x \in X \exists \varepsilon > 0 \exists \alpha \in \mathbb{N} \exists \varepsilon_\alpha \in \mathbb{N} \forall i, j \in \mathbb{N}[d(T^i(x), T^j(x)) < \varepsilon + \varepsilon_\alpha \Rightarrow d(T^{i+r}(x), T^{j+r}(x)) < \varepsilon]$. Moreover, if $x, \varepsilon, \eta$ and $r$ are as in (L2) and if $\lim_{m \to \infty} T^m(x) = w$, then $\forall \varepsilon_\alpha \in \mathbb{N}[d(T^i(x), T^{i+r}(x)) < \eta \Rightarrow d(T^{i+r}(x), w) < \varepsilon]$.

Recall, that the maps satisfying the above conditions (L1) and (L2) are called in literature Leader contractions and weak Leader contractions, respectively.

It is well-known that in complete metric spaces this result generalizes Banach [2], Boyd and Wong [3], Browder [4], Burton [5], Caccioppoli [6], Dugundji [7], Dugundji and Granas [8], Geragthy [9, 10], Krasnosel’skiĭ et al. [16], Matkowski [19–21], Meir and Keeler [22], Mukherjea [23], Rakotch [24], Tasković [32], Walter [35] and many others’ results not mentioned in this paper; for details, see Jachymski [11, 12] and Jachymski and Jóźwik [13]. In the complete metric spaces with $\tau$-distances, beautiful generalizations of Leader’s result [17, Theorem 3] are established by Suzuki [29, Theorem 4] and [30].

Recently, we introduced the concept of generalized pseudodistances in uniform spaces and showed that they provide a natural tool to obtain a natural generalizations of the results of [17, Theorem 3], [29, Theorem 4] and [30] in uniform spaces without sequentially complete assumptions and without complete graph assumptions about maps; for details see [36] and examples therein. These generalized pseudodistances generalize metrics, $\mathcal{w}$-distances of Kada et al. [14], $\tau$-functions of Lin and Du [18], $\tau$-distances of Suzuki [31] and distances of Tataru [33] in metric spaces and distances of Vályi [34] in uniform spaces; for details, see [37, 38].

In this paper, we show how the introduced here generalized quasi-pseudodistances $\mathcal{w}$-distances of Kada et al. [14], $\tau$-functions of Lin and Du [18], $\tau$-distances of Suzuki [31] and distances of Tataru [33] in metric spaces and distances of Vályi [34] in uniform spaces; for details, see [37, 38].

It remains to note that, in the literature, the investigations of periodic points of contractions of Leader or Leader type are not known.

This paper is a continuation of [36–41].

2. Definitions and notations

We first record the definition of left (right) $\mathcal{J}$-families needed in the sequel.

Definition 2.1. Let $(X, \mathcal{P})$ be a quasi-gauge space. The family $\mathcal{J} = \{J_\alpha: \alpha \in A\}$ of maps $J_\alpha: X \times X \to [0, \infty)$, $\alpha \in A$, is said to be a left (right) $\mathcal{J}$-family of generalized quasi-pseudodistances on $X$ (left (right) $\mathcal{J}$-family on $X$, for short) if the following two conditions hold:

(J1) $\forall \alpha \in A \forall x, y, z \in X[J_\alpha(x, z) \leq J_\alpha(x, y) + J_\alpha(y, z)]$; and

(J2) For any sequences $(u_m: m \in \mathbb{N})$ and $(v_m: m \in \mathbb{N})$ in $X$ satisfying

\[
\forall \alpha \in A \left\{ \lim_{m \to \infty} \sup_{n > m} J_\alpha(u_m, u_n) = 0 \right\} \quad (2.1)
\]

\[
\left( \forall \alpha \in A \left\{ \lim_{m \to \infty} \sup_{n > m} J_\alpha(u_n, u_m) = 0 \right\} \right) \quad (2.2)
\]
and
\[\forall \alpha \in A \left\{ \lim_{m \to \infty} J_\alpha (v_m, u_m) = 0 \right\} \] (2.3)
\[\forall \alpha \in A \left\{ \lim_{m \to \infty} p_\alpha (v_m, u_m) = 0 \right\} \] (2.4)
the following holds
\[\forall \alpha \in A \left\{ \lim_{m \to \infty} p_\alpha (v_m, u_m) = 0 \right\} \] (2.5)
\[\forall \alpha \in A \left\{ \lim_{m \to \infty} p_\alpha (v_m, u_m) = 0 \right\} \] (2.6)

Remark 2.1. If \((X, \mathcal{P})\) is a quasi-gauge space, then \(\mathcal{P} \in \mathcal{J}_X^{\mathbb{R}}(X, \mathcal{P})\) and \(\mathcal{P} \in \mathcal{J}_X^{\mathbb{P}}(X, \mathcal{P})\), where \(\mathcal{J}_X^{\mathbb{R}}(X, \mathcal{P}) = \{ \mathcal{J}: \mathcal{J} \text{ is a left } \mathcal{J}\text{-family on } X \} \) and \(\mathcal{J}_X^{\mathbb{P}}(X, \mathcal{P}) = \{ \mathcal{J}: \mathcal{J} \text{ is a right } \mathcal{J}\text{-family on } X \} \), respectively.

One can prove the following proposition:

Proposition 2.1. Let \((X, \mathcal{P})\) be a Hausdorff quasi-gauge space and let \(\mathcal{J} = \{ J_\alpha : \alpha \in A \}\) be a left (right) \(\mathcal{J}\text{-family on } X\). Then
\[\forall x, y \in X \{ x \neq y \Rightarrow \exists \alpha \in A \{ J_\alpha (x, y) > 0 \lor J_\alpha (y, x) > 0 \} \} .\]

Proof. Assume that \(\mathcal{J}\) is a left \(\mathcal{J}\)-family and that there are \(x \neq y, x, y \in X\), such that \(\forall \alpha \in A \{ J_\alpha (x, y) = J_\alpha (y, x) = 0 \}\). Then \(\forall \alpha \in A \{ J_\alpha (x, y) = J_\alpha (y, x) = 0 \} \) since, by using property \((\mathcal{J}1)\) in Definition 2.1, it follows that \(\forall \alpha \in A \{ J_\alpha (x, y) \leq J_\alpha (x, y) + J_\alpha (y, x) = J_\alpha (x, y) \} \), Defining the sequences \((u_m, m \in \mathbb{N})\) and \((v_m, m \in \mathbb{N})\) in \(X\) by \(u_m = x\) and \(v_m = y\) or by \(u_m = y\) and \(v_m = x\) for \(m \in \mathbb{N}\), observing that \(\forall \alpha \in A \{ J_\alpha (x, y) = J_\alpha (y, x) = 0 \}\), and using property \((\mathcal{J}2)\) of Definition 2.1 to these sequences we see that \((2.1)\) and \((2.3)\) hold, and therefore \((2.5)\) is satisfied which gives \(\forall \alpha \in A \{ J_\alpha (u_m, v_m) = 0 \}\). But this is a contradiction since \((X, \mathcal{P})\) is Hausdorff and thus, by Definition 1.1, \(x \neq y \Rightarrow \exists \alpha \in A \{ J_\alpha (u_m, v_m) > 0 \lor J_\alpha (v_m, u_m) > 0 \}\). When \(\mathcal{J}\) is a right \(\mathcal{J}\)-family, then the proof is based on the analogous technique.

Remark 2.2. Let \((X, \mathcal{P})\) be a quasi-gauge space and let \(\mathcal{J} = \{ J_\alpha : \alpha \in A \}\) be a left (right) \(\mathcal{J}\text{-family on } X\).

(i) We say that a sequence \((u_m; m \in \mathbb{N})\) in \(X\) is left (right) \(\mathcal{J}\text{-Cauchy sequence in } X\) if \(\forall \alpha \in A \{ \lim_{m \to \infty} \sup_{n \geq m} J_\alpha (u_m, u_n) = 0 \} \) \(\forall \alpha \in A \{ \lim_{m \to \infty} \inf_{n \geq m} J_\alpha (u_m, u_n) = 0 \} \).

(ii) Let \(u \in X\) and let \((u_m; m \in \mathbb{N})\) be a sequence in \(X\). We say that \((u_m; m \in \mathbb{N})\) is left (right)-convergent to \(u\) if \(\lim_{m \to \infty} J_\alpha (u_m, u) = 0\) \(\exists \alpha \in A \{ \lim_{m \to \infty} J_\alpha (u_m, u) = 0 \} \) \(\exists \alpha \in A \{ \lim_{m \to \infty} J_\alpha (u_m, u) = 0 \} \).

(iii) We say that a sequence \((u_m; m \in \mathbb{N})\) in \(X\) is left (right) \(\mathcal{J}\text{-Cauchy sequence in } X\) if \(S_{\alpha \in A \{ \lim_{m \to \infty} J_\alpha (u_m, u) = 0 \}} \supseteq \mathcal{J}_X^{\mathbb{R}}(X, \mathcal{P})\).

(iv) If every left (right) \(\mathcal{J}\text{-Cauchy sequence in } X\) is left (right) \(\mathcal{J}\text{-convergent in } X\) (i.e., \(S_{\alpha \in A \{ \lim_{m \to \infty} J_\alpha (u_m, u) = 0 \}} \supseteq \mathcal{J}_X^{\mathbb{P}}(X, \mathcal{P})\)), then \((X, \mathcal{P})\) is called left (right) \(\mathcal{J}\text{-sequentially complete quasi-gauge space.}

Remark 2.2. (a) There exist examples of quasi-gauge spaces \((X, \mathcal{P})\) and left (right) \(\mathcal{J}\text{-family } \mathcal{J} \text{ on } X, \mathcal{J} \neq \mathcal{P}, \text{ such that } (X, \mathcal{P}) \text{ is left (right) } \mathcal{J}\text{-sequentially complete, but not left (right) } \mathcal{P}\text{-sequentially complete (see Section 5).}

(b) It is clear that if \((w_m; m \in \mathbb{N})\) is left (right) \(\mathcal{J}\text{-convergent in } X\), then \(S_{\alpha \in A \{ \lim_{m \to \infty} J_\alpha (w_m, w) = 0 \}} \supseteq \mathcal{J}_X^{\mathbb{R}}(X, \mathcal{P})\) for each sequence \((v_m; m \in \mathbb{N})\) of \((w_m; m \in \mathbb{N})\).

3. Statement of results

Let \(X\) be a nonempty set. If \(T : X \to X\), then, for each \(w_0 \in X\), we define a sequence \((w_m; m \in \mathbb{N})\) starting with \(w_0\) as follows \(m \in \mathbb{N} = T^{[m]}(w_0)\) where \(T^{[0]} = T \circ T \circ T \ldots \circ T \) (m-times) and \(T^{[0]} = I_X\) is an identity map on \(X\). By \(Fix(T)\) and \(Per(T)\) we denote the sets of all fixed points and periodic points of \(T : X \to X\), respectively, i.e., \(Fix(T) = \{ w \in X : w = T(w) \} \) and \(Per(T) = \{ w \in X : w = T^{[m]}(w) \} \) for some \(q \in \mathbb{N}\).

Definition 3.1. Let \((X, \mathcal{P})\) be a quasi-gauge space. Let the family \(\mathcal{J} = \{ J_\alpha : \alpha \in A \}\) of maps \(J_\alpha : X \times X \to [0, \infty)\), \(\alpha \in A\), be a left (right) \(\mathcal{J}\text{-family on } X\). We say that \(T \circ T \circ T \ldots \circ T \) (m-times) and \(T^{[0]} = I_X\) is an identity map on \(X\). By \(Fix(T)\) and \(Per(T)\) we denote the sets of all fixed points and periodic points of \(T : X \to X\), respectively, i.e., \(Fix(T) = \{ w \in X : w = T(w) \} \) and \(Per(T) = \{ w \in X : w = T^{[m]}(w) \} \) for some \(q \in \mathbb{N}\).

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Remark 3.1. If $(X, \mathcal{P})$ is left (right) $\mathcal{J}$-sequentially complete quasi-gauge space, then each $T : X \to X$ is left (right) $\mathcal{J}$-admissible.

Also, using Definition 2.2, we can define the following generalization of continuity.

Definition 3.2. Let $(X, \mathcal{P})$ be a quasi-gauge space, let $T : X \to X$ and let $q \in \mathbb{N}$. The map $T^{[q]}$ is said to be a left (right) $(S_{-}^{R-P}(m_{\mathcal{P}}; m_{\mathcal{Q}}) \neq \emptyset)$ and having subsequences $(w_{m} : m \in \mathbb{N})$ and $(u_{m} : m \in \mathbb{N})$ satisfying $\forall m\in\mathbb{N}\{v_{m} = T^{[q]}(u_{m})\}$, has the property $\exists_{w \in S_{-}^{R-P}(m_{\mathcal{P}}; m_{\mathcal{Q}})}\{w = T^{[q]}(w)\}$ (\exists_{w \in S_{-}^{R-P}(m_{\mathcal{P}}; m_{\mathcal{Q}})}\{w = T^{[q]}(w)\})

Using the above, we can now state the main results of this paper.

Theorem 3.1. Let $(X, \mathcal{P})$ be a quasi-gauge space, let the family $\mathcal{J} = \{J_{\alpha} : \alpha \in A\}$ of maps $J_{\alpha} : X \times X \to [0, \infty)$, $\alpha \in A$, be a left (right) $\mathcal{J}$-family on $X$ and let $T : X \to X$. Assume that:

(H1) $T$ is left (right) $\mathcal{J}$-admissible on $X$; and

(H2) $T$ is a quasi $\mathcal{J}$-contraction of Leader type on $X$, i.e. $\forall x, y \in X \forall \alpha \in A \exists \varepsilon > 0 \exists k \in \mathbb{N} \exists \eta > 0 \exists \alpha \in A \exists T^{[\alpha]}(x, T^{[\alpha]}(y)) < \varepsilon + \eta \Rightarrow J_{\alpha}(T^{[\alpha]}(x), T^{[\alpha]}(y)) < \varepsilon$.

The following statements hold:

(A) For each $w^{0} \in X$ the sequence $(w^{m} : m \in \{0\} \cup \mathbb{N})$ is left (right) $\mathcal{P}$-convergent in $X$; i.e., $(a_{1}) \forall w^{0} \in X \{S_{-}^{R-P}(m_{\mathcal{P}}; m_{\mathcal{Q}}) \neq \emptyset\}$

(B) Assume that: (b1) $T^{[q]}$ is left (right) $\mathcal{P}$-quasi-convergent on $X$ for some $q \in \mathbb{N}$. Then: (b1) $\exists \eta \in X \exists \varepsilon \in \mathbb{R} \exists k \in \mathbb{N} \exists J_{\alpha}(T^{[\alpha]}(x), T^{[\alpha]}(y)) < \varepsilon + \eta \Rightarrow J_{\alpha}(T^{[\alpha]}(x), T^{[\alpha]}(y)) < \varepsilon$.

(C) Assume that: (C1) $(X, \mathcal{P})$ is a Hausdorff space; and (C2) there exists $q \in \mathbb{N}$ such that $\forall J_{\alpha}(T^{[\alpha]}(x), T^{[\alpha]}(y)) < \varepsilon$. Then: (c1) $F(x, y) = T^{[k]}(x)$ for some $w \in X$; (c2) $\forall w \in X \{S_{-}^{R-P}(m_{\mathcal{P}}; m_{\mathcal{Q}}) \neq \emptyset\}$

Theorem 3.2. Let $(X, \mathcal{P})$ be a quasi-gauge space, let the family $\mathcal{J} = \{J_{\alpha} : \alpha \in A\}$ of maps $J_{\alpha} : X \times X \to [0, \infty)$, $\alpha \in A$, be a left (right) $\mathcal{J}$-family on $X$ and let $T : X \to X$. Assume that:

(H3) $T$ is left (right) $\mathcal{J}$-admissible on $X$; and

(H4) $T$ is a weak quasi $\mathcal{J}$-contraction of Leader type on $X$, i.e., $\exists \epsilon \in X \exists \alpha \in A \exists \varepsilon \in \mathbb{R} \exists k \in \mathbb{N} \exists \eta \in \mathbb{R} \exists \epsilon \in X \exists T^{[\alpha]}(x, T^{[\alpha]}(y)) < \varepsilon + \eta \Rightarrow J_{\alpha}(T^{[\alpha]}(x), T^{[\alpha]}(y)) < \varepsilon$.

The following statements hold:

(D) For each $w^{0} \in X$ the sequence $(w^{m} : m \in \{0\} \cup \mathbb{N})$ is left (right) $\mathcal{P}$-convergent in $X$; i.e., $(d_{1}) S_{-}^{R-P}(m_{\mathcal{P}}; m_{\mathcal{Q}}) \neq \emptyset$.

(E) Assume that: (E1) $T^{[q]}$ is left (right) $\mathcal{P}$-quasi-convergent on $X$ for some $q \in \mathbb{N}$. Then: (e1) $\exists \eta \in \mathbb{R} \exists \alpha \in A \exists J_{\alpha}(T^{[\alpha]}(x), T^{[\alpha]}(y)) < \varepsilon$.

Remark 3.2. (i) It is worth noticing that each map $T$ satisfying (H2) satisfies (H4). (ii) If the condition (B1) holds, then, by (b1), the condition (C2) holds.

4. Proofs of Theorems 3.1 and 3.2

We prove Theorems 3.1 and 3.2 in the case of “left”; we omit the proofs in the case of “right” since they are based on the analogous technique.

Proof of Theorem 3.1. For each $w^{0}, v^{0} \in X$, $\alpha \in A$ and $k \in \mathbb{N}$, we define

\[ \delta_{\mathcal{J}}^{\alpha} : [0, \infty) \to [0, \infty) \]
\[ \gamma_{\mathcal{J}}^{\alpha} : [0, \infty) \to [0, \infty) \]
\[ \Delta_{\mathcal{J}}^{\alpha} : [0, \infty) \to [0, \infty) \]
\[ \Gamma_{\mathcal{J}}^{\alpha} : [0, \infty) \to [0, \infty) \]

where $w^{m} = T^{[m]}(w^{0})$ and $v^{m} = T^{[m]}(v^{0})$, $m \in \{0\} \cup \mathbb{N}$.
Part (A). The proof will be broken into six steps.

Step A.I. The following property holds

\[ \forall \omega, \omega' \in X \forall \alpha, \epsilon > 0 \exists \eta > 0 \{ \exists \gamma \in \mathbb{N} \forall \epsilon > 0 \{ J_\alpha (w, v) < \epsilon + \eta \Rightarrow J_\alpha (w^{r+1}, v^{r+1}) < \epsilon \} \} \wedge \exists \gamma \in \mathbb{N} \forall \epsilon > 0 \{ J_\alpha (v, w) < \epsilon + \eta \Rightarrow J_\alpha (v^{r+2}, w^{r+2}) < \epsilon \} \].

(4.5)

Indeed, let \( w^0, v^0 \in X \) be arbitrary and fixed. If we define the sequences \( (w^m = T_{m}^{\omega}(w^0); \ m \in \{0\} \cup \mathbb{N}) \) and \( (v^m = T_{m}^{\omega}(v^0); \ m \in \{0\} \cup \mathbb{N}) \) and assume that \( \alpha \in A \) and \( \epsilon > 0 \) are arbitrary and fixed, then, using (H2) for \( x = w^0 \) and \( y = v^0 \), we obtain \( \exists \eta \geq 0 \forall \epsilon > 0 \{ J_\alpha (w^0, v^0) < \epsilon + \eta \Rightarrow J_\alpha (w^{r+1}, v^{r+1}) < \epsilon \} \) and, using (H2) for \( x = v^0 \) and \( y = w^0 \), we obtain \( \exists \eta \geq 0 \forall \epsilon > 0 \{ J_\alpha (v^0, w^0) < \epsilon + \eta \Rightarrow J_\alpha (v^{r+2}, w^{r+2}) < \epsilon \} \). Hence, putting \( \eta = \min(\eta_1, \eta_2) \), we have \( \exists \gamma \in \mathbb{N} \forall \epsilon > 0 \{ J_\alpha (w^0, v^0) < \epsilon + \eta \Rightarrow J_\alpha (w^{r+1}, v^{r+1}) < \epsilon \} \) and \( \exists \gamma \in \mathbb{N} \forall \epsilon > 0 \{ J_\alpha (v^0, w^0) < \epsilon + \eta \Rightarrow J_\alpha (v^{r+2}, w^{r+2}) < \epsilon \} \). This gives (4.5).

Step A.II. We show that

\[ \forall \omega, \omega' \in X \forall \alpha, \epsilon > 0 \exists \eta > 0 \{ J_\alpha (w^0, v^0) = 0 \} \].

(4.6)

and

\[ \forall \omega, \omega' \in X \forall \alpha, \epsilon > 0 \exists \eta > 0 \{ J_\alpha (w^0, v^0) = 0 \} \].

(4.7)

Indeed, suppose that (4.6) does not hold; that is,

\[ \exists \omega, \omega' \in X \exists \alpha, \epsilon > 0 \exists \eta > 0 \{ J_\alpha (w^0, v^0) = \epsilon > 0 \}. \]

(4.8)

With this choice of \( u^0, z^0, \alpha_0 \) and \( \epsilon_0 \) we can use (4.5) and then there exists \( \eta_0 > 0 \) and \( r_0 \in \mathbb{N} \), such that

\[ \forall \epsilon > 0 \{ J_\alpha (u^0, z^0) < \epsilon + \eta_0 \Rightarrow J_\alpha (u^{r_0}, z^{r_0}) < \epsilon + \eta_0 \} \].

(4.9)

Additionally, (4.8) and (4.1) imply that there exists \( n_0 \in \mathbb{N} \) such that \( \Delta_{\alpha} (\alpha_0, k_0) (u_0, z_0, n_0) \in (0, \epsilon_0) \). By (4.3), gives

\[ \forall n_0 \in \mathbb{N} \exists \epsilon_0 > 0 \{ J_\alpha (u^0, z^0) < \epsilon_0 + \eta_0 \}. \]

Consequently, by (4.9), we get \( \forall n_0 \in \mathbb{N} \exists \epsilon_0 > 0 \{ J_\alpha (u^{r_0}, z^{r_0}) < \epsilon_0 + \eta_0 \} \) which we can write as \( \forall n_0 \in \mathbb{N} \exists \epsilon_0 > 0 \{ J_\alpha (u^0, z^0) < \epsilon_0 \}. \) This, by (4.3), gives that \( \Delta_{\alpha} (\alpha_0, k_0) (u^0, z^0, n_0) \in (0, \epsilon_0) \). However, hence and from (4.8) and (3.1) it follows that \( \epsilon_0 = \delta_{\alpha}(\alpha_0, k_0) (u_0, z_0) = \inf(\Delta_{\alpha} (\alpha_0, k_0) (u_0, z_0), n) \leq \Delta_{\alpha} (\alpha_0, k_0) (u_0, z_0, n_0 + r_0) < \epsilon_0 \) which is impossible. Therefore, (4.6) holds. Now, suppose that (4.7) does not hold, i.e.

\[ \exists \omega, \omega' \in X \exists \alpha, \epsilon > 0 \exists \eta > 0 \{ J_\alpha (u^0, z^0) = \epsilon > 0 \}. \]

(4.10)

Of course, for this \( u^0, z^0, \alpha_0 \) and \( \epsilon_0 \), by (4.5), there exist \( \eta_0 > 0 \) and \( r_0 \in \mathbb{N} \), such that

\[ \forall \epsilon > 0 \{ J_\alpha (z^0, u^0) < \epsilon + \eta_0 \Rightarrow J_\alpha (z^{r_0}, u^{r_0}) < \epsilon + \eta_0 \} \].

(4.11)

In addition, by (4.10) and (4.2), there exists \( n_0 \in \mathbb{N} \) such that \( \Delta_{\alpha} (\alpha_0, k_0) (u_0, z_0, n_0) \in (0, \epsilon_0) \). Hence, using (4.4), we conclude that \( \forall n_0 \in \mathbb{N} \exists \epsilon_0 > 0 \{ J_\alpha (z^0, u^0) < \epsilon_0 + \eta_0 \} \). This, using (4.11), gives that \( \forall n_0 \in \mathbb{N} \exists \epsilon_0 > 0 \{ J_\alpha (z^{r_0}, u^{r_0}) < \epsilon_0 + \eta_0 \} \). This means, by (4.3), that \( \Delta_{\alpha} (\alpha_0, k_0) (u_0, z_0, n_0 + r_0) < \epsilon_0 \). Consequently, \( \epsilon_0 \) does not hold, i.e.

\[ \exists \omega, \omega' \in X \exists \alpha, \epsilon > 0 \exists \eta > 0 \{ J_\alpha (u^0, z^0) = \epsilon > 0 \}. \]

(4.7)

Step A.III. Let \( w^0, v^0 \in X \), \( \alpha \in A \) and \( \epsilon > 0 \) be arbitrary and fixed and let \( \eta_0 > 0 \) and \( r_1, r_2 \in \mathbb{N} \) satisfy (4.5). Denote \( r = \max(r_1, r_2) \). We show that if there exists \( n_0 \in \mathbb{N} \) such that

\[ \max(\Delta_{\alpha}, \eta_0 (w_0, v_0, n_0), \Delta_{\alpha} (w_0, v_0, n_0)) < \min(\epsilon, \eta/2) \].

(4.12)

then

\[ \forall \epsilon > 0 \{ J_\alpha (w^0, v^0) < 3 \epsilon \}. \]

(4.13)

Let \( n_0 \) satisfy (4.12) and let us write \( \Delta' = \Delta_{\alpha} (\alpha_0, k_0) (w_0, v_0, n_0) \) and \( \Gamma^i = \Gamma_{\alpha, \alpha_0} (w_0, v_0, n_0) \), \( i = 1, 2 \). Then, by (4.3), (4.4) and definition of \( r \), we obtain that \( \max(\Delta_{\alpha} (w_0, v_0, n_0), \Delta_{\alpha} (w_0, v_0, n_0)) < \min(\Delta_{\alpha} (w_0, v_0, n_0), \Gamma_{\alpha, \alpha_0} (w_0, v_0, n_0)) \) and taking this into account, we see that (4.12) implies

\[ \max(\Delta^1, \Delta^2, \Gamma^1, \Gamma^2) < \min(\epsilon, \eta/2) \].

(4.14)

To establish

\[ \forall \epsilon > 0 \{ J_\alpha (w^{r_0+1}, v^0) < \epsilon \}. \]

(4.15)
it suffices to show that
\[ L = \emptyset \]  
where
\[ L = \{ l \in \mathbb{N} : l \geq n_0 \land J_\alpha (w^{n_0+t_1}, v^l) \geq \varepsilon \}. \]  
Suppose that \( L \neq \emptyset \) and let \( l_0 = \min L \); of course \( l_0 \geq n_0 \). It is clear that, by (4.17),
\[ \forall n_0 \leq l_0, J_\alpha (w^{n_0+t_1}, v^l) < \varepsilon. \]  
Now, we see that \( l_0 > n_0 + r_1 \). Otherwise, \( l_0 \leq n_0 + r_1 \) and, by virtue of (4.3) and (4.14), we get \( J_\alpha (w^{n_0+t_1}, v^{l_0}) \leq \max\{J_\alpha (w^i, v^j) : n_0 \leq i, j \leq n_0 + r_1\} = \Delta_{J_\alpha, \alpha, r_1}(w^0, v^0, n_0) < \min\{\varepsilon, \eta/2\} \leq \varepsilon \), which, by the definitions of \( l_0 \) and \( L \), is impossible. Hence it follows that \( n_0 < l_0 - r_1 < l_0 \) and, consequently, using (4.18), we conclude that
\[ J_\alpha (w^{n_0+t_1}, v^{l_0-r_1}) < \varepsilon. \]  
Next, using (J1), (4.3), (4.4), (4.19) and (4.14), we get \( J_\alpha (w^{n_0}, v^{l_0-r_1}) \leq J_\alpha (w^{n_0}, v^{n_0}) + J_\alpha (v^{n_0}, w^{n_0+r_1}) + J_\alpha (w^{n_0}, v^{l_0-r_1}) = \Delta_{J_\alpha, \alpha, r_1}(w^0, v^0, n_0) + \Delta_{J_\alpha, \alpha, r_1}(w^0, v^0, n_0) + \varepsilon < \eta/2 + \eta/2 + \varepsilon = \varepsilon + \eta \). Hence, since, by assumption, \( r_1 \) satisfies (4.5), we get \( J_\alpha (w^{n_0-r_1}, v^0) < \varepsilon \), which, by definitions of \( l_0 \) and \( L \), is impossible. Consequently, (4.16) holds which implies (4.15).

We can show in a similar way that
\[ \forall s \geq s_0 \{ J_\alpha (w^s, v^{n_0+t_2}) < \varepsilon \}. \]  
In fact, suppose that \( S \neq \emptyset \) where
\[ S = \{ s \in \mathbb{N} : s \geq n_0 \land J_\alpha (w^s, v^{n_0+t_2}) \geq \varepsilon \} \]  
and let \( s_0 = \min S \); of course \( s_0 \geq n_0 \). Then, by (4.21),
\[ \forall n_0 \leq s < s_0 \{ J_\alpha (w^s, v^{n_0+t_2}) < \varepsilon \}. \]  
We see that \( s_0 > n_0 + r_2 \). Indeed, if \( s_0 \leq n_0 + r_2 \), then, since \( s_0 \geq n_0 \), we see that \( J_\alpha (w^{n_0}, v^{n_0+t_2}) \leq \max\{J_\alpha (w^i, v^j) : n_0 \leq i, j \leq n_0 + r_2\} = \Delta_{J_\alpha, \alpha, r_2}(w^0, v^0, n_0) < \min\{\varepsilon, \eta/2\} \leq \varepsilon \) which, by (4.21) and definition of \( s_0 \), is impossible. Therefore, \( n_0 < s_0 - r_2 < s_0 \) and, by (4.22),
\[ J_\alpha (w^{s_0-t_2}, v^{n_0+t_2}) < \varepsilon. \]  
Consequently, using (J1), (4.23), (4.4), (4.3) and (4.14), we have \( J_\alpha (w^{s_0-t_2}, v^{n_0}) \leq J_\alpha (w^{s_0-t_2}, v^{s_0+t_2}) + J_\alpha (v^{s_0+t_2}, v^{n_0}) + J_\alpha (w^{s_0-t_2}, v^{s_0}) < \varepsilon + \Delta_{J_\alpha, \alpha, r_2}(w^0, v^0, n_0) + \Delta_{J_\alpha, \alpha, r_2}(w^0, v^0, n_0) + \varepsilon < \eta/2 + \eta/2 + \varepsilon = \varepsilon + \eta \). Hence, using (4.5) we get \( J_\alpha (w^{s_0}, v^{n_0}) < \varepsilon \). This, by the definitions of \( s_0 \) and \( S \), is impossible. Consequently, \( S = \emptyset \) which gives (4.20).

Let now \( s, l \geq n_0 \) be arbitrary and fixed. Then, by (J1), (4.20), (4.15), (4.4) and (4.12), we obtain \( J_\alpha (w^s, v^l) \leq J_\alpha (w^s, v^{n_0+t_2}) + J_\alpha (v^{n_0+t_2}, v^{n_0+t_2}) + J_\alpha (w^{n_0+t_1}, v^l) \leq \varepsilon + \max\{J_\alpha (w^s, v^l) : n_0 \leq s, l \leq n_0 + r_1\} = 2\varepsilon + \Gamma_{J_\alpha, \alpha, r_1}(w^0, v^0, n_0) < 3\varepsilon. \) Therefore, (4.13) holds.

**Step A.IV. We show that**
\[ \forall w^u \in X \forall \alpha, \varepsilon \geq 0 \exists n_0 \in \mathbb{N} \forall u, l \geq n_0 \{ J_\alpha (w^u, v^l) < \varepsilon/2 \}. \]  
Indeed, let \( w^0 \in X \) be arbitrary and fixed and let \( (v^m) : m \in [0] \cup \mathbb{N} \) by a sequence defined by formulae \( v^m = w^m, m \in [0] \cup \mathbb{N} \). We see that for sequences \( (w^m) : m \in [0] \cup \mathbb{N} \) and \( (v^m) : m \in [0] \cup \mathbb{N} \) the property (4.5) holds, i.e.
\[ \forall \alpha, \varepsilon \exists s > 0 \exists n_0 \in \mathbb{N} \forall u, l \geq n_0 \{ J_\alpha (w^u, v^l) < \varepsilon \} \]  
and, by (4.3) and (4.4), we have
\[ \forall \alpha, \varepsilon \exists s > 0 \exists n_0 \in \mathbb{N} \{ \Delta_{J_\alpha, \alpha, k}(w^0, v^0, n) = \Gamma_{J_\alpha, \alpha, k}(w^0, v^0, n) \}. \]  
Moreover, by Step A.II, we have
\[ \forall \alpha, \varepsilon \exists n_0 \{ \delta_{J_\alpha, \alpha, k}(w^0, v^0) = \gamma_{J_\alpha, \alpha, k}(w^0, v^0) = 0 \}. \]  
Let now \( w^0 \in X \), \( \alpha_0 \in A \) and \( \varepsilon_0 > 0 \) be arbitrary and fixed. By (4.25) there exist \( \eta_0 > 0 \) and \( r_0 \in \mathbb{N} \) such that \( \forall s, t \geq 0 \{ J_{\alpha_0, \alpha_0, r_0}(w^s, w^t) < \varepsilon_0 + n_0 \Rightarrow J_{\alpha_0, \alpha_0, r_0}(w^{s+t}, v^{t+r}) < \varepsilon_0 \} \) and, in particular, (4.27) implies
\[ \delta_{J_\alpha, \alpha_0, r_0}(w^0, w^0) = \gamma_{J_\alpha, \alpha_0, r_0}(w^0, w^0) = 0. \]
By (4.28), using (4.26), (4.1) and (4.2), there exists $n_0 \in \mathbb{N}$, such that

$$\Delta_{J; \alpha_0, \eta_0}(w^0, w^0, n_0) = \Gamma_{J; \alpha_0, \eta_0}(w^0, w^0, n_0) < \min\{\varepsilon/6, \eta_0/2\}. \quad (4.29)$$

From (4.29), using Step A.III, we get $\forall \omega_i \exists n_0 \{J_{\alpha_0}(w^i, w^i) < \varepsilon_0/2\}$. This proved that (4.24) holds.

Step A.V. We show that

$$\forall \omega_i \exists n_0 \{\limsup_{n \to \infty} J_{\alpha_0}(w^n, w^m) = 0\}. \quad (4.30)$$

Indeed, (4.24) implies, in particular, that $\forall \omega_i \exists n_0 \{\sup_{m \geq n} J_{\alpha_0}(w^i, w^m) \leq \varepsilon / 2 < \varepsilon\}$. Therefore, (4.30) holds.

Step A.VI. For each $w^0 \in X$, $S_{L^{-P}}^{(w^0) \in \{m(0) | \infty\} \neq \emptyset}$.

Indeed, let $w^0 \in X$ be arbitrary and fixed. By Step A.V, Definition 2.2(iii) and hypothesis (H1), we get that $(w^m): m \in (0 \cup \mathbb{N})$ is left $J$-convergent in $X$, i.e., there exists a nonempty set $S_{J^{-P}}^{(w^m) \in \{m(0) | \infty\} \subseteq X}$, such that for all $w \in S_{J^{-P}}^{(w^m) \in \{m(0) | \infty\} \subseteq X}$, we have $\forall m \in \{m(0) | \infty\} \subseteq X$, and using Step A.V and Definition 2.1 for these sequences we conclude that $\forall \omega_i \exists n_0 \{\sup_{m \geq n} J_{\alpha_0}(w^i, w^m) < \varepsilon\}$, i.e., $\lim_{m \to \infty} w^m = w$. This means that $S_{J^{-P}}^{(w^m) \in \{m(0) | \infty\} \neq \emptyset}$. Thus, (a1) holds.

Part (B). The proof will be broken into two steps.

Step B.I. We show that (b1) and (b2) hold. Indeed, let $w^0 \in X$ be arbitrary and fixed. By (a1), $S_{J^{-P}}^{(w^m) \in \{m(0) | \infty\} \neq \emptyset}$. Next, we have $w^{m+q} = J_{\omega_0}(w, w) \in \mathbb{N}$ for $k = 1, 2, \ldots, q$ and $m \in \mathbb{N}$. Let, in the sequel, $k = 1, 2, \ldots, q$ be arbitrary and fixed. Defining $(w_m) = w^{m+q} = J_{\omega_0}(w, w)$, we see that $(w_m, m \in \mathbb{N}) \subseteq T^Q(X)$, such that $S_{J^{-P}}^{(w^m) \in \{m(0) | \infty\} \neq \emptyset}$, for all $m \in \mathbb{N}$. Therefore, (4.31) gives

$$\forall \omega_i \exists n_0 \{J_{\alpha_0}(T^{[\eta_0]}(w), T^{[\eta_0]}(w)) < \varepsilon_0 \Rightarrow J_{\alpha_0}(T^{[\eta_0]}(w), T^{[\eta_0]}(w)) < \varepsilon_0\}. \quad (4.31)$$

Since $w \in F_X(T^{[\eta_0]}(w))$ (which gives $\forall \omega_0 \exists \eta_0 \{J_{\alpha_0}(T^{[\eta_0]}(w), T^{[\eta_0]}(w)) = J_{\alpha_0}(T^{[\eta_0]}(w), T^{[\eta_0]}(w)) < \varepsilon_0\}$, hence, using (4.31) for $s = q$ and $l = q + 1$, we get $J_{\alpha_0}(T^{[\eta_0]}(w), T^{[\eta_0]}(w)) < \varepsilon_0$, and using (4.31) for $s = q + r_0$ and $l = q + r_0 + 1$, we have $J_{\alpha_0}(T^{[\eta_0]}(w), T^{[\eta_0]}(w)) < \varepsilon_0$, and by induction, we deduce that (4.31) gives

$$\forall \omega_i \exists n_0 \{J_{\alpha_0}(T^{[\eta_0]}(w), T^{[\eta_0]}(w)) < \varepsilon_0 \Rightarrow J_{\alpha_0}(T^{[\eta_0]}(w), T^{[\eta_0]}(w)) < \varepsilon_0\}. \quad (4.32)$$

Now, we see that (4.32) implies, in particular, for $m = q$, that $\varepsilon_0 = J_{\alpha_0}(w, w) \in \mathbb{N}$, such that $J_{\alpha_0}(w, w) < \varepsilon_0$. It is absurd. Therefore, $J_{\alpha_0}(w, w) = 0$. Similarly, we prove that $J_{\alpha_0}(T(w), T(w)) = 0$. We proved that (b3) holds.

Part (C). Assertions of (c1)-(c2) hold. The proof will be broken into four steps.

Step C.I. We show that $F_X(T^{[\eta_0]}(w)) = F_X(T) \neq \emptyset$. Indeed, let $w \in F_X(T^{[\eta_0]}(w))$. Then, by (b3), $\forall \omega_i \exists \eta_0 \{J_{\alpha_0}(w, w) = J_{\alpha_0}(T(w), T(w)) = 0\}$. By (C1) and Proposition 2.1, this gives $w = T(w)$, i.e., $w \in F_X(T)$. Consequently, $F_X(T^{[\eta_0]}(w)) = F_X(T) \neq \emptyset$.

Step C.II. We show that (c1) holds. Indeed, we first see that $F_X(T) = \{w\}$ for some $w \in X$. Otherwise, $u, v \in F_X(T)$ and $u \neq v$ for some $u, v \in X$. Therefore, $u, v \neq v$ for some $u, v \in X$. To see that by Step C.I, $F_X(T) = \{w\}$. Then, by Proposition 2.1, there exists $\alpha_0 \in \mathcal{A}$ such that $J_{\alpha_0}(u, v) > 0$ or $J_{\alpha_0}(u, v) < 0$. Suppose $J_{\alpha_0}(u, v) > 0$. Then, for $\varepsilon_0 = J_{\alpha_0}(u, v) > 0$, by (H2), there exist $\eta_0 > 0$ and $r_0 \in \mathbb{N}$ such that

$$\forall \omega_i \exists n_0 \{J_{\alpha_0}(T^{[\eta_0]}(u), T^{[\eta_0]}(v)) < \varepsilon_0 \Rightarrow J_{\alpha_0}(T^{[\eta_0]}(u), T^{[\eta_0]}(v)) < \varepsilon_0\}. \quad (4.33)$$

However, for each $s, l \in \mathbb{N}$, we have $J_{\alpha_0}(T^{[\eta_0]}(u), T^{[\eta_0]}(v)) = J_{\alpha_0}(u, v) = \varepsilon_0 < \varepsilon_0$, and thus, by (4.33), we get $0 < \varepsilon_0 = J_{\alpha_0}(u, v) < J_{\alpha_0}(T^{[\eta_0]}(u), T^{[\eta_0]}(v)) < \varepsilon_0$, which is impossible. We obtain a similar conclusion in the case when $J_{\alpha_0}(u, v) < 0$. Therefore, $F_X(T) = \{w\}$ for some $w \in X$.

The above mean that the assertion (c1) holds.

Step C.III. Assertion (c2) holds. This is a consequence of (c1) and (b1).

Step C.IV. We show that $\forall \omega_i \exists \eta_0 \{J_{\alpha_0}(w, w) = 0\}$ where $F_X(T) = \{w\}$. Indeed, if we assume that there exists $\alpha_0 \in \mathcal{A}$ such that $J_{\alpha_0}(w, w) > 0$, then, denoting $\varepsilon_0 = J_{\alpha_0}(w, w) > 0$, by (H2), there exist $\eta_0 > 0$ and $r_0 \in \mathbb{N}$, such that

$$\forall \omega_i \exists \eta_0 \{J_{\alpha_0}(w, w) < \varepsilon_0 \Rightarrow J_{\alpha_0}(T^{[\eta_0]}(w), T^{[\eta_0]}(w)) < \varepsilon_0\}. \quad (4.34)$$
However, for each $s, t \in \mathbb{N}$, we have $J_{\alpha_0}(T[s](w), T[t](w)) = J_{\alpha_0}(w, w) = \varepsilon_0 < \varepsilon_0 + \eta_0$. Thus, using (4.34), we obtain that $0 < \varepsilon_0 = J_{\alpha_0}(w, w) = J_{\alpha_0}(T[s+\alpha_1](w), T[t+\alpha_1](w)) < \varepsilon_0$, which is impossible. This gives (C3).

The proof of Theorem 3.1 is complete. □

**Proof of Theorem 3.2.** Assume that the condition (H4) holds. Then, defining $(w^m: m \in [0] \cup \mathbb{N})$ where $w^0 = x \in X$ and $x$ is such as in (H4) and next, using a similar argumentation as in the proof of Theorem 3.1 for this sequence $(w^m: m \in [0] \cup \mathbb{N})$, we have the assertions. □

5. Examples

Now, we provide some examples to illustrate the concepts introduced so far.

**Example 5.1.** Let $X = [0, 1]$, let $A = \{1/2^n: n \in \mathbb{N}\}$ and let $\mathcal{P} = \{p\}$ where $p : X \times X \to [0, \infty)$ is of the form

$$p(x, y) = \begin{cases} 0 & \text{if } x = y \text{ or } (x, y) \cap A = \{x, y\}, \\ 1 & \text{if } x \neq y \text{ and } (x, y) \cap A \neq \{x, y\}. \end{cases} \quad (5.1)$$

(I.1) The map $p$ is quasi-pseudometric on $X$ and $(X, \mathcal{P})$ is the quasi-gauge space. Indeed, from (5.1), we have that $p(x, x) = 0$ for each $x \in X$ and thus the condition $(P1)$ holds.

Now, it is worth noticing that the condition $(P2)$ does not hold only if there exist $x_0, y_0, z_0 \in X$ such that $p(x_0, z_0) > p(x_0, y_0) + p(y_0, z_0)$. This inequality is equivalent to $1 > 0 = p(x_0, y_0) + p(y_0, z_0)$, where

$$p(x_0, z_0) = 1, \quad p(x_0, y_0) = 0 \quad \text{and} \quad p(y_0, z_0) = 0. \quad (5.2)$$

By (5.1), condition (5.3) implies $[(x_0 = y_0) \lor ((x_0, y_0) \subset A)]$ and $[(y_0 = z_0) \lor ((y_0, z_0) \subset A)]$. We consider the following two cases:

Case 1. If $x_0 = y_0$ and $y_0 = z_0$, then $x_0 = z_0$ which, by (5.1), implies $p(x_0, z_0) = 0$. By (5.2) this is absurd.

Case 2. If $(x_0 = y_0) \land (y_0 = z_0) \subset A$ or $(x_0 = y_0) \subset A \land y_0 = z_0$ or $(x_0, y_0) \subset A \land (y_0, z_0) \subset A)$, then $x_0, y_0 \in A = \{x_0, z_0\}$. Hence, by (5.1), $p(x_0, z_0) = 0$. By (5.2) this is absurd.

Thus, the condition $(P2)$ holds.

We proved that $p$ is quasi-pseudometric on $X$ and $(X, \mathcal{P})$ is the quasi-gauge space.

(I.2) The quasi-gauge space $(X, \mathcal{P})$ is not Hausdorff. Indeed, for $x = 1/16$ and $y = 1/4$ we have $x \neq y$ and $(x, y) \cap A = \{x, y\}$. Hence, by (5.1), we obtain $p(x, y) = p(y, x) = 0$. This, by Definition 1.1(v), means that $(X, \mathcal{P})$ is not Hausdorff.

**Example 5.2.** Let $X = [0, 1] \subset \mathbb{R}$, let $\mathcal{P} = \{p\}$ where $p$ is defined as in Example 5.1 and let $T : X \to X$ where

$$T(x) = \begin{cases} 1/2 & \text{if } x \in [0, 1/4], \\ 1/4 & \text{if } x \in (1/4, 1]. \end{cases} \quad (5.4)$$

(I.1) The pair $(X, \mathcal{P})$ is a not a Hausdorff quasi-gauge space. This is a consequence of (I.1) and (I.2).

(I.2) The space $(X, \mathcal{P})$ is a left $\mathcal{P}$-sequentially complete. Indeed, let $(u_m: m \in \mathbb{N})$ be a left $\mathcal{P}$-Cauchy sequence in $X$. By (5.1), without loss of generality, we may assume that

$$\forall_0 < \varepsilon < 1 \exists k_0 \in \mathbb{N} \forall m, n \in \mathbb{N} : k_0 < m, n \in \mathbb{N} \{p(u_m, u_n) = 0 < \varepsilon < 1\}. \quad (5.5)$$

Now, we have the following two cases:

Case 1. Let $\forall m \geq k_0 \in \mathbb{N}$. By (5.1), in particular, we have that $\forall m > k_0 \in \mathbb{N}$, $p(1/2, u_m) = 0$. This gives, 1/2 $\in S_{u_m}^{\mathcal{P}}(m \in [0] \cup \mathbb{N})$, i.e. $S_{u_m}^{\mathcal{P}}(m \in [0] \cup \mathbb{N}) \neq \emptyset$.

Case 2. Let $\forall m \in \mathbb{N} : k_0 < \alpha_1 \in \mathbb{N}$, $u_m \neq \emptyset$. Then we have the following two subcases: Subcase 2(a) If $\forall m \in \mathbb{N} : k_0 < \alpha_1 \in \mathbb{N}$, then, by (5.1), we get $\forall m \in \mathbb{N} : k_0 < \alpha_1 \neq u_m$, which implies $u_m \in S_{u_m}^{\mathcal{P}}(m \in [0] \cup \mathbb{N})$, i.e. $S_{u_m}^{\mathcal{P}}(m \in [0] \cup \mathbb{N}) \neq \emptyset$; Subcase 2(b) if $\forall m \in \mathbb{N} : k_0 < \alpha_1 \in \mathbb{N}$, then, by (5.1), $p(u_m, u_m) = 1$. However, since $k_0 < m_0$ and $m_0 < m_1$, this, by (5.5), implies $p(u_m, u_m) = 0$. This is absurd.

We proved that if (5.5) holds, then $S_{u_m}^{\mathcal{P}}(m \in [0] \cup \mathbb{N}) \neq \emptyset$. By Definition 2.2(iii), the sequence $(u_m: m \in \mathbb{N})$ is left $\mathcal{P}$-convergent in $X$.

(I.3) For $\mathcal{F} = \mathcal{P}$ the assumption (H1) of Theorem 3.1 holds, i.e. the map $T$ is $\mathcal{P}$-admissible. This follows from Definition 3.1, Remark 3.1 and (I.2).

(I.4) For $\mathcal{F} = \mathcal{P}$ the assumption (H2) of Theorem 3.1 holds. Indeed, from (5.4) we get $\forall x \in X \forall y \in \mathbb{N} : T[y](x) \in [1/4, 1/2] \subset A$. This, by (5.1), implies $\forall x, y \in X \forall y \in \mathbb{N} : p(T[y](x), T[y](y)) = 0$.

(I.5) The map $T$ is not left $\mathcal{P}$-quasi-closed on $X$. Indeed, let a sequence $(w_m: m \in \mathbb{N})$ in $X$ be of the form: $w_m = 1/4$ if $m$ is even; $w_m = 1/2$ if $m$ is odd. Since $\forall m \in \mathbb{N} : w_m \in A$ thus, by (5.1), $\forall w \in A : p(w, w_m) = 0$ and $\forall w \in X \setminus A : p(w, w_m) = 1$. Hence
\[ S_{(w_m; m \in \mathbb{N})} \subseteq X \text{ and let (}w_m; m \in \mathbb{N}\text{)} be an arbitrary and fixed sequence in } T^2(X) = \{1/4, 1/2\} \text{ which is left } T\text{-convergent to each point of a nonempty set } S_{(w_m; m \in \mathbb{N})} \subseteq X \text{ and having subsequences } (v_m; m \in \mathbb{N}) \text{ satisfying } \forall m \in \mathbb{N}\{v_m = T(u_m)\}. \]

By (5.6) and (5.1), we conclude that \(1/4, 1/2 \subseteq S_{(w_m; m \in \mathbb{N})} \subseteq X \). Next, we see that \( w \in (1/4, 1/2) \subseteq S_{(w_m; m \in \mathbb{N})} \).

By Definition 3.2, \( T^2 \) is left \( T\)-quasi-closed on \( X \).

(II.7) For \( J \subseteq \mathcal{P} \) the statement (A) and (B) of Theorem 3.1 hold. By (II.1)–(II.6), we have: \( \text{Fix}(T^2) = \{1/4, 1/2\}; \)

\[ \forall w \in (1/4, 1/2) \subseteq S_{(w_m; m \in \mathbb{N})} \subseteq \{1/4, 1/2\}; \]

\[ \forall w \in (1/4, 1/2) \subseteq S_{(w_m; m \in \mathbb{N})} \subseteq \{1/4, 1/2\}. \]

Moreover, since \( \text{Fix}(T^2) = \{1/4, 1/2\} \subseteq A \), thus, by (5.5) and (5.1), we get \( p(1/4, 1/2) = p(1/2, 1/4) = 0 \), so (b) holds.

(II.8) For \( J \subseteq \mathcal{P} \) the statement (A) and (B) of Theorem 3.1 does not hold. We then have that: the assumption (C1) does not hold; for \( q = 2 \) the assumption (C2) holds; \( \text{Fix}(T^2) \neq \emptyset; \) properties (c1)–(c3) do not hold since \( \text{Fix}(T) = \emptyset \).

References