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Sufficient conditions for strongly Carathéodory functions

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ABSTRACT

Recently, by applying the principle of subordination for analytic functions in the open unit disk $\mathbb{U}$, Kim and Cho [I. H. Kim, N. E. Cho, Sufficient conditions for Carathéodory functions, Comput. Math. Appl. 59 (2010), 2067–2073] considered several sufficient conditions for a family of Carathéodory functions. The main purpose of this paper is to investigate some (presumably new) sufficient conditions for the class of strongly Carathéodory functions in $\mathbb{U}$. One illustrative example and several corollaries of the main results presented here are also considered.

1. Introduction, definitions and preliminaries

Let $\mathcal{H}[a_0, n]$ denote the class of functions $p(z)$ of the form:

$$p(z) = a_0 + \sum_{k=n}^{\infty} a_k z^k \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\};\ a_0 \in \mathbb{C}),$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

$\mathbb{C}$ being, as usual, the set of complex numbers.

Definition 1. If the function $p(z) \in \mathcal{H}[a_0, n]$ satisfies the following argument inequality:

$$|\arg\{p(z)\}| < \frac{\pi}{2}\mu \quad (z \in \mathbb{U};\ 0 < \mu \leqq 1),$$

then we say that $p(z)$ is a *strongly Carathéodory function of order* μ in $\mathbb{U}$ and we write $p(z) \in \mathcal{STP}(\mu)$ (see, for example, [1–4]; see also [5]).

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Definition 2. Let $\mathcal{A}_n$ denote the class of functions of the form:

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots \quad (n \in \mathbb{N}),$$

which are analytic in $\mathbb{U}$. Also let

$$\mathcal{A} \equiv \mathcal{A}_1.$$

The class $\mathcal{STS}(\mu)$ of functions $f(z) \in \mathcal{A}_n$ is then defined defined by

$$\mathcal{STS}(\mu) := \left\{ f : f(z) \in \mathcal{A}_n \text{ and } \left| \arg\left(\frac{zf'(z)}{f(z)}\right) \right| < \frac{\pi}{2}\mu \quad (0 < \mu \leqq 1) \right\}.$$

We say that $f(z) \in \mathcal{STS}(\mu)$ is a *strongly starlike function of order* μ in $\mathbb{U}$. We also write

$$\mathcal{S}^* \equiv \mathcal{STS}(1).$$

The above-defined function class $\mathcal{STS}(\mu)$ was considered by Shiraishi and Owa [6].

Definition 3. For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$, if there exists an analytic (Schwarz) function $w(z)$ in $\mathbb{U}$, satisfying the following conditions:

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)).$$

We denote this subordination by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

In particular, if $g(z)$ is univalent in $\mathbb{U}$, then the subordination

$$f(z) \prec g(z) \quad (z \in \mathbb{U})$$

is equivalent to the following conditions:

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U})$$

(see, for details, [7,8]; see also [9]).

We denote by $\mathcal{Q}$ the class of functions $q(z)$ which are analytic and injective on $\overline{\mathbb{U}} \setminus \mathbb{E}(q)$, where

$$\mathbb{E}(q) = \left\{ \zeta : \zeta \in \partial\mathbb{U} \quad \text{and} \quad \lim_{z \to \zeta}\{q(z)\} = \infty \right\},$$

and are such that

$$q'(\zeta) \neq 0 \quad \big(\zeta \in \partial\mathbb{U} \setminus \mathbb{E}(q)\big).$$

Here, as usual, we write

$$\overline{\mathbb{U}} := \mathbb{U} \cup \partial\mathbb{U} \quad \text{and} \quad \partial\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| = 1\}.$$

Finally, let the subclass of $\mathcal{Q}$ for which $q(0) = a_0$ be denoted by $\mathcal{Q}(a_0)$.

In order to investigate our problems involving (for example) sufficient conditions for the above-defined function classes, we need the following lemma due to Miller and Mocanu [8].

Lemma 1. *Let $q(z) \in \mathcal{Q}(a_0)$ and let $h(z) \in \mathcal{H}[a_0, n]$ with $h(z) \not\equiv a_0$. If $h(z) \not\prec q(z)$, then there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus \mathbb{E}(q)$ for which*

$$h(z_0) = q(\zeta_0)$$

and

$$z_0 h'(z_0) = m\zeta_0 q'(\zeta_0) \quad (m \geqq n \geqq 1).$$

2. Conditions associated with strongly Carathéodory functions

Applying Lemma 1, we derive our first main result.

Theorem 1. *Let the function $g(z)$ be analytic in $\mathbb{U}$ with*

$$A := \inf_{z\in\mathbb{U}} \{\Re\{g(z)\}\cos\alpha - |\Im\{g(z)\}\sin\alpha|\} > 0 \quad \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right). \tag{1}$$

If $p(z) \in \mathcal{H}[1, n]$ satisfies the following conditions:

$$p(z) \not\equiv 1 \quad (z \in \mathbb{U})$$

and

$$\Re\{p(z) + g(z)zp'(z)\} > \frac{1}{2nA}[(\cos\alpha + 2nA)\sin^2\alpha - n^2A^2\cos\alpha] \quad (z \in \mathbb{U}),$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - |\alpha| \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right).$$

Proof. First of all, let us define the function $h_1(z)$ by

$$h_1(z) = e^{i\alpha}p(z) \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right) \tag{2}$$

and the function $q_1(z)$ by

$$q_1(z) = \frac{e^{i\alpha} + \overline{e^{i\alpha}}z}{1 - z} \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right). \tag{3}$$

Then, clearly, the functions $h_1(z)$ and $q_1(z)$ are analytic in $\mathbb{U}$ with

$$h_1(0) = q_1(0) = e^{i\alpha} \in \mathbb{C}$$

and

$$q_1(\mathbb{U}) = \{w : w \in \mathbb{C} \quad \text{and} \quad \Re(w) > 0\}.$$

We now suppose that the function $h_1(z)$ is not subordinate to the function $q_1(z)$. Lemma 1 would then show that there exist points $z_1 \in \mathbb{U}$ and $\zeta_1 \in \partial\mathbb{U} \setminus \{1\}$ such that

$$h_1(z_1) = q_1(\zeta_1) = i\rho_1 \quad (\rho_1 \in \mathbb{R}) \tag{4}$$

and

$$z_1h_1'(z_1) = m\zeta_1q_1'(\zeta_1) \quad (m \geqq n \geqq 1). \tag{5}$$

Here we note that

$$\zeta_1 = q_1^{-1}(h_1(z_1)) = \frac{h_1(z_1) - e^{i\alpha}}{h_1(z_1) + \overline{e^{i\alpha}}} \tag{6}$$

and

$$\zeta_1q_1'(\zeta_1) = -\frac{\rho_1^2 - 2\rho_1\sin\alpha + 1}{2\cos\alpha} \equiv \sigma_1(\rho_1) < 0. \tag{7}$$

For such points $z_1 \in \mathbb{U}$ and $\zeta_1 \in \partial\mathbb{U} \setminus \{1\}$, we obtain

$$\begin{aligned}
\Re\{p(z_1) + g(z_1)z_1p'(z_1)\} &= \Re\{e^{-i\alpha}h_1(z_1) + g(z_1)e^{-i\alpha}z_1h_1'(z_1)\} \\
&= \Re\{e^{-i\alpha}q_1(\zeta_1) + g(z_1)e^{-i\alpha}m\zeta_1q_1'(\zeta_1)\} \\
&= \Re\{e^{-i\alpha}i\rho_1 + g(z_1)e^{-i\alpha}m\sigma_1(\rho_1)\} \\
&= \rho_1\sin\alpha + m[\Re\{g(z_1)\}\cos\alpha - \Im\{g(z_1)\}\sin\alpha]\sigma_1(\rho_1) \\
&\leqq \rho_1\sin\alpha + nA\sigma_1(\rho_1) \\
&= \rho_1\sin\alpha - nA\left(\frac{\rho_1^2 - 2\rho_1\sin\alpha + 1}{2\cos\alpha}\right)
\end{aligned}$$

$$= -\frac{nA}{2\cos\alpha}\left(\rho_1 - \frac{\sin\alpha(\cos\alpha + nA)}{nA}\right)^2 + \frac{\sin^2\alpha(\cos\alpha + 2nA) - n^2A^2\cos\alpha}{2nA}$$
$$\leqq \frac{1}{2nA}[\sin^2\alpha(\cos\alpha + 2nA) - n^2A^2\cos\alpha],$$

where A is given by (1). This evidently contradicts the assumption of Theorem 1. Therefore, we obtain

$$\Re\{h(z)\} = \Re\{e^{i\alpha}p(z)\} > 0 \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right). \tag{8}$$

We next put

$$h_2(z) = e^{-i\alpha}p(z) \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right) \tag{9}$$

and

$$q_2(z) = \frac{e^{-i\alpha} + \overline{e^{-i\alpha}}z}{1 - z} \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right). \tag{10}$$

We then see that the functions $h_2(z)$ and $q_2(z)$ are analytic in $\mathbb{U}$ with

$$h_2(0) = q_2(0) = e^{-i\alpha} \in \mathbb{C}$$

and

$$q_2(\mathbb{U}) = \{w : w \in \mathbb{C} \quad \text{and} \quad \Re(w) > 0\} = q_1(\mathbb{U}).$$

If we suppose that the function $h_2(z)$ is not subordinate to the function $q_2(z)$, then Lemma 1 would show that there exist points $z_2 \in \mathbb{U}$ and $\zeta_2 \in \partial\mathbb{U} \setminus \{1\}$ such that

$$h_2(z_2) = q_2(\zeta_2) = i\rho_2 \quad (\rho_2 \in \mathbb{R}) \tag{11}$$

and

$$z_2h_2'(z_2) = m\zeta_2q_2'(\zeta_2) \quad (m \geqq n \geqq 1). \tag{12}$$

Furthermore, we note that

$$\zeta_2 = q_2^{-1}(h_1(z_2)) = \frac{h_2(z_2) - e^{-i\alpha}}{h_2(z_2) + \overline{e^{-i\alpha}}} \tag{13}$$

and

$$\zeta_2q_2'(\zeta_2) = -\frac{\rho_2^2 + 2\rho_2\sin\alpha + 1}{2\cos\alpha} \equiv \sigma_2(\rho_2) < 0. \tag{14}$$

For such points $z_2 \in \mathbb{U}$ and $\zeta_2 \in \partial\mathbb{U} \setminus \{1\}$, we see that

$$\begin{aligned}
\Re\{p(z_2) + g(z_2)z_2p'(z_2)\} &= \Re\{e^{i\alpha}h_2(z_2) + g(z_2)e^{i\alpha}z_2h_2'(z_2)\} \\
&= \Re\{e^{i\alpha}q_2(\zeta_2) + g(z_2)e^{i\alpha}m\zeta_2q_2'(\zeta_2)\} \\
&= \Re\{e^{i\alpha}i\rho_2 + g(z_2)e^{i\alpha}m\sigma_2(\rho_2)\} \\
&= -\rho_2\sin\alpha + m[\Re\{g(z_2)\}\cos\alpha + \Im\{g(z_2)\}\sin\alpha]\sigma_2(\rho_2) \\
&\leqq -\rho_2\sin\alpha + nA\sigma_2(\rho_2) \\
&= -\rho_2\sin\alpha - nA\left(\frac{\rho_2^2 + 2\rho_2\sin\alpha + 1}{2\cos\alpha}\right) \\
&= -\frac{nA}{2\cos\alpha}\left(\rho_2 + \frac{\sin\alpha(\cos\alpha + nA)}{nA}\right)^2 + \frac{\sin^2\alpha(\cos\alpha + 2nA) - n^2A^2\cos\alpha}{2nA} \\
&\leqq \frac{1}{2nA}[\sin^2\alpha(\cos\alpha + 2nA) - n^2A^2\cos\alpha],
\end{aligned}$$

where A is given by (1). This also contradicts the assumption of Theorem 1. Therefore, we have

$$\Re\{h_2(z)\} = \Re\{e^{-i\alpha}p(z)\} > 0 \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right). \tag{15}$$

Hence, by suitably combining the inequalities (8) and (15), we complete the proof of Theorem 1. □

By setting

$$\alpha = \frac{\pi}{2}(1-\mu) \quad (0 < \mu \leqq 1)$$

in Theorem 1, we obtain the following corollary.

Corollary 1. *Let the function $g(z)$ be analytic in $\mathbb{U}$ with*

$$A := \inf_{z\in\mathbb{U}} \left\{ \Re\{g(z)\} \sin\frac{\pi}{2}\mu - \left| \Im\{g(z)\} \cos\frac{\pi}{2}\mu \right| \right\} > 0 \quad (z \in \mathbb{U};\ 0 < \mu \leqq 1).$$

If $p(z) \in \mathcal{H}[1, n]$ satisfies the following conditions:

$$p(z) \not\equiv 1 \quad (z \in \mathbb{U})$$

and

$$\Re\{p(z) + g(z)zp'(z)\} > \frac{1}{2nA}\left[\left(\sin\frac{\pi}{2}\mu + 2nA\right)\cos^2\frac{\pi}{2}\mu - n^2A^2 \sin\frac{\pi}{2}\mu\right] \quad (z \in \mathbb{U}),$$

then $p(z) \in \mathcal{STP}(\mu)$.

If we put

$$p(z) = \frac{zf'(z)}{f(z)} = 1 + na_{n+1}z^n + \cdots \quad (z \in \mathbb{U})$$

for $f(z) \in \mathcal{A}_n$ in Corollary 1, we are led to the following result.

Corollary 2. *Let the function $g(z)$ be analytic in $\mathbb{U}$ with*

$$A := \inf_{z\in\mathbb{U}} \left\{ \Re\{g(z)\} \sin\frac{\pi}{2}\mu - \left| \Im\{g(z)\} \cos\frac{\pi}{2}\mu \right| \right\} > 0 \quad (z \in \mathbb{U}; 0 < \mu \leqq 1).$$

If $f(z) \in \mathcal{A}_n$ satisfies the following conditions:

$$\frac{zf'(z)}{f(z)} \not\equiv 1 \quad (z \in \mathbb{U})$$

and

$$\Re\left\{\frac{zf'(z)}{f(z)} + g(z)\frac{zf'(z)}{f(z)}\left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right)\right\}$$
$$> \frac{1}{2nA}\left[\left(\sin\frac{\pi}{2}\mu + 2nA\right)\cos^2\frac{\pi}{2}\mu - n^2A^2\sin\frac{\pi}{2}\mu\right] \quad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{STS}(\mu)$.

Upon letting

$$n = 1 \quad \text{and} \quad 0 \leqq \alpha < \frac{\pi}{2}$$

in Theorem 1, we get the following corollary.

Corollary 3. *Let the function $g(z)$ be analytic in $\mathbb{U}$ with*

$$A := \inf_{z\in\mathbb{U}} \{\Re\{g(z)\}\cos\alpha - \Im\{g(z)\}\sin\alpha\} > 0 \quad \left(z \in \mathbb{U}; 0 \leqq \alpha < \frac{\pi}{2}\right).$$

If $p(z) \in \mathcal{H}[1, 1]$ satisfies the following conditions:

$$p(z) \not\equiv 1 \quad (z \in \mathbb{U})$$

and

$$\Re\{p(z) + g(z)zp'(z)\} > \frac{1}{2A}[(\cos\alpha + 2A)\sin^2\alpha - A^2\cos\alpha] \quad (z \in \mathbb{U}),$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \quad \left(z \in \mathbb{U}; 0 \leqq \alpha < \frac{\pi}{2}\right).$$

We find it to be of interest to illustrate Theorem 1 by the following example.

Example. Let us consider the function $p(z)$ given by

$$p(z) = 1 + kz^n \quad \left(n \in \mathbb{N};\, k \leqq \frac{n^2+6n-9}{4n(4n+3)}\right).$$

Then it is easy to observe that the function $p(z)$ is analytic in $\mathbb{U}$ and maps $\mathbb{U}$ onto the disk with the center at $p(0) = 1$ and the radius equal to k. We thus find that

$$|\arg\{p(z)\}| < \sin^{-1}\left(\frac{n^2+6n-9}{4n(4n+3)}\right) < \frac{\pi}{4} \quad (z \in \mathbb{U}).$$

Moreover, if we consider

$$g(z) = 1 + \frac{1}{3}z \qquad \text{and} \qquad \alpha = \frac{\pi}{4},$$

we obtain

$$\begin{aligned} A &= \inf_{z\in\mathbb{U}} \{\Re\{g(z)\}\cos\alpha - |\Im\{g(z)\}\sin\alpha|\} \\ &= \frac{1}{3\sqrt{2}} > 0 \quad (z \in \mathbb{U}). \end{aligned}$$

This implies that

$$\frac{1}{2nA}[(\cos\alpha + 2nA)\sin^2\alpha - n^2A^2\cos\alpha] = \frac{-n^2+6n+9}{12n}.$$

On the other hand, we also have

$$\begin{aligned} \Re\{p(z) + g(z)zp'(z)\} &= \Re\left(1 + (1+n)kz^n + \frac{n}{3}kz^{n+1}\right) \\ &> 1 - (1+n)k - \frac{n}{3}k \\ &\geqq \frac{-n^2+6n+9}{12n} \quad (z \in \mathbb{U}). \end{aligned}$$

Thus, in view of Theorem 1, we conclude that

$$|\arg\{p(z)\}| < \frac{\pi}{4} \quad (z \in \mathbb{U}).$$

3. Further sets of conditions associated with strongly Carathéodory functions

Our main result in this section is contained in Theorem 2.

Theorem 2. *If the function $p(z) \in \mathcal{H}[1, n]$ satisfies each of the following conditions:*

$$p(z) \neq 0 \quad (z \in \mathbb{U}), \qquad p(z) \not\equiv 1 \quad (z \in \mathbb{U})$$

and

$$\begin{aligned} -\frac{\sqrt{2n\cos^2\alpha + n^2} + n\sin\alpha}{\cos\alpha} &< \Im\left(p(z) + \frac{zp'(z)}{p(z)}\right) \\ &< \frac{\sqrt{2n\cos^2\alpha + n^2} - n\sin\alpha}{\cos\alpha} \quad \left(z \in \mathbb{U};\, 0 \leqq \alpha < \frac{\pi}{2}\right), \end{aligned}$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \quad \left(z \in \mathbb{U};\, 0 \leqq \alpha < \frac{\pi}{2}\right).$$

Proof. We define the function $h_1(z)$ by (2) and the function $q_1(z)$ by (3) for the parameter α constrained by

$$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}.$$

If the function $h_1(z)$ is not subordinate to the function $q_1(z)$, then there exist points $z_1 \in \mathbb{U}$ and $\zeta_1 \in \partial\mathbb{U} \setminus \{1\}$ satisfying (4) and (5). Thus, by using the Eqs. (2)–(7), we have

$$\Im\left(p(z_1) + \frac{z_1 p'(z_1)}{p(z_1)}\right) = \Im\left(\mathrm{e}^{-\mathrm{i}\alpha} h_1(z_1) + \frac{z_1 h_1'(z_1)}{h_1(z_1)}\right) = \Im\left(\mathrm{e}^{-\mathrm{i}\alpha} q_1(\zeta_1) + \frac{m\zeta_1 q_1'(\zeta_1)}{q_1(\zeta_1)}\right) = \rho_1 \cos\alpha - \frac{m\sigma_1(\rho_1)}{\rho_1} \qquad (\rho_1 \in \mathbb{R} \setminus \{0\})$$

for such points as $z_1 \in \mathbb{U}$ and $\zeta_1 \in \partial\mathbb{U} \setminus \{1\}$.

For the case when $\rho_1 > 0$, since $\sigma_1(\rho_1) < 0$ and $m \geqq n$, we obtain

$$\rho_1 \cos\alpha - \frac{m\sigma_1(\rho_1)}{\rho_1} \geqq \rho_1 \cos\alpha - \frac{n\sigma_1(\rho_1)}{\rho_1} = \frac{\rho_1^2(2\cos^2\alpha + n) - 2n\rho_1 \sin\alpha + n}{2\rho_1 \cos\alpha} \qquad (\rho_1 \in \mathbb{R} \setminus \{0\}).$$

Moreover, since the function $g_1(\rho_1)$ given by

$$g_1(\rho_1) = \frac{\rho_1^2(2\cos^2\alpha + n) - 2n\rho_1 \sin\alpha + n}{2\rho_1 \cos\alpha} \qquad (\rho_1 \in \mathbb{R} \setminus \{0\})$$

takes on the minimum value at ρ_1^* given by

$$\rho_1^* = \sqrt{\frac{n}{2\cos^2\alpha + n}},$$

we have

$$\rho_1 \cos\alpha - \frac{m\sigma_1(\rho_1)}{\rho_1} \geqq g_1(\rho_1^*) = \frac{\sqrt{2n\cos^2\alpha + n^2} - n\sin\alpha}{\cos\alpha} \qquad (\rho_1 \in \mathbb{R} \setminus \{0\}),$$

which is a contradiction for the assumption of Theorem 2.

For the case when $\rho_1 < 0$, we put

$$\rho_1 = -\rho_1' \quad (\rho_1' > 0).$$

Then, by using the same method as above, we have

$$\begin{aligned}\rho_1 \cos\alpha - \frac{m\sigma_1(\rho_1)}{\rho_1} &\leqq -\rho_1' \cos\alpha + \frac{n\sigma_1(-\rho_1')}{\rho_1'} \\ &= -\frac{{\rho_1'}^2(2\cos^2\alpha + n) + 2n\rho_1' \sin\alpha + n}{2\rho_1' \cos\alpha} \\ &\equiv g_1(-\rho_1') \\ &\leqq g_1\left(-\sqrt{\frac{n}{2\cos^2\alpha + n}}\right) \\ &= -\frac{\sqrt{2n\cos^2\alpha + n^2} + n\sin\alpha}{\cos\alpha},\end{aligned}$$

which obviously contradicts the assumption of Theorem 2. Hence we have

$$\Re\left\{\mathrm{e}^{\mathrm{i}\alpha} p(z)\right\} > 0 \quad \left(z \in \mathbb{U};\, 0 \leqq \alpha < \frac{\pi}{2}\right). \tag{16}$$

Next, by considering the function $h_2(z)$ defined by (9) and the function $q_2(z)$ defined by (10) and using the above method *mutatis mutandis*, we also get

$$\Re\left\{\mathrm{e}^{-\mathrm{i}\alpha} p(z)\right\} > 0 \quad \left(z \in \mathbb{U};\, 0 \leqq \alpha < \frac{\pi}{2}\right). \tag{17}$$

Therefore, if we make use of the inequalities in (16) and (17), we complete the proof of Theorem 2. □

First of all, upon setting

$$\alpha = \frac{\pi}{2}(1-\mu) \quad (0 < \mu \leqq 1)$$

in Theorem 2, we obtain Corollary 4.

Corollary 4. *If the function $p(z) \in \mathcal{H}[1, n]$ satisfies the following conditions:*

$$p(z) \neq 0 \quad (z \in \mathbb{U}), \qquad p(z) \not\equiv 1 \quad (z \in \mathbb{U})$$

and

$$-\frac{\sqrt{2n\sin^2\frac{\pi}{2}\mu + n^2} + n\cos\frac{\pi}{2}\mu}{\sin\frac{\pi}{2}\mu} < \Im\left(p(z) + \frac{zp'(z)}{p(z)}\right) < \frac{\sqrt{2n\sin^2\frac{\pi}{2}\mu + n^2} - n\cos\frac{\pi}{2}\mu}{\sin\frac{\pi}{2}\mu} \qquad (z \in \mathbb{U}; 0 < \mu \leqq 1),$$

then $p(z) \in \mathcal{STP}(\mu)$.

We next let

$$p(z) = \frac{zf'(z)}{f(z)} = 1 + na_{n+1}z^n + \cdots \quad (z \in \mathbb{U})$$

for $f(z) \in \mathcal{A}_n$ in Corollary 4. We thus obtain the following result.

Corollary 5. *If the function $f(z) \in \mathcal{A}_n$ satisfies the following conditions:*

$$\frac{zf'(z)}{f(z)} \neq 0 \quad (z \in \mathbb{U}), \qquad f(z) \not\equiv 1 \quad (z \in \mathbb{U})$$

and

$$-\frac{\sqrt{2n\sin^2\frac{\pi}{2}\mu + n^2} + n\cos\frac{\pi}{2}\mu}{\sin\frac{\pi}{2}\mu} < \Im\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{\sqrt{2n\sin^2\frac{\pi}{2}\mu + n^2} - n\cos\frac{\pi}{2}\mu}{\sin\frac{\pi}{2}\mu} \qquad (z \in \mathbb{U}; 0 < \mu \leqq 1),$$

then $f(z) \in \mathcal{STS}(\mu)$.

In its special case when $n = 1$, Theorem 2 yields the following result due to Kim and Cho [2].

Corollary 6. *If the function $p(z) \in \mathcal{H}[1, 1]$ satisfies the following conditions:*

$$p(z) \neq 0 \quad (z \in \mathbb{U}), \qquad p(z) \not\equiv 1 \quad (z \in \mathbb{U})$$

and

$$-\frac{\sqrt{2\cos^2\alpha + 1} + \sin\alpha}{\cos\alpha} < \Im\left(p(z) + \frac{zp'(z)}{p(z)}\right) < \frac{\sqrt{2\cos^2\alpha + 1} - \sin\alpha}{\cos\alpha} \qquad \left(z \in \mathbb{U}; 0 \leqq \alpha < \frac{\pi}{2}\right),$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \qquad \left(z \in \mathbb{U}; 0 \leqq \alpha < \frac{\pi}{2}\right).$$

Finally, we derive the following result.

Theorem 3. *If the function $p(z) \in \mathcal{H}[1, n]$ satisfies the following conditions:*

$$p(z) \neq 0 \quad (z \in \mathbb{U}), \qquad p(z) \not\equiv 1 \quad (z \in \mathbb{U})$$

and

$$\left|p(z) + \frac{zp'(z)}{p(z)} - 1\right| < \left(\frac{n}{2} + 1\right)|p(z)|\cos\alpha \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right),$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - |\alpha| \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right).$$

Proof. Let us define the function $h_1(z)$ by

$$h_1(z) = \frac{e^{i\alpha}}{p(z)} \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right)$$

and let the function $q_1(z)$ be defined as in the proof of Theorem 1. If the function $h_1(z)$ is not subordinate to the function $q_1(z)$, then there exist points $z_1 \in \mathbb{U}$ and $\zeta_1 \in \partial\mathbb{U} \setminus \{1\}$ satisfying (4) and (5). Thus, by using the Eqs. (3)–(7), we have

$$\begin{aligned}
\frac{\left|p(z_1) + \frac{z_1 p'(z_1)}{p(z_1)} - 1\right|}{|p(z_1)|} &= \left|1 - e^{-i\alpha} z_1 h_1'(z_1) - e^{-i\alpha} h_1(z_1)\right| \\
&= \left|h_1(z_1) + z_1 h_1'(z_1) - e^{-i\alpha}\right| \\
&= \left|q_1(\zeta_1) + m\zeta_1 q_1'(\zeta_1) - e^{-i\alpha}\right| \\
&= |[m\sigma_1(\rho_1) - \cos\alpha] + i(\rho_1 - \sin\alpha)| \\
&= \left[\left(m\sigma_1(\rho_1) - \cos\alpha\right)^2 + \left(\rho_1 - \sin\alpha\right)^2\right]^{\frac{1}{2}} \\
&= \left[\left(m\frac{\rho_1^2 - 2\rho_1 \sin\alpha + 1}{2\cos\alpha} + \cos\alpha\right)^2 + (\rho_1 - \sin\alpha)^2\right]^{\frac{1}{2}} \\
&\geqq \left[\left(n\frac{\rho_1^2 - 2\rho_1 \sin\alpha + 1}{2\cos\alpha} + \cos\alpha\right)^2 + (\rho_1 - \sin\alpha)^2\right]^{\frac{1}{2}} \\
&= \left[\left(n\frac{(\rho_1 - \sin\alpha)^2 + \cos^2\alpha}{2\cos\alpha} + \cos\alpha\right)^2 + (\rho_1 - \sin\alpha)^2\right]^{\frac{1}{2}} \\
&\geqq \left(\frac{n}{2} + 1\right)\cos\alpha,
\end{aligned}$$

which is a contradiction for the assumption of Theorem 3. Hence we have

$$\Re\{h_1(z)\} = \Re\left(\frac{e^{i\alpha}}{p(z)}\right) > 0 \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right). \tag{18}$$

We next consider the function $h_2(z)$ defined by

$$h_2(z) = \frac{e^{-i\alpha}}{p(z)} \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right).$$

Then, by virtue of (10), we also obtain

$$\Re\{h_2(z)\} = \Re\left(\frac{e^{-i\alpha}}{p(z)}\right) > 0 \quad \left(z \in \mathbb{U}; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right). \tag{19}$$

Therefore, in view of (18) and (19), we complete our proof of Theorem 3. □

By setting

$$\alpha = \frac{\pi}{2}(1 - \mu) \quad (0 < \mu \leqq 1)$$

in Theorem 3, we obtain the following corollary.

Corollary 7. *If the function $p(z) \in \mathcal{H}[1, n]$ satisfies the following conditions:*

$$p(z) \neq 0 \quad (z \in \mathbb{U}), \qquad p(z) \not\equiv 1 \quad (z \in \mathbb{U})$$

and

$$\left|p(z) + \frac{zp'(z)}{p(z)} - 1\right| < \left(\frac{n}{2} + 1\right)|p(z)| \sin\frac{\pi}{2}\mu \quad (z \in \mathbb{U}; 0 < \mu \leqq 1),$$

then $p(z) \in \mathcal{STP}(\mu)$.

Considering the function

$$p(z) = \frac{zf'(z)}{f(z)} = 1 + na_{n+1}z^n + \cdots \quad (z \in \mathbb{U}; n \in \mathbb{N})$$

for $f(z) \in \mathcal{A}_n$ in Corollary 7, we have the following result.

Corollary 8. *If the function* $f(z) \in \mathcal{A}_n$ *satisfies the following conditions:*

$$\frac{zf'(z)}{f(z)} \neq 0 \quad (z \in \mathbb{U}), \qquad f(z) \not\equiv 1 \quad (z \in \mathbb{U})$$

and

$$\left|\frac{zf''(z)}{f'(z)}\right| < \left(\frac{n}{2} + 1\right) \left|\frac{zf'(z)}{f(z)}\right| \sin \frac{\pi}{2}\mu \quad (z \in \mathbb{U}; 0 < \mu \leqq 1),$$

then $f(z) \in \mathcal{STS}(\mu)$.

In its special case when

$$n = 1 \quad \text{and} \quad 0 \leqq \alpha < \frac{\pi}{2},$$

in Theorem 3, we are easily led to the following result due to Kim and Cho [2].

Corollary 9. *If the function* $p(z) \in \mathcal{H}[1, 1]$ *satisfies the following conditions:*

$$p(z) \neq 0 \quad (z \in \mathbb{U}), \qquad p(z) \not\equiv 1 \quad (z \in \mathbb{U})$$

and

$$\left|p(z) + \frac{zp'(z)}{p(z)} - 1\right| < \frac{3}{2}|p(z)| \cos \alpha \quad \left(z \in \mathbb{U}; 0 \leqq \alpha < \frac{\pi}{2}\right),$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \quad \left(z \in \mathbb{U}; 0 \leqq \alpha < \frac{\pi}{2}\right).$$

We conclude our present investigation by remarking that several *further* corollaries and consequences of each of our main results (Theorems 1 to 3) can be deduced in a similar manner.

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