ELSEVIER

Discrete Mathematics 181 (1998) 31–36

DISCRETE MATHEMATICS

Neighborhood conditions for balanced independent sets in bipartite graphs

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Received 19 April 1993; received in revised form 10 January 1997; accepted 27 January 1997

Abstract

Let G be a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geqslant 3$. If, for every balanced independent set S of four vertices, $|N(S)| > n$ then G is traceable, the circumference is at least $2n - 2$ and G contains a 2-factor (with only small order exceptional graphs for the latter statement). If the neighborhood union condition is replaced by $|N(S)| > n + 2$ then G is hamiltonian.

Keywords: Hamiltonian properties; Neighborhood union; Bipartite graph

AMS classification: 05C38; 05C45; 05C70

The investigation of certain extremal problems involving neighborhood union conditions was initiated by Faudree, Gould, Jacobson and Schelp [3].

Theorem A (Faudree et al. [3]). *Let G be a 2-connected graph of order n. If for every pair of non-adjacent vertices u and v*
(a) $|N(u) \cup N(v)| \geqslant \frac{1}{3}(2n - 1)$ *then G is hamiltonian.*
(b) $|N(u) \cup N(v)| \geqslant \frac{1}{2}(n - 1)$ *then G is traceable.*
(c) $|N(u) \cup N(v)| \geqslant \frac{1}{3}(2n + 1)$ *and G is 3-connected then G is hamilton-connected.*

Let $\mathcal{G}$ be the class of 2-connected graphs, which contain three vertices which do not lie on a common cycle (in particular, these graphs cannot be hamiltonian). This class

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[1] Partially supported by French PCP. Info (CEFI-CONICIT).
[2] Supported by Deutsche Forschungsgemeinschaft (DFG), Grant We 1265.

PII S0012-365X(97)00042-3

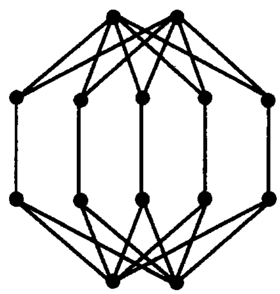

Fig. 1. The minimal graph of $\mathscr{H}_{14}$.

was structurally characterized by Watkins and Mesner [6]. Answering a conjecture of Jackson [4] affirmatively, Broersma, van den Heuvel and Veldman strengthened Theorem A(a).

Theorem B (Broersma, van den Heuvel and Veldman [2]). *Let G be a 2-connected graph of order n. If for every pair of non-adjacent vertices u and v*

$$|N(u)\cup N(v)|\geqslant \tfrac{1}{2}n,$$

then G is either hamiltonian, or the Petersen graph, or $G\in\mathscr{G}$.

The purpose of this paper is to investigate a certain neighborhood union condition for bipartite graphs. Let G be a balanced bipartite graph, i.e. a graph with a bipartition into two independent vertex sets of the same cardinality (if the graph is connected, which is implied by most of our conditions, then the bipartition is unique). We investigate conditions on the neighborhood union $N(S)$ of any balanced independent set S of four vertices, i.e. an independent set containing two vertices from each side of the bipartition.

Theorem 1. *Let G be a balanced bipartite graph of order $2n$ with $\delta(G)\geqslant 3$. If for every balanced independent set S with $|S|=4$ we have $|N(S)|>n+2$ then G is hamiltonian.*

We conjecture that the neighborhood union condition can be slightly relaxed. Let $\mathscr{H}_{14}$ denote the class of graphs obtained from the graph depicted in Fig. 1, where some (or all) of the four possible edges joining the top to the bottom vertices might be present as well.

Conjecture 1. Let G be a balanced bipartite graph of order $2n$ with $\delta(G)\geqslant 3$. If for every balanced independent set S with $|S|=4$ we have $|N(S)|>n$ then G is hamiltonian or $G\in\mathscr{H}_{14}$.

Note that this would be a considerable generalization of Moon and Moser's famous result [5], saying that every balanced bipartite graph of order $2n$ with minimum degree exceeding n is hamiltonian.

If true then the conjecture is best possible as none of the conditions can be relaxed. Indeed, take two complete balanced bipartite graphs, choose from each of them one side of the bipartiton and join these sides completely. The resulting graph satisfies $|N(S)|=n$ for every balanced independent set S but it is not hamiltonian. This shows that imposing a stronger minimum degree or even connectivity condition would not help in this case. The following construction shows that also the minimum degree condition cannot be reduced: Take a complete balanced bipartite graph, duplicate an edge and subdivide both duplicates twice. This is a non-hamiltonian graph with minimum degree two, which satisfies the neighborhood union condition.

The following results for the same neighborhood union condition give some support to the conjecture, by showing that graphs satisfying the hypothesis of the conjecture are at least very close to being hamiltonian.

Theorem 2. *Let G be a balanced bipartite graph of order $2n$ with $\delta(G)\geqslant 3$. If for every balanced independent set S with $|S|=4$ we have $|N(S)|>n$ then G is hamiltonian or G contains a spanning subgraph consisting of a cycle and an isolated edge.*

As an immediate consequence we get:

Corollary 1. *Let G be a balanced bipartite graph of order $2n$ with $\delta(G)\geqslant 3$. If for every balanced independent set S with $|S|=4$ we have $|N(S)|>n$ then G is traceable.*

Theorem 3. *Let G be a balanced bipartite graph of order $2n$ with $\delta(G)\geqslant 3$. If for every balanced independent set S with $|S|=4$ we have $|N(S)|>n$ then G has a 2-factor, unless $G\in\mathscr{H}_{14}$.*

The graph consisting of two vertices which are joined by four internally disjoint paths of length 3 shows that the degree condition in the preceeding three results cannot simply be dropped. Anyway, if n is sufficiently large, then the statements might hold for $\delta(G)\geqslant 2$ as well.

We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of a (simple, finite, undirected) graph G. If $X\subseteq V(G)$ then $G[X]$ denotes the subgraph of G induced by the vertices of X. For a fixed subgraph H of G we simply write $G-H$ for $G[V(G)\setminus V(H)]$. For a cycle C of G we always assume an orientation, which is arbitrary but fixed unless the orientation is explicitly specified. The successor and predecessor of a vertex v of C according to the orientation of C is v^+ and v^-, respectively. The cyclic distance of two vertices of C is their distance in C (and not in G). For undefined graph theoretical concepts the reader is referred to introductory literature, e.g. [1].

In the proofs we need that the graphs are connected and contain a perfect matching.

Lemma 1. *Let G be a balanced bipartite graph of order $2n$ with $\delta(G)\geqslant 2$. If for every balanced independent set S with $|S|=4$ we have $|N(S)|>n$ then G is connected.*

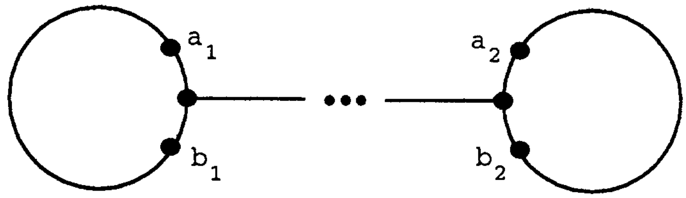

Fig. 2. The dumb-bell.

Proof. Take a balanced bipartition $\{X, Y\}$ of $V(G)$ and suppose that there are distinct components G_1 and G_2 of G. As $\delta(G)\geqslant 2$ each component has at least two vertices in X and in Y. We may assume that $|V(G_1)\cap X|\geqslant|V(G_1)\cap Y|$. Then every set S containing two vertices of $V(G_1)\cap X$ and two vertices of $V(G_2)\cap Y$ is a balanced independent set with $|N(S)|\leqslant n$, contradicting the hypothesis. □

With a slightly more careful reasoning it can be shown that the graphs are actually 2-connected.

Lemma 2. *Let G be a balanced bipartite graph of order $2n$ with $\delta(G)\geqslant 1$. If for every balanced independent set S with $|S|=4$ we have $|N(S)|\geqslant n$ then G has a perfect matching.*

Proof. Let $\{X, Y\}$ be a balanced bipartition of $V(G)$. By the König-Hall Theorem (see e.g. [1, p. 128]) it suffices to show, that for every subset $A\subseteq X$ we have $|N(A)|\geqslant|A|$. Indeed, if $|N(A)|<|A|$ then $2\leqslant|A|\leqslant n-1$ as $\delta(G)\geqslant 1$. So every set S containing two vertices of A and two vertices of $Y-N(A)$ is a balanced independent set of four vertices with $|N(S)|<n$, a contradiction. □

In the proofs the following subgraph, called *dumb-bell*, consisting of two disjoint cycles Γ_1 and Γ_2 which are joined by a path P (cf. Fig. 2) will play a significant role. The neighbors in Γ_i of the vertex shared by Γ_i and P are denoted by a_i and b_i, respectively. Moreover, we call a spanning dumb-bell of a graph *maximal*, if the path P is as long as possible among all spanning dumb-bells.

Lemma 3. *Let G be a balanced bipartite graph of order $2n$ and let D be a maximal spanning dumb-bell of G with $S=\{a_1, b_1, a_2, b_2\}$. If $|N(S)|>n$ and G is not hamiltonian, then S is independent and G contains a spanning subgraph consisting of a cycle and an isolated edge. If $|N(S)|>n+2$ then G is hamiltonian.*

Proof. Fix an orientation of Γ_i such that its intersection vertex with P is $a_i^-=b_i^+$. Fix a perfect matching M of D which contains the edges $a_ia_i^+$. Assume that G is not hamiltonian, so S is an independent set. Since $|N(S)|>n$, there is an edge of M, both ends of which are adjacent to S. If for any edge $e\in M$ different from $b_i^-b_i^{--}$ both ends are adjacent to S then we obtain a contradiction to the maximality of D ($e \in E(P)$), or a hamiltonian cycle results ($e \in E(\Gamma_i)$). So $|N(S)| \leqslant n+2$ and, for some $i \in \{1,2\}$, b_i^{--} has a neighbor in S. Thus we obtain the indicated subgraph with $b_i b_i^-$ being the isolated edge. □

We will start with the proof of Theorem 3.

Proof of Theorem 3. As G contains a perfect matching by Lemma 2, it certainly has a spanning subgraph consisting of vertex disjoint even length cycles and odd length paths. Let H be such a spanning subgraph, which has

(i) the largest number of vertices contained in cycle components,

(ii) subject to (i) the least number of components, and

(iii) subject to (i) and (ii) a path component of largest order.

Let M be a perfect matching of H and let H' be the subgraph of H which is spanned by the cycles.

It remains to show that $H' = H$. Assume, to the contrary, that $H' \neq H$. Let Q be a path component of largest order of $H - H'$, and let v_1 and v_2 be the endvertices of Q. By (i) and (iii) both v_1 and v_2 have exactly one neighbor in $G - H'$, so each of them has at least two neighbors in H' as $\delta(G) \geqslant 3$. If v_1 and v_2 have neighbors in different cycles of H' then we obtain a dumb-bell containing Q and two cycles C and C' of H'. Let D be a maximal dumb-bell with vertex set $V(Q) \cup V(C) \cup V(C')$ then $S = \{a_1, b_1, a_2, b_2\}$ is a balanced independent set of D. Observe that:

$$\text{one end of every edge } e \in M \cap E(G-D) \text{ has no neighbor in } S. \qquad (*)$$

Otherwise, if $e = xy \in E(H - H')$ then $G[V(D) \cup \{x,y\}]$ is hamiltonian, and if $e \in E(H')$ then $G[V(D) \cup V(K_e)]$ is hamiltonian, where K_e is the component of H' containing e. In any case (i) is violated. Hence we conclude $|N(S) \cap V(D)| > \frac{1}{2}|V(D)|$, and by Lemma 3 applied to $G[D]$ this implies, that (i) or (ii) is violated.

So we may assume that all neighbors of v_1 and v_2 in H' belong to one cycle of H', say C. Let w_1 and w_2 be neighbors of v_1 and v_2, resp., which have largest cyclic distance on C. By (i) the cyclic distance must be at least $|Q| + 1 \geqslant 3$. If the cyclic distance exceeds 3 then $S = \{w_1^+, w_1^-, w_2^+, w_2^-\}$ is a balanced independent set, as any edge between two vertices of S causes one or two cycles spanning $G[V(Q) \cup V(C)]$, which contradicts (i). Now fix a perfect matching M' of G which contains the edges $v_1 w_1$ and $v_2 w_2$ and takes the remaining edges from $E(H)$. It is easy to see that such a matching exists. On the other hand, if for any edge of M' both ends have a neighbor in S then we obtain a subgraph of disjoint cycles covering H' and Q, which again contradicts (i).

So, finally, assume that w_1 and w_2 have cyclic distance 3. It is easy to observe that in this case $|Q| = 2$, $|C| = 12$, $C = x_1, x_2, \ldots, x_{12}, x_1$, such that v_1 is adjacent to x_1 and x_7 and v_2 is adjacent to x_4 and x_{10}. Consider the edges $E' = \{v_1 v_2, x_2 x_3, x_5 x_6, x_8 x_9, x_{11} x_{12}\}$. Note that for each edge $e' \in E'$ there is a 12-cycle in $G[V(Q) \cup V(C)]$ avoiding e'. Furthermore, any edge joining two edges of E' would cause a 2-factor of

$G[V(Q) \cup V(C)]$, contradicting (i). So, in particular, we may assume that $S = \{x_2, x_5, x_8, x_{11}\}$ is a balanced independent set and $|N(S) \cap (V(Q) \cup V(C))| \leqslant 8$.

If $n \geqslant 8$ then there must be an edge f joining S to another component of H. If f joins S to a path Q' in $H - H'$ then $G[V(Q) \cup V(Q') \cup V(C)]$ has a spanning subgraph consisting of a 12-cycle and a length 3 path since $|Q'| \leqslant |Q| = 2$. This contradicts (ii). If f joins S to another cycle C' then we obtain a spanning dumb-bell of $G[V(Q) \cup V(C) \cup V(C')]$. As above, for a maximal dumb-bell $(*)$ holds, and we again conclude with Lemma 3 that (i) or (ii) is violated. So we may assume that $n = 7$, and since $\delta(G) \geqslant 3$, every vertex of $\{x_1, x_4, x_7, x_{10}\}$ has a neighbor in each edge of E'. So the resulting graph is indeed in $\mathscr{H}_{14}$. $\square$

We prove the remaining statements simultaneously.

Proof of Theorems 1 and 2. It remains to show that if G is a non-hamiltonian with $\delta(G) \geqslant 3$ and $|N(S)| > n$ for every balanced independent set S of four vertices, then G contains a spanning dumb-bell, as the results can be immediately derived from Lemma 3.

Indeed, if $G \in \mathscr{H}_{14}$ then G contains a spanning dumb-bell. Otherwise, by Theorem 3, G contains a spanning subgraph consisting of vertex disjoint cycles. Since G is non-hamiltonian and connected, it contains a spanning subgraph, consisting of a dumb-bell and vertex disjoint cycles. Let H be such a spanning subgraph, where the order of the dumb-bell D is as large as possible, and subject to this requirement, the path P of D is as long as possible. Set $S = \{a_1, b_1, a_2, b_2\}$. If $H - D$ contains a cycle then the vertices of S have all their neighbors in D, since the order of D is maximal. By Lemma 3 this implies that the subgraph induced by D is hamiltonian, which again contradicts the maximality of the order of D. So D must be a spanning dumb-bell, as required. $\square$

Acknowledgements

We thank the anonymous referee for the excellent report, which helped to improve the presentation and avoid an inaccuracy.

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