Integral Geometry on Grassmann Manifolds and Calculus of Invariant Differential Operators

Tomoyuki Kakehi

Institute of Mathematics, University of Tsukuba, Tsukuba-shi, Ibaraki 305, Japan
E-mail: kakehi@math.tsukuba.ac.jp

Communicated by Irving Segal
Received November 24, 1997

DEDICATED TO PROFESSOR SIGURDUR HELGASON ON HIS 70TH BIRTHDAY

In this paper, we mainly deal with two problems in integral geometry, the range characterizations and construction of inversion formulas for Radon transforms on higher rank Grassmann manifolds. The results will be described explicitly in terms of invariant differential operators. We will also study the harmonic analysis on Grassmann manifolds, using the method of integral geometry. In particular, we will give eigenvalue formulas and radial part formulas for invariant differential operators.

Key Words: eigenvalue formula; Grassmann manifold; integral geometry; invariant differential operator; inversion formula; radial part; Radon transform; range-characterization.

1. INTRODUCTION

In this paper, we study the detailed calculus of invariant differential operators on Grassmann manifolds and apply it to the integral geometry on Grassmann manifolds. The main purpose of this paper is to characterize the ranges of the Radon transforms on real and complex Grassmann manifolds by means of invariant differential operators and to construct the explicit inversion formulas for these Radon transforms.

Let $\mathbb{F}$ be the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$. We denote by $\text{Gr}(p,n;\mathbb{F})$ the Grassmann manifold of all $p$-dimensional subspaces in $\mathbb{F}^n$. As is well known, $\text{Gr}(p,n;\mathbb{F})$ is a compact symmetric space of rank $\min\{p,n-p\}$. Then, the Radon transform $R^p_q: C^\infty(\text{Gr}(q,n;\mathbb{F})) \to C^\infty(\text{Gr}(p,n;\mathbb{F}))$ is defined as
for a \( p \)-dimensional subspace \( \zeta \in \text{Gr}(p, n; F) \) and for \( f \in C^\infty(\text{Gr}(q, n; F)) \).

Here in (1.1) (resp. in (1.2)), \( dv_{\zeta}(\gamma) \) denotes the normalized measure on the submanifold \( \{ \gamma \in \text{Gr}(q, n; F); \gamma \in \zeta \} \) (resp. \( \{ \gamma \in \text{Gr}(q, n; F); \xi \in \gamma \} \) induced from the canonical measure on \( \text{Gr}(q, n; F) \).

Let \( s := \min\{q, n-q\} = \text{rank}\ \text{Gr}(q, n; F) \) and \( r := \min\{p, n-p\} = \text{rank}\ \text{Gr}(p, n; F) \). If \( s < r \), that is, if \( \dim\ \text{Gr}(q, n; F) < \dim\ \text{Gr}(p, n; F) \), we can no longer expect the surjectivity of the Radon transform \( R_q^p \). Therefore, it is natural to ask if there is a nice characterization of the range of \( R_q^p \).

A similar problem for Radon transforms on Euclidean spaces was first studied by F. John. In [J], he showed that the range of the X-ray transform on \( \mathbb{R}^3 \) is identical with the kernel of some ultrahyperbolic second order differential operator. Gelfand \textit{et al.} [GeGiGr] extended John’s result to the case of the \( k \)-plane Radon transform \( (1 \leq k \leq n-2) \) on \( \mathbb{R}^n \); they characterized the range by a (locally defined) second order system of John type differential equations. (See also [GeGr, GeGrRo].) Roughly speaking, their range-characterizing system is of the form

\[
\left( \frac{\partial^2}{\partial x_\alpha \partial x_\beta} - \frac{\partial^2}{\partial x_\beta \partial x_\alpha} \right) f = 0, \tag{1.3}
\]

where \((x_\alpha)\) denotes some suitable local coordinate system.

After that, Richter [R] gave a complete proof of the extension of John’s result from [GeGiGr]. His range-characterizing system is globally defined in terms of left-invariant vector fields on \( M(n) \). Finally, for the above Radon transform, Gonzalez [Gon1] gave a simple range characterization by a single fourth order invariant differential operator on the corresponding affine Grassmann manifold, which is also of a similar form to John’s operator.

On the other hand, the range characterization of Radon transforms on projective spaces was investigated by Grinberg [Gri2] for the first time. He showed the existence of second order system which characterizes the range of the projective Radon transform. After that, our previous papers [K1, 2] gave the explicit form of the range characterizing operator which is a fourth order invariant differential operator of John’s type. In addition, recently Gonzalez [Gon3] proved that there exists a \( 2s + 2 \) order invariant differential operator on \( \text{Gr}(p, n; F) \) which characterizes the range of the Radon transform \( R_q^p \). However, he did not give the explicit form of the
range characterizing operator. Nevertheless, taking into consideration the above situation, we can expect that such a range characterizing operator is, in some sense, similar to John’s operator.

To describe the range-characterization for $R^q_p$, we will define a generalized John type differential operator $\Phi^{(n, p)}_k$ ($1 \leq k \leq r$) on the Grassmann manifold $Gr(p, n; F)$, which consists of certain determinantal type of differential operators of order $k$. (For the definition, see (2.5) in Section 2 for the complex case, and see (7.4) and (7.5) in Section 7 for the real case.) Then, our first main theorem is as follows.

**Theorem A.** We assume that

$$s = \text{rank } Gr(q, n; F) < r = \text{rank } Gr(p, n; F).$$

Then the range of the Radon transform $R^q_p : C^\infty(Gr(q, n; F)) \to C^\infty(Gr(p, n; F))$ is identical with the kernel of the generalized John type invariant differential operator $\Phi^{(n, p)}_{s+1}$ of order $2s + 2$ on $Gr(p, n; F)$. (See Theorem 5.1(D) for the complex case, and see Theorem 9.1(D) and Theorem 9.6 for the real case.)

Moreover, we will discuss the uniqueness and the non-uniqueness of the range-characterizing operator for the Radon transform $R^q_p$. (See Theorem 5.1(E), Theorem 9.1(E), and Proposition 9.7)

Here it should be remarked that similar range characterizations to ours were recently obtained by Oshima [O] and Tanisaki [Tan]. (See Section 12, Final Remarks (IV).)

Next, let us consider the second subject, namely, the construction of inversion formulas.

In the case $q = 1$ and $p = n - 1$, there is the famous Helgason’s inversion formula. For example, for the Radon transform $R^1_{n-1}$ on the complex projective space $\mathbb{P}^{n-1} \mathbb{C}$, Helgason gave the inversion formula

$$\prod_{\alpha=2}^{n-1} \left( \frac{1}{(\alpha - 1)(n - \alpha)} A_{\mathbb{P}^{\alpha-1} \mathbb{C}} + 1 \right) R^1_{n-1} R^1_{n-1} = I,$$  \hspace{1cm} (1.4)

where $A_{\mathbb{P}^{\alpha-1} \mathbb{C}}$ denotes the standard Laplacian on $\mathbb{P}^{n-1} \mathbb{C}$. (See [H4, Chap. I, Theorem 4.11].)

After that, Grinberg [Gri3] proved the existence of the inversion formula for a general $R^q_p$. In the complex case, his result is stated as follows.

Assume that $q < p \leq n - p$. Then, there exists an $\mathcal{S}(n)$-invariant differential operator $D_{(q)}$ on $Gr(q, n; \mathbb{C})$ such that $D_{(q)} R^q_p R^q_p = I$.

However, he did not give the explicit form of the above operator $D_{(q)}$.

In general, harmonic analysis on a symmetric space of higher rank is quite difficult. In particular, it is hard to calculate the eigenvalue of a given
invariant differential operator on a higher rank Grassmann manifold. For this reason, the explicit inversion formula for $R^q_p$ is still unknown. Using our range theorems and related results (Theorem 5.1 and Theorem 9.1), we are able to express explicitly the operator $D_{(q)}$ above in terms of the generalized John type differential operators $\Phi^{(n-q)}_k (1 \leq k \leq s)$ on $Gr(q, n; \mathbb{C})$. (As we see later in Sections 2 and 7, generalized John type differential operators on $Gr(q, n; F)$ can be defined in the same way.) As a result, we are able to construct the explicit inversion formulas.

For example, in the complex case, our second main theorem is given by the following.

**Theorem B.** We assume that $s \leq r$. Then the following inversion formula holds for the Radon transform $R^q_p$ on the compact complex Grassmann manifold $Gr(q, n, \mathbb{C})$.

$$\left\{ \prod_{j=1}^s \frac{(\alpha - 1 - k)! (n - \alpha - k)!}{(\alpha - 1)! (n - \alpha)!} \Phi^{(n-q)}_k \right\} R^q_p R^q_p = I,$$

on $C^\infty (Gr(q, n, \mathbb{C}))$,

where $\Phi^{(n-q)}_1 = 1$. (See Theorem 6.6.)

In the real case, we have to assume that $p - q$ is even. Then the analogous inversion formula holds. (See Theorem 10.4.) In particular, our results generalize the Helgason’s inversion formulas.

This paper is organized as follows. Sections 2 and 7, we construct generalized John type differential operators. In Sections 3 and 8, we study the spherical representations of the Grassmann manifold. Section 4 is devoted to prove several important equalities on symmetric polynomials which are the keys to prove the range theorem. In Sections 5 and 9, we deal with the problem of the range characterization. Sections 6 and 10 are devoted to the construction of the explicit inversion formulas. Finally in Section 11, as an application of our results, we give the radial part formula for invariant differential operators on complex Grassmann manifolds.

2. INVARIANT DIFFERENTIAL OPERATORS ON COMPLEX GRASSMANN MANIFOLDS

From this section to Section 6 (excluding Section 4), we deal with the case of complex Grassmann manifolds.

The special unitary group $G := SU(n)$ acts on the complex Grassmann manifold $Gr(p, n; \mathbb{C})$ transitively. The stabilizer of the $p$-dimensional
subspace $C_1 \oplus \cdots \oplus C_{p_r}$ is $K_p := S(U(p) \times U(n - p))$. Then, $Gr(p, n; \mathbb{C})$ can be identified with the compact symmetric space $G/K_p$.

In this section, we construct generalized John type invariant differential operators on the complex Grassmann manifold $G/K_p = SU(n)/SU(U(p) \times U(n - p))$, which are expressed in terms of determinental type of differential operators.

\textbf{Notation.} (1) In general, for a Lie group $G$ and its closed subgroup $H$, we denote by $C^\infty(G, H)$ the function space $\{ f \in C^\infty(G); f(gh) = f(g) \forall g \in G \text{ and } \forall h \in H \}$. We identify $C^\infty(G, H)$ with $C^\infty(G/H)$. (2) We define a left action $\rho_L(g)$ of $G$ on $C^\infty(G)$ by $\rho_L(g) f(x) := f(g^{-1}x)$ for $x \in G$ and $f \in C^\infty(G)$. Similarly, we define a right action $\rho_R(g)$ by $\rho_R(g) f(x) := f(xg)$. (3) A differential operator $\Phi$ on $G$ is called left $G$-invariant if $\rho_L(g) \Phi = \Phi \rho_L(g)$ for all $g \in G$. Similarly, a differential operator $\Phi$ on $G$ is called right $H$-invariant if $\rho_R(h) \Phi = \Phi \rho_R(h)$ for all $h \in H$. We identify a right $H$-invariant differential operator on $G$ with a differential operator on $G/H$.

Let $\mathfrak{q}$ and $\mathfrak{l}_p$ denote the Lie algebras of $G$ and of $K_p$, respectively. Then $\mathfrak{q}$ and $\mathfrak{l}_p$ are given by

$$\mathfrak{q} = \{ X \in M_n(\mathbb{C}); X + X^* = 0, \text{tr } X = 0 \},$$

$$\mathfrak{l}_p = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \in \mathfrak{q}; \ X_1 \in M_p(\mathbb{C}), \ X_2 \in M_{n-p}(\mathbb{C}) \right\}.$$ 

Let $\mathfrak{g} = \mathfrak{l}_p \oplus \mathfrak{u}_p$ be the Cartan decomposition of the symmetric space $G/K_p$, where $\mathfrak{u}_p$ is the space of all the matrices $X$ of the form

$$X = \begin{pmatrix} 0 & -Z^* \\ Z & 0 \end{pmatrix} \in \mathfrak{g}, \quad Z = (z_{i\alpha}); \text{ complex } (n-p) \times p \text{ matrix} \quad (1 \leq i \leq n - 1, 1 \leq \alpha \leq p). \quad (2.1)$$

Let $r := \text{rank } G/K_p = \min \{ p, n - p \}$. Let $I = \{ i(1), i(2), \ldots, i(d) \}; \ 1 \leq i(1) < i(2) < \cdots < i(d) \leq n - p \}$ and $A = \{ \alpha(1), \alpha(2), \ldots, \alpha(d) \}; \ 1 \leq \alpha(1) < \alpha(2) < \cdots < \alpha(d) \leq p \}$ be two ordered sets. (Here we assume that $1 \leq d \leq r$.)

For the submatrix $Z$ of $X$ in (2.1) and the above two ordered sets $I$ and $A$, we define $d \times d$ matrix valued differential operators $\partial Z_{(i, A)}$ and $\partial Z_{(I, A)}$ by

$$\partial Z_{(i, A)} := \left( \frac{\partial}{\partial z_{i\alpha}} \right)_{i \in I, \alpha \in A}, \quad \partial Z_{(I, A)} := \left( \frac{\partial}{\partial z_{i\alpha}} \right)_{i \in I, \alpha \in A}. \quad (2.2)$$
Next, we define $d$th order differential operators $L^{(d)}_{(I,A)}$ and $L^{(d)\ast}_{(I,A)}$ acting on $C^\infty(G)$ by
\begin{align}
L^{(d)}_{(I,A)}(f) &:= \det \partial Z_{(I,A)} f(g \exp X)|_{X=0} \quad (2.3) \\
L^{(d)\ast}_{(I,A)}(f) &:= (-1)^d \det \bar{\partial} \bar{Z}_{(I,A)} f(g \exp X)|_{X=0}, \quad (2.4)
\end{align}
for $f \in C^\infty(G)$. Here $X$ is a matrix of the form (2.1).

**Example.** If $I = \{i, j : 1 \leq i < j \leq n-p\}$ and $A = \{x, \beta : 1 \leq x < \beta \leq p\}$, the second order differential operator $L^{(2)}_{(I,A)}$ is given by
\begin{align}
L^{(2)}_{(I,A)}(f) &= \left( \frac{\partial^2}{\partial z_{ix} \partial z_{j\beta}} - \frac{\partial^2}{\partial z_{jx} \partial z_{i\beta}} \right) f(g \exp X)|_{X=0}.
\end{align}

Finally, using $L^{(d)}_{(I,A)}$ and $L^{(d)\ast}_{(I,A)}$, we define a generalized John type differential operator $\Phi^{(n,p)}_{d}$ of order $2d$ acting on $C^\infty(G)$ as
\begin{align}
\Phi^{(n,p)}_{d} := \sum_{\substack{I \subseteq \{1, 2, \ldots, n-p\} \\
A \subseteq \{1, 2, \ldots, p\} \\
|I| = |A| = d}} L^{(d)\ast}_{(I,A)} L^{(d)}_{(I,A)} \quad (1 \leq d \leq r), \quad \Phi^{(n,p)}_{0} := 1. \quad (2.5)
\end{align}

In fact, taking account of the above example, we can regard the operator $\Phi^{(n,p)}_{d}$ as a generalization of John's differential operator. Here we note that $L^{(d)\ast}_{(I,A)}$ is the formal adjoint operator of $L^{(d)}_{(I,A)}$ with respect to the standard inner product of $L^2(G)$, which means that the differential operator $\Phi^{(n,p)}_{d}$ is a non-negative operator and formally self-adjoint. Moreover, the following holds.

**Proposition 2.1.** The generalized John type differential operator $\Phi^{(n,p)}_{d}$ defined by (2.5) is left $G$-invariant and right $K_p$-invariant. Therefore, $\Phi^{(n,p)}_{d}$ is well defined as an invariant differential operator on the symmetric space $G/K_p$.

**Sketch of the Proof of Proposition 2.1.** By definition, the left $G$-invariance of $\Phi^{(n,p)}_{d}$ is obvious. The right $K_p$ invariance of $\Phi^{(n,p)}_{d}$ is due to the fact that the following polynomial $F^{(n,p)}_{d}(X)$ on $\mathfrak{m}_p$ is $\text{Ad}K_p$-invariant. Here
\begin{align}
F^{(n,p)}_{d}(X) := \sum_{\substack{I \subseteq \{1, 2, \ldots, n-p\} \\
A \subseteq \{1, 2, \ldots, p\} \\
|I| = |A| = d}} \det \overline{Z^{(d)}_{(I,A)}} \det Z^{(d)}_{(I,A)} \quad (2.6)
\end{align}
for $X = \begin{pmatrix} 0 & -Z^* \\ Z & 0 \end{pmatrix} \in \mathfrak{m}_p$. 

6 TOMOYUKI KAKEHI
where \( Z^{(d)}_{iL,A} \) is a \( d \times d \) submatrix of \( Z \) defined by \( Z^{(d)}_{iL,A} := (z_{m})_{i \in L, a \in A} \). We see easily that the above polynomial \( F^{(\alpha, \rho)}(X) \) on \( \mathfrak{M}_p \) is Ad-\( K_p \)-invariant. Indeed we have the expansion

\[
\det(I + \lambda X) = 1 + F^{(\alpha, \rho)}(X) \lambda^2 + F^{(\alpha, \rho)}(X) \lambda^3 + \cdots + F^{(\alpha, \rho)}(X) \lambda^n,
\]

for \( X \in \mathfrak{M}_p \),

(2.7)

where \( r = \text{rank} \ G/K_p = \min \{ p, n - p \} \).

Remark 2.2. (1) Since \( \partial/\partial z_m = (1/2)(\partial/\partial x_m - \sqrt{-1}(\partial/\partial y_m)) \) for \( z_m = x_m + \sqrt{-1}y_m \), we see that the polynomial \( 2^{-2d}F^{(\alpha, \rho)}(X) \) on \( \mathfrak{M}_p \) can be regarded as the principal symbol of the invariant differential operator \( \Phi^{(\alpha, \rho)} \). (For the correspondence between an invariant differential operator and its principal symbol, see Takeuchi [Tak, Theorem 3.3 and Corollary 1] or Helgason [H4, Chap. II, Theorem 4.9]). (2) We denote by \( \mathcal{D}(G/K_p) \) the algebra of all invariant differential operators on \( G/K_p \). Then we see that the set of invariant differential operators \( \{ \Phi^{(\alpha, \rho)} \} \) generates the algebra \( \mathcal{D}(G/K_p) \). (3) We define a \( G \)-invariant inner product \( \langle \cdot, \cdot \rangle \) on \( g \) by \( \langle X, Y \rangle := -\frac{1}{2} \text{tr}(XY) \) for \( X, Y \in g \). This invariant inner product defines \( G \)-invariant metrics on \( G \), on \( K_p \), and on \( G/K_p \) respectively. Then, the second order operator \( \Phi^{(\alpha, \rho)} \) coincides with the Laplacian on \( G/K_p \) with respect to this metric.

Proposition 2.3. We assume that \( s := \text{rank} \ G/K_p < r := \text{rank} \ G/K_p \). Then the image \( \text{Im} \ R^s_p \) of the Radon transform \( R^s_p : C^{\infty}(G/K_p) \to C^{\infty}(G/K_p) \) is included in the kernel \( \text{Ker} \Phi^{(\alpha, \rho)} \) of the generalized John type invariant differential operator \( \Phi^{(\alpha, \rho)} \) on \( G/K_p \). Namely, we have \( \text{Im} \ R^s_p \subset \text{Ker} \Phi^{(\alpha, \rho)} \).

Proof. Taking account of the identification \( C^{\infty}(G; K_q) \cong C^{\infty}(G/K_p) \cong C^{\infty}(G/\text{Gr}(q, n, \mathbb{C})) \) and \( C^{\infty}(G; K_q) \cong C^{\infty}(G/K_p) \cong C^{\infty}(G/\text{Gr}(p, n, \mathbb{C})) \), we can regard the Radon transform \( R^s_p \) as a mapping from \( C^{\infty}(G; K_q) \) to \( C^{\infty}(G; K_p) \). Namely, we can rewrite the Radon transform \( R^s_p \) as

\[
R^s_p f(g) = \frac{1}{\text{Vol}(K_p)} \int_{k \in K_p} f(\rho_k(g)) \, dk,
\]

for \( f \in C^{\infty}(G; K_q) \).

Since \( \Phi^{(\alpha, \rho)} \) is right-\( K_p \)-invariant, we have for a function \( f \in C^{\infty}(G; K_q) \)

\[
\text{Vol}(K_p) \Phi^{(\alpha, \rho)} R^s_p f(g) = \int_{k \in K_p} \Phi^{(\alpha, \rho)}(\rho_K(k) f)(g) \, dk
\]

\[
= \int_{k \in K_p} \rho_K(k)(\Phi^{(\alpha, \rho)} f)(g) \, dk.
\]

(2.9)
By (2.9) and the expression (2.5), we have only to prove that
\[ L_{i,A}^{s+1} f = 0, \quad \text{for } f \in C^\infty(G; K_p), \ I \subset \{1, \ldots, n - p\}, \]
\[ A \subset \{1, \ldots, p\}, \ \# I = \# A = s + 1. \ (2.10) \]

In addition, by the density argument, we may assume that a function \( f \) in (2.10) is a polynomial function. Then \( f \) can be extended to a function on \( G^C = SL(n, \mathbb{C}) \). We note that such a function \( f \) satisfies that \( Xf = 0 \) for \( X \in t_q^C \).

Let \( I = \{i(1), i(2), \ldots, i(s + 1); 1 \leq i(1) < i(2) < \cdots < i(s + 1) \leq n - p\} \) and \( A = \{\alpha(1), \alpha(2), \ldots, \alpha(s + 1); 1 \leq \alpha(1) < \alpha(2) < \cdots < \alpha(s + 1) \leq p\} \). Then, by the definition 2.4., the differential operator \( L_{i,A}^{s+1} \) is rewritten as
\[ L_{i,A}^{s+1} = \sum_{\sigma \in \Xi_{s+1}} \text{sgn } \sigma \ E_{ji, \sigma(1), n(\sigma(1))} E_{ji, \sigma(2), n(\sigma(2))} \cdots \]
\[ \tilde{E}_{ji, \sigma(s+1), n(\sigma(s+1))}, \quad (2.11) \]
where \( \Xi_{s+1} \) denotes the symmetric group of degree \( s + 1 \) and where \( E_{ji, m} \) is defined by
\[ E_{ji, m}(g) := \frac{d}{dt} f(g \exp t E_{ji, m}) |_{t=0}. \quad (2.12) \]

Here \( E_{ji, m} = (\delta_{ij}\delta_{km})_{k=1,2,\ldots,n} \) denotes the matrix whose \( (j, m) \) entry is 1 and whose other entries are 0. We note that \( E_{ji, m} \in g^C \) if \( j \neq m \). Since \( s = \text{rank } G/K_q = \min\{q, n - q\} < s + 1 \), we see easily that for any \( I, A, \ (\# I = \# A = s + 1) \) and for any \( \sigma \in \Xi_{s+1} \) there exists an integer \( m (1 \leq m \leq s + 1) \) such that \( E_{ji, \sigma(m), n(\sigma(m))} \in t_q^C \). Thus we have \( E_{ji, \sigma(m), n(\sigma(m))} f = 0 \) for the above \( m \). Since \( s + 1 \) vector fields \( E_{ji, \sigma(1), n(\sigma(1))}, E_{ji, \sigma(2), n(\sigma(2))}, \ldots, E_{ji, \sigma(s+1), n(\sigma(s+1))} \) commute, we obtain
\[ E_{ji, \sigma(1), n(\sigma(1))} E_{ji, \sigma(2), n(\sigma(2))} \cdots E_{ji, \sigma(s+1), n(\sigma(s+1))} f = 0, \]
for \( f \in C^\infty(G; K_p) \),
which proves (2.10).

As we mentioned in the Introduction, we will prove later (in Section 5) that the above invariant differential operator \( \Phi^{(s+1)}_{(n,p)} \) on \( G/K_p \) characterizes the range \( \text{Im } R^g_{(n,p)} \) of the Radon transform \( R^g_{(n,p)} \). Namely, we will prove that \( \text{Im } R^g_{(n,p)} = \text{Ker } \Phi^{(s+1)}_{(n,p)} \).
3. REPRESENTATION OF COMPLEX GRASSMANN MANIFOLDS

In this section, we discuss the representation of the complex Grassmann manifold \( G/K_p = SU(n)/S(U(p) \times U(n-p)) \) from the point of view of the theory of radial parts.

As is well known, any highest weight of \( G = SU(n) \) is written in the form \((k_1, k_2, \ldots, k_n)\) where \( k_j \in \mathbb{Z}, k_1 \geq k_2 \geq \cdots \geq k_n \) and \( k_1 + k_2 + \cdots + k_n = 0 \). In particular, any highest weight of the complex Grassmann manifold \( G/K_p \) is written in the form \((l_1, \ldots, l_r, 0, \ldots, 0, l_r, \ldots, l_1)\) where \( l_j \in \mathbb{Z}, (1 \leq j \leq r) \) and \( l_1 \geq \cdots \geq l_r \geq 0 \). We denote the above highest weight by \( \lambda(l_1, \ldots, l_r) \).

Then the following theorem holds.

**Theorem 3.1** (Grinberg [Gri2], Gonzalez [Gon3]). Let \( F(x_1, \ldots, x_n) \) be a symmetric polynomial. Then, there exists an invariant differential operator \( D \) on \( SU(n) \) such that the eigenvalue of \( D \) corresponding to the highest weight \((k_1, k_2, \ldots, k_n)\) is given by \( F(k_1 + n - 1, k_2 + n - 2, \ldots, k_n + 1) \).

The above theorem follows from the radial part formula in Helgason [H4, Chap. V, Theorem 1.9].

**Proposition 3.2.** Let \( r = \text{rank } G/K_p = \min\{p, n-p\} \) and let \( F(t_1, \ldots, t_r) \) be a symmetric polynomial. Then, there exists an invariant differential operator \( P \) on \( G/K_p = SU(n)/S(U(p) \times U(n-p)) \) such that

\[
P|_{V^{(n,p)}(t_1, \ldots, t_r)} = F(\chi_1 + a_1, \chi_2 + a_2, \ldots, \chi_r + a_r),
\]

where \( \chi_j \) and \( a_j \) are given respectively by

\[
\chi_j := l_j(l_j + n + 1 - 2j), \quad a_j := -(j-1)(n-j) = j^2 - (n+1)j + n.
\]

In particular, for the \( k \)th elementary symmetric polynomial \( S_k(t_1, \ldots, t_r) \) defined by

\[
S_k(t_1, \ldots, t_r) := \sum_{1 \leq s_1 < s_2 < \cdots < s_k \leq r} t_{s_1} \cdots t_{s_k} \quad (1 \leq k \leq r),
\]

\[
S_0(t_1, \ldots, t_r) := 1, \quad S_k(t_1, \ldots, t_r) := 0 \quad (k < 0, r < k),
\]

there exists an invariant differential operator \( \Box_k^{(n,p)} \) on \( G/K_p \) such that

\[
\Box_k^{(n,p)}|_{V^{(n,p)}(t_1, \ldots, t_r)} = S_k(\chi_1 + a_1, \chi_2 + a_2, \ldots, \chi_r + a_r).
\]
Proof. Let us consider the symmetric polynomial $f(\mu; x_1, \ldots, x_n)$ defined by

$$f(\mu; x_1, \ldots, x_n) := \prod_{j=1}^{n} (\mu + x_j) \quad (\mu: \text{parameter}).$$

Then, by Theorem 3.1, there exists an invariant differential operator $D_\mu$ on $G$ such that the eigenvalue of $D_\mu$ corresponding to the highest weight $(k_1, k_2, \ldots, k_n)$ is given by

$$f(\mu; k_1 + n - 1, k_2 + n - 2, \ldots, k_n).$$

Taking $(k_1, k_2, \ldots, k_n) = (l_1, \ldots, l_r, 0, \ldots, 0, -l_r, \ldots, -l_1)$, we have

$$D_\mu|_{\mathfrak{p}^n} \lambda_{(l_1, \ldots, l_r)} = \prod_{j=r+1}^{n} (\mu + n - j) \times \prod_{j=1}^{r} (\mu + l_j + n - j)(\mu - l_j + j - 1).$$

If we put $\tau = -\mu^2 - (n - 1)\mu$ and $c(\mu) = (-1)^r \prod_{j=r+1}^{n} (\mu + n - j)^{-1}$, we have

$$c(\mu) D_\mu|_{\mathfrak{p}^n} \lambda_{(l_1, \ldots, l_r)} = \prod_{j=1}^{r} (\tau + x_j + a_j)$$

$$= \sum_{k=0}^{r} S_k(x_1 + a_1, x_2 + a_2, \ldots, x_r + a_r, \ldots, \tau^{-k}).$$

Since any symmetric polynomial can be written as a polynomial of the elementary symmetric polynomials $S_k(1 \leq k \leq r)$, the assertion of Proposition 3.2 follows from the above equality. \qed

Next, we consider the radial part of the above differential operator $\Box^{[n, p]}$ and of the generalized John type differential operator $\Phi^{[n, p]}_\mu$ defined in Section 2.

Let $a_p$ be the maximal abelian subalgebra of $\mathfrak{m}_p$ defined by

$$a_p := \{ [H(t) = H(t_1, \ldots, t_r) = t_1 H_1 + \cdots + t_r H_r] \in \mathfrak{m}_p:$$

$$l = (t_1, \ldots, t_r) \in \mathbb{R}^r \} \quad (3.5)$$

where $H_j := \sqrt{-1}(E_{j+p, j} + E_{j, j+p})$. We introduce the lexicographical order $<$ on $a_p$ such that $H_1 > H_2 > \cdots > H_r > 0$. We identify $a_p$ with $\mathbb{R}^r$ by the mapping $a_p \ni H(t) \mapsto t \in \mathbb{R}^r$. Moreover, we can regard the highest weight $A(t_1, \ldots, t_r)$ as an element of $a_p$ by the mapping $A(t_1, \ldots, t_r) \mapsto l_1 H_1 + \cdots + l_r H_r$. (See Takeuchi [Tak].) Then, the weight lattice $Z(G, K_p)$ is given by

$$Z(G, K_p) = \{ \lambda_1 H_1 + \cdots + \lambda_r H_r; \lambda_j \in \mathbb{Z}, 1 \leq j \leq r \}. \quad (3.6)$$
Let $W(G, K_p)$ be the Weyl group of the symmetric space $G/K_p = SU(n)/S(U(p) 	imes U(n-p))$. Then, $W(G, K_p)$ is the set of all the linear transformations $w: \mathbb{R}^{r} \to \mathbb{R}^{r}$ of the form

$$w: (t_1, ..., t_r) \mapsto (\varepsilon_1 t_{\sigma(1)}, ..., \varepsilon_r t_{\sigma(r)}),$$

where $\varepsilon_j = \pm 1$ and $\sigma$ is an element of the $r$th symmetric group $S_r$.

It is well known that to each invariant differential operator $P$ on $G/K_p$ there corresponds a unique $W(G, K_p)$-invariant differential operator on Weyl chambers. (For example, see Helgason [H4, Chap. V] or Takeuchi [Tak, Chap. 10].) This operator is called the radial part of $P$ and we denote it by $\text{rad}(P)$.

Let us choose a Weyl chamber $\mathcal{A}^+ / a_p$ as

$$\mathcal{A}^+ := \left\{ (t_1, ..., t_r) \in \mathbb{R}^{r}; 0 < t_j < \frac{\pi}{2}, 1 \leq j \leq r, t_1 > t_2 > \cdots > t_r \right\}.$$  \hspace{1cm} (3.8)

According to Helgason [H4], or Takeuchi [Tak], the following theorem holds.

**Theorem 3.3** [H4, Chap. V, Theorem 1.9; Tak, Theorems 10.1, 10.2].

1. Let us denote by $\mathcal{D}(\mathcal{A}^+)$ the algebra of all $W(G, K_p)$-invariant differential operators on the Weyl chamber $\mathcal{A}^+$. Then the mapping $P \mapsto \text{rad}(P)$ is an injective ring homomorphism from $\mathcal{D}(G/K_p)$ into $\mathcal{D}(\mathcal{A}^+) W(G, K_p)$.

2. Let $P$ be an invariant differential operator on $G/K_p$ and let $F(X) \in S(\mathfrak{g} K_p)$ be the principal symbol of $P$. Then the principal symbol of $\text{rad}(P)$ is given by $F(H(t))$ the restriction of $F$ onto $a_p$. Here we denote by $S(\mathfrak{g} K_p)$ the algebra of all $\text{Ad}-K_p$-invariant polynomials on $\mathfrak{g} K_p$.

3. Let $\lambda$ be a highest weight of a spherical representation of $G/K_p$ and let $\Theta_\lambda(g K_p) \in C^\infty(G/K_p)$ be the zonal spherical function corresponding to the highest weight $\lambda$. Then, we have $\{ P \Theta_\lambda \}|_{\mathcal{A}^+} = \text{rad}(P)|_{\mathcal{A}^+}$.

Let us denote by $\theta(t; l_1, ..., l_r)$ the restriction of the zonal spherical function $\Theta_{\lambda_1, ..., \lambda_r}$ to $\mathcal{A}^+$. Then, making use of Theorem 8.1 in Takeuchi [Tak], we have the following.

**Theorem 3.4.** The function $\theta(t; l_1, ..., l_r)$ is written in the form

$$\theta(t; l_1, ..., l_r) = C(t_1, ..., t_r) \exp \left\{ 2 \sqrt{-1} (l_1 t_1 + \cdots + l_r t_r) \right\}$$

$$+ \sum_{R, l_1, ..., l_r < R(l_1, ..., l_r)} \sum_{R, l_1, ..., l_r < R(\lambda_1, ..., \lambda_r)} \sum_{R, l_1, ..., l_r < R(G, K_p)} \text{finite sum}$$

$$\times \exp \left\{ 2 \sqrt{-1} (\lambda_1 t_1 + \cdots + \lambda_r t_r) \right\},$$
where \( C_{(l_1, \ldots, l_r)} \) is a constant depending on the weight \( A(l_1, \ldots, l_r) \) and \( C_{(l_1, \ldots, l_r)} \neq 0 \), and where \( Z(G, K_p) \) denotes the weight lattice given by (3.6).

First, we discuss the radial part of the invariant differential operator \( \Box_k^{(\nu, \rho)} \) on \( G/K_p \).

Since the principal symbol of \( \text{rad}(\Box_k^{(\nu, \rho)}) \) is a \( W(G, K_p) \)-invariant polynomial on \( e_p \), we can put

\[
\text{rad}(\Box_k^{(\nu, \rho)}) = F(\frac{\partial^2}{\partial t_1^2}, \ldots, \frac{\partial^2}{\partial t_r^2}) + \text{lower order terms},
\]

for some homogeneous symmetric polynomial \( F(t_1, \ldots, t_r) \). By Theorem 3.3 and Theorem 3.4, we have

the eigenvalue of \( \Box_k^{(\nu, \rho)} \) on \( V^{(\nu, \rho)}(l_1, \ldots, l_r) \) is

\[
= \text{rad}(\Box_k^{(\nu, \rho)}) \cdot 0(t; l_1, \ldots, l_r) = (-1)^d 2^{2d} F(l_1^2, \ldots, l_r^2) + \text{lower order polynomial of } (l_1, \ldots, l_r). \tag{3.10}
\]

Here \( d \) is the degree of \( F \).

On the other hand, by (3.3) and (3.4),

\[
\Box_k^{(\nu, \rho)} \mid V^{(\nu, \rho)}(l_1, \ldots, l_r) = S_k(l_1^2 + a_1, \ldots, l_r^2 + a_r) = S_k(l_1^2, \ldots, l_r^2) + \text{lower order polynomial of } (l_1, \ldots, l_r). \tag{3.11}
\]

Comparing (3.10) with (3.11), we have

\[
(-1)^d 2^{2d} F(l_1^2, \ldots, l_r^2) = S_k(l_1^2, \ldots, l_r^2). \tag{3.12}
\]

Since both \( F \) and \( S_k \) are symmetric polynomials, (3.12) holds for any \( (l_1, \ldots, l_r) \in \mathbb{Z}^r \). Thus we have \( d = k \) and \( F \equiv (-1)^d 2^{-2k} S_k \).

Summarizing the above argument, we obtain the following proposition.

**Proposition 3.5.** The invariant differential operator \( \Box_k^{(\nu, \rho)} \) on \( G/K_p \) defined by (3.4) is of order \( 2k \). The radial part of \( \Box_k^{(\nu, \rho)} \) is of the form

\[
\text{rad}(\Box_k^{(\nu, \rho)}) = (-1)^k 2^{-2k} S_k \left( \frac{\partial^2}{\partial t_1^2}, \ldots, \frac{\partial^2}{\partial t_r^2} \right) + \text{lower order terms}.
\]

Finally, we consider the radial part of the invariant differential operator \( \Phi_k^{(\nu, \rho)} \).
As we mentioned in Remark 2.2, the principal symbol of \( \Phi_k^{(n, p)} \) is given by the \( \text{Ad-K}_p \)-invariant polynomial \((-1)^k 2^{-2k} F_k^{(n, p)} \). Thus, by Theorem 3.4, the principal symbol of \( \text{rad}(\Phi_k^{(n, p)}) \) is given by \((-1)^k 2^{-2k} F_k^{(n, p)}(H(t)) \).

On the other hand, by (2.7),

\[
\sum_{j=0}^{r} F_j^{(n, p)}(H(t)) \lambda^{2j} = \det(I + \lambda H(t)) = \sum_{j=0}^{r} S_j(t_1^2, ..., t_r^2) \lambda^{2j}.
\]

Hence, \( F_0^{(n, p)}(H(t)) = S_k(t_1^2, ..., t_r^2) \). Thus, we have the following proposition.

**Proposition 3.6.** The radial part of the generalized John type invariant differential operator \( \Phi_k^{(n, p)} \) is written in the form

\[
\text{rad}(\Phi_k^{(n, p)}) = (-1)^k 2^{-2k} S_k \left( \frac{\partial^2}{\partial t_1^2}, ..., \frac{\partial^2}{\partial t_r^2} \right) + \text{lower order terms}.
\]

### 4. Several Results on Symmetric Polynomials

In this section, we will show several equalities on symmetric polynomials, which play an essential role in the proof of our range theorem.

We mainly study the property of the following type of symmetric polynomial.

\[
F(s; a_1, ..., a_r; x_1, ..., x_r)
:= \sum_{k=0}^{s+1} (-1)^{s+1-k} T_{s+1-k}(a_{s+1}, ..., a_r) S_k(x_1 + a_1, ..., x_r + a_r),
\]

where \( 1 \leq s \leq r - 1 \) and where \( T_k(t_1, ..., t_N) \) is a homogeneous symmetric polynomial of \((t_1, ..., t_N)\) defined by

\[
T_k(t_1, ..., t_N) := \sum_{1 \leq \sigma_1 \leq \sigma_2 \leq ... \leq \sigma_k \leq N} t_{\sigma_1} t_{\sigma_2} \cdots t_{\sigma_k},
\]

\[
T_0(t_1, ..., t_N) := 1.
\]

We introduce the following several notations.

**Notation A.** (1) We denote the set \( \{1, 2, ..., r\} \) by \( I^r \). (2) For \((x_1, ..., x_r)\) and for a subset \( J = \{j_1, ..., j_m\} \subset I^r = \{1, 2, ..., r\} \), we denote by \( x^J \) the product \( x_{j_1} \cdots x_{j_m} \). (3) For a function \( f(y_1, ..., y_m) \), for \((x_1, ..., x_r)\),
and for a subset $J = \{j_1, ..., j_m\} \subset \{1, ..., r\}$, we denote $f(x_{(J)}) := f(x_{j_1}, ..., x_{j_m})$.

Let

$$F(s; a_1, ..., a_r; x_1, ..., x_r) = \sum_{J = \{1, 2, ..., r\}} A_J x^J = \sum_{J = \{1, 2, ..., r\}} A_J(s; a_1, ..., a_r) x^J.$$  \hspace{1cm} (4.3)

Then, the following proposition holds.

**Proposition 4.1.** For $s$, and for a finite sequence $\{a_1, ..., a_r\}$, the coefficient $A_J(s; a_1, ..., a_r)$ defined by (4.3) is expressed as

$$A_J(s; a_1, ..., a_r) = \sum_{j=0}^{l} (-1)^{j-l} T_{l-1}(a_I(s)) S_j(a_{I(s)}),$$

where $m = \#J$, $l = s + 1 - m$, $J_1(s) = J \cap I^{(s)} = J \cap \{1, 2, ..., s\}$, $J_2(s) = J \setminus J_1(s)$, $J_0(s) = I^{(s)} \setminus J_1(s)$. Moreover $T_{l-1}(a_I(s))$ and $S_j(a_{I(s)})$ are as in Notation (A3). In particular, if $J \subset \{1, 2, ..., s\}$, $A_J(s; a_1, ..., a_r) = 0$ and if $\#J = s + 1$, $A_J(s; a_1, ..., a_r) = 1$.

We need the following two lemmas in order to prove Proposition 4.1

**Lemma 4.2.** The following equality holds,

$$S_k(x_1 + a_1, ..., x_r + a_r) = \sum_{J \subset I^{(r)}} S_k - \#J(a_{I^{(r)}}) x^J.$$  \hspace{1cm} (4.4)

**Proof.**

$$S_k(x_1 + a_1, ..., x_r + a_r)$$

$$= \sum_{1 \leq i_1 < \cdots < i_k \leq r} (a_{i_1} + x_{i_1}) \cdots (a_{i_k} + x_{i_k})$$

$$= \sum_{I \subset I^{(r)}, \#I = k} \sum_{J \subset I} a_J^I x^J \quad \text{(here we put } I = \{i_1, ..., i_k\} \text{)}$$

$$= \sum_{J \subset I^{(r)}} \left\{ \sum_{I \subset I^{(r)}, \#I = k} a_I^J \right\} x^J \quad \text{(let } \tilde{I} = I \setminus J \text{)}$$

$$= \sum_{J \subset I^{(r)}} \left\{ \sum_{I \subset I^{(r)}, \#I = k} a_I^J \right\} x^J = \sum_{J \subset I^{(r)}} S_k - \#J(a_{I^{(r)}}) x^J.$$
Lemma 4.3. Let \( m = \# J, \ l = s + 1 - m. \) Then

\[
\sum_{k=0}^{l} (-1)^{l-k} T_{l-k}(a_{(t^r \cap J^s)}) S_k(a_{(t^r \cap J^s)}) \\
= \sum_{j=0}^{l} (-1)^{l-j} T_{l-j}(a_{(t^r \cap J^s)}) S_j(a_{(t^r \cap J^s)}). \tag{4.5}
\]

In particular, if \( J \subset \{1, 2, \ldots, s\}, \) then the above quantity equals 0.

**Proof.** For the sake of simplicity, we put

\[
J_0 = J_0(s), \quad J_1 = J_1(s), \quad J_2 = J_2(s), \quad J_3 = I^{r_0} \backslash (J_0 + J_1 + J_2). \text{ Then } I^{r_0} = J_0 + J_1 + J_2. \text{ Moreover, } I^{r_0} \backslash I^{s_0} = J_2 + J_3, \ I^{r_0} \backslash J^{s_0} = J_0 + J_3 \text{ and } l = s + 1 - m = \# J_0 + 1 - \# J_2.
\]

We note that \( r - m = \# (I^{r_0} \backslash J) = \# (J_0 + J_3) \) and that

\[
\sum_{j=0}^{r-m} S_j(a_{(t^r_2 + J_3)}) t^j = \prod_{r^t_2 + J_3} (1 + a_r t) \\
= \prod_{a \in J_2} (1 + a_r t) \times \prod_{a \in J_3} (1 + a_r t),
\]

\[
\sum_{k=0}^{\infty} (-1)^k T_k(a_{(t^r_2 + J_3)}) t^k = \prod_{t^r_2 + J_3} (1 + a_r t)^{-1} \\
= \prod_{a \in J_2} (1 + a_r t)^{-1} \times \prod_{a \in J_3} (1 + a_r t)^{-1}.
\]

Thus we have

\[
\left\{ \sum_{j=0}^{r-m} S_j(a_{(t^r_2 + J_3)}) t^j \right\} \left\{ \sum_{k=0}^{\infty} (-1)^k T_k(a_{(t^r_2 + J_3)}) t^k \right\} \\
= \prod_{a \in J_2} (1 + a_r t) \times \prod_{a \in J_3} (1 + a_r t)^{-1} \\
= \left\{ \sum_{j=0}^{r-m} S_j(a_{(t^r_2)}) t^j \right\} \left\{ \sum_{k=0}^{\infty} (-1)^k T_k(a_{(t^r_2)}) t^k \right\}. \tag{4.6}
\]

Comparing the coefficients of \( t^l = t^{s+1-m} = t^{\# J_2 + 1 - \# J_2} \) of the both sides of (4.6), we obtain (4.5). If \( J \subset \{1, 2, \ldots, s\}, \) \( J_2 \) is an empty set and the right hand side of (4.6) is a polynomial of \( t \) of degree \( s \). Then we see that the coefficient of \( t^l \) \( (t^{s+1-m} = t^{\# J_2 + 1 - \# J_2}) \) equals 0.

Now we proceed to the proof of Proposition 4.1.
Proof of Proposition 4.1. By Lemma 4.2, we have

\[ \sum_{k=0}^{s+1} (-1)^{s+1-k} T_{s+1-k}(a_{s+1}, \ldots, a_s) S_k(x_1 + a_1, \ldots, x_r + a_r) \]

\[ = \sum_{k=0}^{s+1} (-1)^{s+1-k} T_{s+1-k}(a_{I \setminus \{\rho_0\}}) \]

\[ \times \sum_{J \in I^0, \#J \leq s+1} S_{k - \#J}(a_{I \setminus \{\rho_0,J\}}) x^J \]

\[ = \sum_{J \in I^0, \#J \leq s+1} x^J \]

\[ \times \left\{ \sum_{k=0}^{s+1} (-1)^{s+1-k} T_{s+1-k}(a_{I \setminus \{\rho_0,J\}}) S_k(a_{I \setminus \{\rho_0,J\}}) \right\} . \]

If $k - \#J < 0$, $S_{k - \#J}(a_{I \setminus \{\rho_0,J\}}) = 0$. Thus, by Lemma 4.3,

\[ A_J = \text{the coefficient of } x^J \]

\[ = \sum_{k \neq J} (-1)^{s+1-k} T_{s+1-k}(a_{I \setminus \{\rho_0,J\}}) S_{k - \#J}(a_{I \setminus \{\rho_0,J\}}) \]

\[ = \sum_{j=0}^{s+1-m} (-1)^{s+1-m-j} T_{s+1-m-j}(a_{I \setminus \{\rho_0,J\}}) S_j(a_{I \setminus \{\rho_0,J\}}) . \]

In particular, if $J \in \{1, 2, \ldots, s\}$, by Lemma 4.3 we see that $A_J (x; a_1, \ldots, a_s) = 0$. The last assertion is obvious. Therefore the assertion is proved.

Furthermore, we introduce several notations to state the next Proposition.

**Notation B.** (1) We denote by $P_+^\mu$ the set of all the strictly increasing sequences $I = \{i_1, i_2, \ldots, i_\mu\}$ contained in the set $I^\mu = \{1, 2, \ldots, \mu\}$, namely,

\[ P_+^\mu := \{I = \{i_1, i_2, \ldots, i_\mu\} \in I^\mu; 1 \leq i_1 < i_2 < \cdots < i_\mu \leq \mu \} . \quad (4.7) \]

(2) Similarly, we denote by $P_-^\mu$ the set of all the non-decreasing sequences $J = \{j_1, j_2, \ldots, j_\mu\}$ such that $j_\mu \in \mathbb{Z}$, $(1 \leq \mu \leq l)$ and $1 \leq j_1 \leq j_2 \leq \cdots \leq j_\mu \leq v$. Namely, we put

\[ P_-^\mu := \{J = \{j_1, j_2, \ldots, j_\mu\} \in I^\mu; 1 \leq j_1 \leq j_2 \leq \cdots \leq j_\mu \leq \nu, j_\mu \in \mathbb{Z} \}, \]

\[ \{1 \leq \mu \leq l\} . \quad (4.8) \]
We denote \( f(x) := i_a \) (resp. \( J(x) := j_a \)).

We define a lexicographical order \( < \) on \( \mathcal{P} \) and on \( \mathcal{B} \) in the usual manner, that is, \( I = \{ i_1, i_2, \ldots, i_n \} < I' = \{ i'_1, i'_2, \ldots, i'_n \} \Leftrightarrow \exists x \) such that \( i_p = i'_p \), (1 \( \leq \beta \leq n - 1 \)) and \( i_n < i'_n \).

(3) Under the assumption that \( v + l = \mu + 1 \), we can define a mapping \( M \) \( \exists I \mapsto I^* \in \mathcal{B} \) as follows. \( I = \{ i_1, i_2, \ldots, i_n \} \mapsto I^* := \{ i_1, i_2 - 1, \ldots, i_n - 1 \} \). Then we see easily that the above mapping is an order preserving bijection.

**Proposition 4.4.** We assume that \( v + l = \mu + 1 \). For any two sequences \( \{ u_1, u_2, \ldots, u_n \} \) and \( \{ v_1, v_2, \ldots, v_n \} \), we have

\[
\sum_{j=0}^{l} T_{l-j}(u_1, u_2, \ldots, u_n) S_j(v_1, v_2, \ldots, v_n) = \sum_{I \in \mathcal{P}} \prod_{s=1}^{l} (v_{r(s)} + u_{j(s)}). \tag{4.9}
\]

**Proof.** For simplicity, let

\[
A_j := T_{l-j}(u_1, u_2, \ldots, u_n) S_j(v_1, v_2, \ldots, v_n), \quad A := \sum_{j=0}^{l} A_j,
\]

\[
B(I) := \prod_{s=1}^{l} (v_{r(s)} + u_{j(s)}), \quad B := \sum_{I \in \mathcal{P}} B(I).
\]

For a polynomial \( f \) of \( u_a \)'s and \( v_a \)'s, we define a set \( \mathcal{M}(f) \) by the set of all the monomials which appear in the expansion of \( f \).

**Example (i).** If \( f = (u_1 + u_2)(v_1 + v_2 + v_3) \), \( \mathcal{M}(f) \) is given by

\[
\mathcal{M}(f) = \{ u_1 v_1, u_1 v_2, u_1 v_3, u_2 v_1, u_2 v_2, u_2 v_3 \}.
\]

**Example (ii).** If \( f = T_2(u_1, u_2) S_3(v_1, v_2, v_3) = (u_1^2 + u_1 u_2 + u_2^2)(v_1 + v_2 + v_3) \), \( \mathcal{M}(f) \) is given by

\[
\mathcal{M}(f) = \{ u_1^2 v_1, u_1 u_2 v_1, u_2^2 v_1, u_1 v_2^2, u_1^2 v_2, u_2 v_3^2, u_1 u_2 v_3, u_2^2 v_3 \}.
\]

Of course, we have to exclude the case when the same monomial appears more than twice in the expansion of a given polynomial \( f \). For example, \( f = (u_1 + u_2)^2 = u_1^2 + 2u_1 u_2 + u_2^2 \). Indeed, in such a case, \( \mathcal{M}(f) \) is not well defined. From now on, we consider only the case when \( \mathcal{M}(f) \) is well defined. In other words, a polynomial \( f \) which we deal with has the property that the coefficient of each monomial in the expansion of \( f \) is 1. In particular, by the definition of \( A_j \) and \( B(I) \), we see easily that \( \mathcal{M}(A_j) \) and \( \mathcal{M}(B(I)) \) are well defined.
We note that, for two polynomials \( f \) and \( g \) of \( u_a \)'s and \( v_g \)'s, \( \mathcal{A}(f + g) \) is well defined if \( \mathcal{A}(f) \) and \( \mathcal{A}(g) \) are well defined and \( \mathcal{A}(f) \cap \mathcal{A}(g) = \phi \). Moreover, we see that in such a case \( \mathcal{A}(f + g) = \mathcal{A}(f) + \mathcal{A}(g) \) (disjoint union).

Let us assume the following (a), (b), and (c).

(a) \( j \neq k \Rightarrow \mathcal{A}(A_j) \cap \mathcal{A}(A_k) = \phi \).

(b) \( I, \bar{I} \in \mathcal{P}_{\mu}^*; I \neq \bar{I} \Rightarrow \mathcal{A}(B(I)) \cap \mathcal{A}(B(\bar{I})) = \phi \).

(c) \( \forall I \in \mathcal{P}_{\mu}^* \) and \( \forall w \in \mathcal{A}(B(I)) \), there exists a number \( j \) such that \( w \in \mathcal{A}(A_j) \).

Then Proposition 4.4 is proved in the following way. Since (a) holds, \( \mathcal{A}(A) \) is well defined and we have \( \mathcal{A}(A) = \bigcup_{j=0}^{l} \mathcal{A}(A_j) \) (disjoint union). Similarly, by (b), \( \mathcal{A}(B) \) is well defined and we have \( \mathcal{A}(B) = \bigcup_{I \in \mathcal{P}_{\mu}^*} \mathcal{A}(B(I)) \) (disjoint union). Therefore, by (c), \( \mathcal{A}(B) \subset \mathcal{A}(A) \). On the other hand, by the assumption that \( v + l = \mu + 1 \), \( \# \mathcal{A}(A_j) = \binom{\mu + 1}{j} \). Thus

\[
\# \mathcal{A}(A) = \sum_{j=0}^{l} \# \mathcal{A}(A_j) = \sum_{j=0}^{l} \binom{\mu + 1}{j} = 2^l \cdot \binom{\mu}{1}.
\]

Moreover, \( \# \mathcal{A}(B) = \sum_{I \in \mathcal{P}_{\mu}^*} \# \mathcal{A}(B(I)) = \# \mathcal{P}_{\mu}^* \cdot 2^l = 2^l \cdot \binom{\mu}{1} \). Therefore we have \( \mathcal{A}(B) = \mathcal{A}(A) \), which proves the assertion.

Finally we give the proofs of (a), (b), and (c).

**Proof of (a).** Each element in \( \mathcal{A}(A_j) \) is of degree \( j \) with respect to \( \{ v_g \}_{g=1}^l \) and of degree \( l - j \) with respect to \( \{ u_a \}_{a=1}^m \). Therefore, (a) is easily seen.

**Proof of (b).** We may assume \( I < \bar{I} \). Then, by definition, we can take \( \alpha \) such that \( I(\beta) = \bar{I}(\beta) \), \( 1 \leq \beta \leq \alpha - 1 \) and \( I(\alpha) < \bar{I}(\alpha) \). Since the mapping \( I \mapsto I^\mu \) is order preserving, we have \( I^\mu(\alpha) < \bar{I}^\mu(\alpha) \). Thus \( v_{r(\alpha)} \neq v_{r(\beta)} \) and \( u_{r(\alpha)} \neq u_{r(\beta)} \). Each monomial in \( \mathcal{A}(B(I)) \) has \( v_{r(\alpha)} \) or \( u_{r(\beta)} \) as a factor. Similarly, each monomial in \( \mathcal{A}(B(\bar{I})) \) has \( v_{r(\beta)} \) or \( u_{r(\alpha)} \) as a factor. Therefore, any element in \( \mathcal{A}(B(I)) \) and any element in \( \mathcal{A}(B(\bar{I})) \) do not coincide.

**Proof of (c).** Any element \( w \in \mathcal{A}(B(I)) \) can be written in the form

\[
w = v_{r_1(\alpha)} \cdots v_{r_l(\alpha)} \cdot u_{r_{1}(\beta)} \cdots u_{r_{l}(\beta)},
\]

\[
\{ r_1, ..., r_l \} \cup \{ \sigma_1, ..., \sigma_{l-j} \} = \{ 1, ..., l \},
\]

for some \( j \). Then we see easily that the above \( w \) arises from the expansion of \( A_j = T_{l-j}(u_1, ..., u_\nu) S_j(v_1, ..., v_g) \). Therefore, we have \( w \in \mathcal{A}(A_j) \).
Proposition 4.5. (1) Let \( J^1 := J_0(s) = \{1, 2, \ldots, s\} \setminus J \) and \( J^{2s} := J_2(s) = J \setminus \{1, 2, \ldots, s\} \) for a given subset \( J \subset \{1, 2, \ldots, r\}, (1 \leq \# J \leq s) \). Moreover, let \( \mu := \# J^1 \) and \( v := \# J^{2s} \). We assume that \( v \geq 1 \). Then, the coefficient
\[
A_J(s; a_1, \ldots, a_s)
\]
is rewritten as
\[
A_J(s; a_1, \ldots, a_s) = \sum_{l \in \mathcal{J}_s} \prod_{s=1}^l (a_{\mu l(t_s)} - a_{\mu l(t_s)}),
\]
(4.10)
where \( l = s + 1 - \# J \). Here we note that \( l = \mu + 1 - v \).

(2) In addition, we assume that \( a_1 > a_2 > \cdots > a_s \). Then \( A_J(s; a_1, \ldots, a_s) \geq 0 \). Moreover, \( A_J(s; a_1, \ldots, a_s) = 0 \) if and only if \( J \subset \{1, 2, \cdots, s\} \).

Proof. By the assumption, we can put
\[
J^1 = J_0(s) = \{\tau(1), \tau(2), \ldots, \tau(\mu)\},
\]
\[
J^{2s} = J_2(s) = \{\sigma(1), \sigma(2), \ldots, \sigma(v)\},
\]
Then we can apply Proposition 4.4 to \( v_j := a_{\mu(j)} \ (1 \leq j \leq \mu) \) and to \( u_k := -a_{\mu(k)} \ (1 \leq k \leq v) \). Thus, by Proposition 4.1,
\[
A_J(s; a_1, \ldots, a_s) = \sum_{j=0}^{\mu} T_j (-a_{\mu(1)}, \ldots, -a_{\mu(v)}) S_j (a_{\mu(1)}, \ldots, a_{\mu(v)})
\]
which proves the assertion (1). By Proposition 4.1, if \( J \subset I^{(s)} = \{1, 2, \ldots, s\} \), then \( A_J(s; a_1, \ldots, a_s) = 0 \). Let us assume that \( J \not\subset I^{(s)} = \{1, 2, \ldots, s\} \). Then neither \( J^1 \) nor \( J^{2s} \) is empty, by the definition of \( J^1 \) and \( J^{2s} \). Hence we have \( a_{J(\mu(s))} = a_{J(\mu(s))} > 0 \). Therefore, by (4.10), we see that \( A_J(s; a_1, \ldots, a_s) > 0 \).

As a consequence of Proposition 4.5, we have the following corollary.

Corollary 4.6. We assume that \( a_1 > a_2 > \cdots > a_s \), and that \( x_j \geq 0 \ (1 \leq j \leq r) \). Then the polynomial \( F(s; a_1, \ldots, a_s; x_1, \ldots, x_r) \) defined by (4.1) equals 0 if and only if \( x_{r+1} = \cdots = x_r = 0 \).

Then above corollary will be used to show the range theorem. (See Theorem 5.1(C) and (D).)
Proposition 4.7. The polynomial $F(s; a_1, ..., a_r; x_1, ..., x_r)$ defined by (4.1) satisfies
\begin{equation}
F(d-1; a_1, ..., a_r; x_1, ..., x_r, \underbrace{0, ..., 0}_s) = F(d-1; a_1 + c, ..., a_s + c; x_1, ..., x_r),
\end{equation}
for any constant $c$ and for $d(1 \leq d \leq s)$.

Proof. Due to the expression (4.3), we have
the left hand side of (4.11)\[= J \big|_{s=1}^{d-1; a_1, ..., a_r} x_j |_{x_{s+1} = \ldots x_r = 0}
= J \big|_{s=1}^{d-1; a_1, ..., a_r} x_j.
\]
Here by Proposition 4.1, the coefficient $A_j(d-1; a_1, ..., a_r)$ is given by
\[
A_j(d-1; a_1, ..., a_r) = \sum_{j=0}^{l} (-1)^{l-j} T_{l-j}(a_j(\delta d - 1)) S_j(a_j(\delta d - 1)),
\]
where $l = (d - 1) + 1 - \# J = d - \# J J_0(s) = I^{d-1} \setminus (J \cap I^{d-1})$, and $J_0(s) = I^{d-1} \setminus (J \cap I^{d-1})$. We note that, since $J \subset \{1, 2, ..., s\}$ and $d \leq s$, $A_j(d-1; a_1, ..., a_r)$ does not depend on $\{a_{s+1}, ..., a_r\}$ any more. Thus we can rewrite
$A_j(d-1; a_1, ..., a_r) = A_j(d-1; a_1 + c, ..., a_s + c)$ for any constant $c$. Therefore we obtain the assertion.

5. RANGE THEOREM—COMPLEX CASE

In this section, we give detailed results about generalized John type differential operators on the complex Grassmann manifolds. As a part of the results, we obtain the range theorem for the Radon transform $R^I_q$ on the complex Grassmann manifold $Gr(p, n, \mathbb{C})$. (See Theorem 5.1(D) below.)

Theorem 5.1. We assume that $s := \text{rank } Gr(q, n, \mathbb{C}) < r := \text{rank } Gr(p, n, \mathbb{C})$. For the generalized John type invariant differential operator $\Phi_{\nu, \eta}^{\mu, \xi}$ on the complex Grassmann manifold $Gr(p, n, \mathbb{C})$, the following (A), (B), (C), (D), and (E) hold.
(A) The eigenvalue of $\Phi^{(n,p)}_{s+1}$ on the irreducible eigenspace $V^{(n,p)}(l_1, ..., l_s)$ is given by the formula

$$\Phi^{(n,p)}_{s+1} \mid V^{(n,p)}(l_1, ..., l_s) = \sum_{k=0}^{s+1} (-1)^{s+1-k} T_{s+1-k}(a_{s+1}, ..., a_s) S_k(z_1 + a_1, ..., z_s + a_s),$$

where $T_j$ and $S_j$ are the symmetric polynomials given respectively by (3.3) and (4.2), and where $z_j = l_j + n + 1 - 2j$ and $a_j = j^2 - (n+1)j + n$. (See (3.2).)

(B) $\Phi^{(n,p)}_{s+1}$ is expressed as

$$\Phi^{(n,p)}_{s+1} = \sum_{k=0}^{s+1} (-1)^{s+1-k} T_{s+1-k}(a_{s+1}, ..., a_s) \square_k^{(n,p)},$$

where $\square_k^{(n,p)}$ is the $2k$th order invariant differential operator on $Gr(p,n,\mathbb{C})$ defined by (3.4).

(C) $\Phi^{(n,p)}_{s+1} \mid V^{(n,p)}(l_1, ..., l_s) = 0$ if and only if $l_{s+1} = \cdots = l_s = 0$.

(D) The range $\operatorname{Im} R^q_p$ of the Radon transform $R^q_p$ is identical with the kernel of $\Phi^{(n,p)}_{s+1}$, that is, $\operatorname{Im} R^q_p = \operatorname{Ker} \Phi^{(n,p)}_{s+1}$.

(E) Let $P$ be an invariant differential operator on $Gr(p,n,\mathbb{C})$ satisfying the following two conditions.

(Ei) $\operatorname{Im} R^q_p \subset \operatorname{Ker} P$.

(Eii) The radial part of $P$, $\operatorname{rad}(P)$, is of the form

$$\operatorname{rad}(P) = (-1)^{s+1} 2^{-2(s+1)} S_{s+1} \left( \frac{\partial^2}{\partial t_1^2}, ..., \frac{\partial^2}{\partial t_r^2} \right) + \text{lower order terms}.$$

Then $P$ coincides with $\Phi^{(n,p)}_{s+1}$.

In order to prove the above theorem, we need the existence of the inversion mapping for the Radon transform $R^q_p: C^\infty(G/K_q) \to C^\infty(G/K_p)$, which is guaranteed by the following theorem.

Theorem 5.2 (Grinberg [Gri3]). We assume that $s = \operatorname{rank} G/K_q \leq r = \operatorname{rank} G/K_p$. Then there exists an invariant differential operator $D_{(q)}$ on the Grassmann manifold $G/K_q$ such that $D_{(q)} R^q_p R^q_p = 1$ on $C^\infty(G/K_q)$.

As we mentioned in the Introduction, Grinberg [Gri3] assumes that $q < p \leq n - p$. Namely, he does not deal with the cases $q < [n/2] \leq p$ and $p < [n/2] \leq q$. We will prove the existence of the inversion formulas in such cases in Theorem 6.4 without using the results in this section.
Proof of Theorem 5.1. We put
\[ \Psi_d^{(n, p)} := \sum_{k=0}^{d} (-1)^{d-k} T_{d-k}(a_1, ..., a_r) \square_k^{(n, p)}. \]  
(5.1)

By the definition of \( \square_k^{(n, p)} \) (see (3.4)), we have
\[ \Psi_{s+1}^{(n, p)} |_{y^{(n, p)(l_0, ..., l_s)}} = \sum_{k=0}^{s+1} (-1)^{s+1-k} T_{s+1-k}(a_{s+1}, ..., a_r) S_k(x_1 + a_1, ..., x_r + a_r) \]
\[ = F(s; a_1, ..., a_r; x_1, ..., x_r). \]  
(5.2)

Here the polynomial \( F(s; a_1, ..., a_r; x_1, ..., x_r) \) is given by (4.1).

Since \( \{a_j\}_{j=1}^{r} \) is a strictly decreasing sequence and \( x_j \geq 0 \), we can apply Corollary 4.6 to the polynomial \( F(s; a_1, ..., a_r; x_1, ..., x_r) \). Thus we have
\[ \Psi_{s+1}^{(n, p)} |_{y^{(n, p)(l_0, ..., l_s)}} = 0 \iff x_{s+1} = \cdots = x_r = 0 \]
\[ \iff l_{s+1} = \cdots = l_r = 0. \]  
(5.3)

Since \( R_{p}^{s} \) is \( G \)-equivariant and injective (due to Theorem 5.2), we see that \( R_{p}^{s} \) isomorphically maps
\[ \bigoplus_{l_1 \geq \cdots \geq l_s \geq 0} V^{(n, p)}(l_1, ..., l_s) \to \bigoplus_{l_1 \geq \cdots \geq l_s \geq 0} V^{(n, p)}(l_1, ..., l_s, 0, ..., 0) \]
which is densely contained in \( \text{Ker} \Psi_{s+1}^{(n, p)} \) with respect to the \( C^\infty \) topology.

By the density argument combined with Theorem 5.2, we obtain
\[ \text{Im} \ R_{p}^{s} = \text{Ker} \Psi_{s+1}^{(n, p)}. \]  
(5.4)

Due to Proposition 2.3 and Proposition 3.6, the invariant differential operator \( \Phi_{s+1}^{(n, p)} \) satisfies the conditions (EI) and (EII) in Theorem 5.1(E).

Therefore, by the above argument, it suffices to show that \( P = \Psi_{s+1}^{(n, p)} \) for an invariant differential operator \( P \) satisfying (EI) and (EII). Let \( Q = P - \Psi_{s+1}^{(n, p)} \). It follows from Proposition 3.5 that \( \text{rad}(\Psi_{d}^{(n, p)}) \) is written in the form
\[ \text{rad}(\Psi_{d}^{(n, p)}) = (-1)^{d} S_0 \left( \frac{\partial^2}{\partial x_1^2}, ..., \frac{\partial^2}{\partial x_r^2} \right) + \text{lower order terms}. \]  
(5.5)

By (5.5) and the condition (EII), the principal part of \( \text{rad}(P) \) coincides with that of \( \text{rad}(\Psi_{s+1}^{(n, p)}) \). Thus, \( Q = P - \Psi_{s+1}^{(n, p)} \) is an invariant differential
operator at most of 2\textsuperscript{st} order. Here we note that any symmetric polynomial of \((x_1, \ldots, x_s)\) of degree \(s\) can be written as a polynomial of \(s\) elementary symmetric polynomials \(S_j(x_1, \ldots, x_r)\) (\(1 \leq j \leq s\)). Due to (5.5), we see that there is a polynomial \(F(w_1, \ldots, w_s)\) such that \(\operatorname{rad}(Q) = F(\operatorname{rad}(\Psi^{(n, p)}_1), \ldots, \operatorname{rad}(\Psi^{(n, p)}_s))\). In view of Theorem 3.3(1), we have \(Q = F(\Psi^{(n, p)}_1, \ldots, \Psi^{(n, p)}_s)\) for the above polynomial \(F\).

By (5.4) and the condition (EI), we have \(\operatorname{Im} R^\phi = \operatorname{Ker} Q\). Hence \(QR^\phi f = 0\) for \(f \in C^\infty(G/K_p)\). The following lemma follows from Proposition 4.7.

\textbf{Lemma 5.3.} \(\Psi^{(n, p)}_d R^\phi = R^\phi \Psi^{(n, q)}_d\) holds for \(d(1 \leq d \leq s)\).

Then, by Lemma 5.3, we have \(R^\phi f(\Psi^{(n, q)}_1, \ldots, \Psi^{(n, q)}_s) f = 0\), for \(f \in C^\infty(G/K_q)\). Thus, by Theorem 5.2, \(F(\Psi^{(n, q)}_1, \ldots, \Psi^{(n, q)}_s) f = 0\), for \(f \in C^\infty(G/K_q)\), which means \(F(\Psi^{(n, q)}_1, \ldots, \Psi^{(n, q)}_s) = 0\). Taking into account that \(s\) operators \(\Psi^{(n, q)}_1, \ldots, \Psi^{(n, q)}_s\) are algebraically independent, we see easily that \(F \equiv 0\). Therefore, \(Q = P - \Psi^{(n, q)}_{d+1} = 0\), which completes the proof of Theorem 5.1.

\textbf{Remark 5.4.} (1) Theorem 5.1(E) implies a kind of uniqueness of the range-characterizing operator. In fact, the explicit form of the range-characterizing operator is determined only by the conditions (EI) and (EII). However, if we remove the condition (EII), there exist infinitely many range-characterizing operators. Indeed, for any positive differential operator \(A\), the differential operator \(A^\phi\Psi^{(n, q)}_{d+1}\) characterizes the range of \(R^\phi\).

(2) Conversely, there exists no invariant range-characterizing operator whose order is less than \(2s+2\). This is easily checked by the same argument as in the proof of Theorem 5.1. (3) By the proof of Theorem 5.1 combined with Lemma 5.3, we see that the Radon transform \(R^\phi\) intertwines generalized John type operators, i.e., \(A^\phi \Psi^{(n, p)}_d R^\phi = R^\phi \phi^\phi \Psi^{(n, q)}_d\) (\(1 \leq d \leq s\)).

6. INVERSION FORMULA—COMPLEX CASE

In this section, we construct the explicit inversion formula for the Radon transform on the complex Grassmann manifold \(R^\phi: C^\infty(G/K_p) \rightarrow C^\infty(G/K_q)\). (Here \(G = SU(n)\) and \(K_q = SU(d) \times U(n-d)\).) Therefore, throughout this section, we assume that \(s := \text{rank } G/K_p \leq r := \text{rank } G/K_q\).

We start with the following result.
Theorem 6.1 (Grinberg [Gr3]). We assume that \(2d + 1 \leq n\). Then
\[
R^d_{d+1} R^d_{d+1} f = \prod_{k=1}^{d} k(n-d-k) \prod_{k=1}^{d} (l_k + d + 1 - k)(l_k + n - d - k) f,
\]
for \(f \in V^{(n,d)}(l_1, \ldots, l_d)\).

First, we remark that he proved the above formula under the assumption that \(d + 1 \leq \lfloor n/2 \rfloor\). However, we see that his method works well up to the case \(n = 2d + 1\), if we read his paper carefully. Second, Grinberg [Gr3] does not deal with the case \(q < \lfloor n/2 \rfloor < p\). In addition, he does not calculate explicitly the generalized inversion formula. However, we need the detailed calculus to get the explicit expression of the inversion formula. Our calculation is divided into two steps.

Step (I). We calculate the value \(\{R^q_{p} R^q_{p} \mid_{V^{(n,q)}(l_1, \ldots, l_q)}\}^{-1}\) in the case \(q < p \leq \lfloor n/2 \rfloor\). We can rewrite the above theorem as
\[
\prod_{k=1}^{q} (x_k + a_k + n - d) \prod_{k=1}^{q} R^d_{d+1} R^d_{d+1} \mid_{V^{(n,q)}(l_1, \ldots, l_q)} = 1,
\]
(6.1)
where \(x_j = l_j(l_j + n + 1 - 2j)\) and \(a_j = j^2 - (n + 1)j + n\). (See (3.2).)

Since the Radon transform \(R^q_{p}\) is decomposed into \(R^q_{p-1} \cdots R^q_{q+1}\), we can apply (6.1) to the case \(d = q, q + 1, \ldots, p - 1\). Then, after a straightforward computation, we have
\[
\prod_{q+1 \leq s \leq p} C_{[s,s]} \prod_{k=1}^{s} (x_k + a_k + r_a) \prod_{k=1}^{s} R^q_{p} R^q_{p} \mid_{V^{(n,q)}(l_1, \ldots, l_q)} = 1,
\]
(6.2)
where
\[
C_{[s,s]} = \frac{(s-1)! (n-s)!}{(s-1)! (n-s)!}, \quad r_a = (s-1)(n-s).
\]
(6.3)

Step (II). Next, we go into the case \(q < \lfloor n/2 \rfloor < p\). What we do here is a kind of “mountain pass argument.”

We need the following two lemmas to use the result in Step (I).

Lemma 6.2. We define a mapping \(\text{Perp}_{d} : G/K_d \to G/K_{n-a}\) by \(\text{Perp}_{d}(\xi) := \alpha^+ (the \ orthogonal \ complement \ of \ \xi)\). Then, under the assumption that \(q < p \leq \lfloor n/2 \rfloor\), we have the commutative diagram
where \( \text{Perp}^*_d \) denotes the canonical mapping from \( C^\infty(G/K_{n-d}) \) to \( C^\infty(G/K_d) \) induced from \( \text{Perp}^*_d \).

**Lemma 6.3.** We assume that \( \lfloor n/2 \rfloor < t < p \). Let \( \mu = \text{rank } G/K_t \). If \( CR_t^p R_t^q |_{V^{\infty}(l_1, \ldots, l_r, 0, \ldots, 0)} = 1 \) for some constant \( C \), then

\[
CR_t^p R_t^q |_{V^{\infty}(l_1, \ldots, l_r, 0, \ldots, 0)} = 1 \quad \text{for the same constant } C.
\]

Since the above two lemmas are easily seen, we omit the proofs.

Let \( t = \lfloor n/2 \rfloor \). By (6.2) and by Lemma 6.2, we have

\[
\left\{ \prod_{r+1 \leq s \leq t} C_{\{r,s\}} \prod_{k=1}^{s} (z_k + a_k + v_p) \right\} \text{ of } V^{\infty}(l_1, \ldots, l_r, 0, \ldots, 0) = 1. \quad (6.4)
\]

By making use of Lemma 6.3, we see that

\[
\left\{ \prod_{r+1 \leq s \leq t} C_{\{r,s\}} \prod_{k=1}^{s} (z_k + a_k + v_p) \right\} \text{ of } V^{\infty}(l_1, \ldots, l_r, 0, \ldots, 0) = 1. \quad (6.5)
\]

On the other hand, by the result in Step (I),

\[
\left\{ \prod_{q+1 \leq s \leq p} C_{\{s,q\}} \prod_{k=1}^{s} (z_k + a_k + v_q) \right\} \text{ of } V^{\infty}(l_1, \ldots, l_q) = 1. \quad (6.6)
\]

Since \( R_q^p R_p^q = R_p^q R_q^p R_p^q R_q^p \), we obtain

\[
\left\{ \prod_{q+1 \leq s \leq p} C_{\{s,q\}} \prod_{k=1}^{s} (z_k + a_k + v_q) \right\} \text{ of } V^{\infty}(l_1, \ldots, l_q) = 1. \quad (6.7)
\]

Hence, by (3.4),

\[
\left\{ \prod_{q+1 \leq s \leq p} C_{\{s,q\}} \sum_{k=0}^s v_{k}^{s-k} \square_{[k]}^{(m,q)} \right\} R_q^p R_p^q = I \quad \text{on } C^\infty(G/K_p). \quad (6.8)
\]

The argument is similar in the two cases \( \lfloor n/2 \rfloor < p < q \) and \( p < \lfloor n/2 \rfloor < q \). Therefore, summarizing the calculations above, we obtain the following theorem.
Theorem 6.4. We assume that \( s \leq r \). Then, we have the following.

1. (Inversion formula—diagonalized version)

\[
\left\{ \prod_{s+1 \leq k \leq \lfloor p-q \rfloor} c_{s,k} \prod_{k=1}^{s} \left( z_k + a_k + v_k \right) \right\}_{R^p} R^q_{\|_{\mathcal{V}(\zeta, \cdots, \zeta)}} = 1,
\]

where \( c_{s,k} \) and \( v_k \) are given by (6.3).

2. (Inversion formula—weak version)

\[
\left\{ \prod_{s+1 \leq k \leq \lfloor p-q \rfloor} c_{s,k} \sum_{k=0}^{s} v_{s-k} \square_k (n,q) \right\}_{R^p} R^q_{\|_{\mathcal{V}(\zeta, \cdots, \zeta)}} = I, \quad \text{on } C^\infty(G/K_q).
\]

Here the invariant differential operator \( \square_k (n,q) \) on \( G/K_q \) is defined by (3.4).

Therefore, Grinberg's theorem (Theorem 5.2) is completely proved under the condition \( s \leq r \). We remark that any result in Section 5 is not used in the proof of the above theorem.

Example 6.5. We apply the above theorem to the case \( q = 1 \) and \( s = 1 \). We see that \( c_{s,1} = \left\{ (\alpha - 1)(n-\alpha) \right\}^{-1} \) and that \( v_1 + \square_1 (n,q) = (\alpha - 1)(n-\alpha) + \Phi_1^{(n,1)} \). Then, we have the following inversion formula for the Radon transform \( R^p_k : C^\infty(\mathbb{P}^{n-1} \mathbb{C}) \rightarrow C^\infty(G/K_p) \equiv C^\infty(Gr(p,n,\mathbb{C})) \).

\[
\prod_{n=2}^{p} \left( \frac{1}{(n-1)(n-\alpha)} \Phi_1^{(n,1)} + 1 \right) R^p_k R^q_k = I \quad \text{on } C^\infty(\mathbb{P}^{n-1} \mathbb{C}).
\]

We note that \( \Phi_1^{(n,1)} \) is the standard Laplacian on \( \mathbb{P}^{n-1} \mathbb{C} \). The above inversion formula was first obtained by Grinberg [Gri2].

Now, at last, we arrive at the explicit inversion formula.

Theorem 6.6 (Inversion Formula—Explicit Version). We assume that \( s := \text{rank } G/K_q \leq r := \text{rank } G/K_p \). The following inversion formula holds for the Radon transform \( R^p_k \) on the compact complex Grassmann manifold \( G/K_q = Gr(q,n,\mathbb{C}) \).

\[
\left\{ \prod_{s+1 \leq k \leq \lfloor p-q \rfloor} (\alpha - 1-k)! (n-\alpha-k)! \Phi_k^{(n,q)} \right\}_{R^p_k} R^q_k = I, \quad \text{on } C^\infty(G/K_q),
\]

where \( \Phi_k^{(n,q)} \) \((1 \leq k \leq s)\) is the generalized John type operator on \( Gr(q,n,\mathbb{C}) \) given by (2.5). (Replace \( p \) by \( q \) and \( d \) by \( k \) in the expression (2.5).)
Example 6.5.) Thus we assume that \( s \) have \( n/s \) where \( A \).

Therefore, by (6.11) and (6.12),

\[
\mathcal{H}(s,n) = \sum_{k=0}^{s} C_{[s,k]} \Phi_{k}^{(n,s)}. \tag{6.9}
\]

By the hypothesis of the induction, we can assume

\[
\mathcal{H}(s,n-1) \cdots \mathcal{H}(n+s) R_{s-1}^p R_{p-1}^s = I, \quad \text{on } C^{\infty}(G/K,s). \tag{a}
\]

On the other hand, by Theorem 5.1(B), the invariant differential operator \( \Box_{k}^{(n,s)} \) is written as a linear combination of \( \Phi_{k}^{(n,s)}, \ldots, \Phi_{k}^{(n,s)} \). In particular, we see that \( \Box_{k}^{(n,s)} \) is written as \( \Phi_{k}^{(n,s)} \) + a linear combination of \( \Phi_{k}^{(n,s)}, \ldots, \Phi_{k}^{(n,s)} \). Thus, we can rewrite Theorem 6.4(2) as

\[
\left\{ \prod_{s+1 \leq s \leq p} \left( \sum_{k=0}^{s-1} A_{k}^{(n,s)} \Phi_{k}^{(n,s)} \right) \right\} R_{p}^s R_{p}^s = I, \quad \text{on } C^{\infty}(G/K,s), \tag{6.10}
\]

where \( A_{k}^{(n,s)} \) is a constant.

We take any element \( \varphi \) in any irreducible eigenspace \( V^{(n,s-1)}(l_1, \ldots, l_{s-1}) = C^{\infty}(G/K,s-1) \) and fix it. Then \( R_{p}^s R_{p}^s R_{p-1}^{s-1} \varphi = R_{p}^s R_{p}^s \varphi \in V^{(n,s)}(l_1, \ldots, l_{s-1}, 0) \). Thus, by Theorem 5.1(D), \( \Phi_{k}^{(n,s)} R_{p}^s R_{p}^s \varphi = 0 \). Therefore, it follows from (6.10) that

\[
\left\{ \prod_{s+1 \leq s \leq p} \left( \sum_{k=0}^{s-1} A_{k}^{(n,s)} \Phi_{k}^{(n,s)} \right) \right\} R_{p}^s R_{p}^s = R_{p}^s \varphi. \tag{6.11}
\]

Here we use the hypothesis of induction (a) for \( p = s \). Then we have

\[
\mathcal{H}(s,n-1) R_{s-1}^p R_{p-1}^s = I, \quad \text{on } C^{\infty}(G/K,s-1). \tag{6.12}
\]

Therefore, by (6.11) and (6.12),

\[
\varphi = \mathcal{H}(s,n-1) R_{s-1}^p R_{p-1}^s \varphi
\]

\[
= \mathcal{H}(s,n-1) R_{s-1}^p \left\{ \prod_{s+1 \leq s \leq p} \left( \sum_{k=0}^{s-1} A_{k}^{(n,s)} \Phi_{k}^{(n,s)} \right) \right\} R_{p}^s R_{p}^s \varphi
\]

\[
= \mathcal{H}(s,n-1) \left\{ \prod_{s+1 \leq s \leq p} \left( \sum_{k=0}^{s-1} A_{k}^{(n,s)} \Phi_{k}^{(n,s-1)} \right) \right\} R_{p}^s R_{p}^s \varphi. \tag{6.13}
\]
In the above computation, we used the property that $\Phi^{[n,s]}_k R_{s-1} = R_{s-1} \Phi^{[n,s]}_k$ (see Remark 5.4 (3)) and the property that $R_{s-1} R_s = R_{s-1}$.

By (6.12) combined with the density argument,

$$\mathcal{H}^{(s,s-1)} \left\{ \prod_{s+1 \leq x \leq s} \left( \sum_{k=0}^{s-1} A_k^{[n,s]} \Phi^{[n,s-1]}_k \right) \right\} R_{s-1} R_{s-1}^{-1} = I$$

on $C^\infty(G/K_{s-1})$. (6.14)

Comparing (6.14) with (a), we have

$$\mathcal{H}^{(s,s-1)} \prod_{s+1 \leq x \leq s} \left( \sum_{k=0}^{s-1} A_k^{[n,s]} \Phi^{[n,s-1]}_k \right) = \mathcal{H}^{(s,s-1)} \mathcal{H}^{(x+1,s-1)} \cdots \mathcal{H}^{(p,s-1)},$$

(6.15)

Since (6.15) holds for any $p$ ($s+1 \leq p \leq n-s$) and since $\mathcal{H}^{(s,s-1)}$ is a positive operator,

$$\sum_{k=0}^{s-1} A_k^{[n,s]} \Phi^{[n,s-1]}_k = \mathcal{H}^{(s,s-1)} \equiv \sum_{k=0}^{s-1} C_{[s,k]} \Phi^{[n,s-1]}_k,$$

for any $s$ ($s+1 \leq s \leq n-s$). Hence

$$A_k^{[n,s]} = C_{[s,k]} \quad \forall s+1 \leq s \leq n-s \quad \text{and} \quad \forall 1 \leq k \leq s-1.$$  (6.16)

As a result, we find that the coefficient $A_k^{[n,s]}$ in (6.10) does not depend on $s$. By (6.10) and (6.16), we obtain

$$\left\{ \prod_{s+1 \leq x \leq s} \left( C_{[s,s]} \Phi^{[n,s]}_s + \sum_{k=0}^{s-1} C_{[s,k]} \Phi^{[n,s]}_k \right) \right\} R_{s}^* R_{s}^* = I,$$

on $C^\infty(G/K_s)$, which completes the proof.

7. INVARIANT DIFFERENTIAL OPERATORS ON REAL GRASSMANN MANIFOLDS

From this section, we study the case of real Grassmann manifolds. Therefore, from now on, we denote by $G$ and by $K_\alpha$ the special orthogonal group $SO(n)$ and its subgroup $SO(p) \times O(n-p)$, respectively. Then the real Grassmann manifold $Gr(p,n; \mathbb{R})$ of all $p$ dimensional subspaces in $\mathbb{R}^n$ can be identified with the compact symmetric space $G/K_\alpha = SO(n)/SO(p) \times O(n-p)$ in the usual manner.
The contents of this section are almost the same as those in Section 2. In other words, in this section, we construct generalized John type differential operators on the real Grassmann manifold $GK^p$, which are of a similar form to the counterparts in the complex Grassmann manifolds.

Let $g$ and $I_p$ denote the Lie algebras of $G$ and of $K_p$, respectively. Then $g$ and $I_p$ are given by

$$g = \{ X \in M_n(\mathbb{R}) ; X + X^t = 0 \},$$

$$I_p = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \in g ; X_1 \in M_p(\mathbb{R}), X_2 \in M_{n-p}(\mathbb{R}) \right\}.$$

Let $g = I_p \oplus \mathfrak{W}_p$ be the corresponding Cartan decomposition. Then $\mathfrak{W}_p$ is the space of all the matrices $X$ of the form

$$X = \begin{pmatrix} 0 & -Y \\ Y & 0 \end{pmatrix} \in g, \quad Y = \{ y_{ij} ; \text{real } (n-p) \times p \text{ matrix} \} \quad (1 \leq i \leq n-p, 1 \leq j \leq p). \tag{7.1}$$

Let $I = \{ i(1), i(2), \ldots, i(d) ; 1 \leq i(1) < i(2) < \cdots < i(d) \leq n-p \}$ and $A = \{ \alpha(1), \alpha(2), \ldots, \alpha(d) ; 1 \leq \alpha(1) < \alpha(2) < \cdots < \alpha(d) \leq p \}$ be two ordered sets. (Here we assume that $1 \leq d \leq \text{rank } G/K_p = \min \{ p, n-p \}.$)

For the submatrix $Y$ of $X$ in (7.1) and the above two ordered sets $I$ and $A$, we define a $d \times d$ matrix valued differential operator $Y_{(I,A)}$ by

$$\partial Y_{(I,A)} := \left( \frac{\partial}{\partial y_{ij}} \right)_{i \in I, j \in A}. \tag{7.2}$$

Next, we define a $d$th order differential operator $M^{(d)}_{(I,A)}$ acting on $C^\infty(G)$ by

$$M^{(d)}_{(I,A)} f(g) := \det \partial Y_{(I,A)} f(g \exp X)|_{X=0}, \tag{7.3}$$

for $f \in C^\infty(G)$. Here $X$ is a matrix of the form (7.1).

**Example.** If $I = \{ i, j ; 1 \leq i < j \leq n-p \}$ and $A = \{ \alpha, \beta ; 1 \leq \alpha < \beta \leq p \}$, the second order differential operator $M^{(2)}_{(I,A)}$ is given by

$$M^{(2)}_{(I,A)} f(g) = \left( \frac{\partial^2}{\partial y_{\alpha \alpha} \partial y_{\beta \beta}} - \frac{\partial^2}{\partial y_{\alpha \beta} \partial y_{\beta \alpha}} \right) f(g \exp X)|_{X=0}.$$

Finally, using $M^{(d)}_{(I,A)}$, we define a generalized John type differential operator $\Phi^{(n,p)}_g$ on $C^\infty(G)$ as follows.
Case I. $2r < n$.

$$\Phi_0^{(n, p)} := ( -1 )^d 2^{-2d} \sum_{\substack{I = \{1, 2, \ldots, n-p\} \atop \# I = \# \Delta = d}} \{ M_{(I, \Delta)}^{(d)} \}^2 \quad (1 \leq d \leq r),$$  

$$\Phi_0^{(n, p)} := 1.$$  

(7.4)

Case II. $n = 2r$.

$$\Phi_2^{(2r, r)} := ( -1 )^d 2^{-2d} \sum_{\substack{I = \{1, 2, \ldots, n-p\} \atop \# I = \# \Delta = d}} \{ M_{(I, \Delta)}^{(d)} \}^2 \quad (1 \leq d \leq r - 1),$$  

$$\Phi_0^{(2r, r)} := (-\sqrt{-1}) r 2^{-r} M_{(1, 2, \ldots, r)}^{(r)} \quad \Phi_0^{(2r, r)} := 1.$$  

(7.5)

We note that, in (7.4) and (7.5), $(-1)^d M_{(I, \Delta)}^{(d)}$ is the formal adjoint operator of $M_{(I, \Delta)}^{(d)}$ with respect to the standard inner product of $L^2(G)$. In other words, the differential operator $\Phi_0^{(n, p)}$ is a non-negative formally self-adjoint operator.

Similarly as in Proposition 2.1, we have the following.

**Proposition 7.1.** The differential operator $\Phi_0^{(n, p)}$ above is left $G$-invariant and right $K_p$-invariant. Therefore, $\Phi_0^{(n, p)}$ is well defined as an invariant differential operator on the symmetric space $G/K_p$.

As we mentioned in the sketch of the proof of Proposition 2.1, this is essentially due to the fact that the polynomial $F_{(n, p)}(X)$ on $\mathfrak{M}_p$ defined below is Ad-$K_p$-invariant. Here

$$F_{(n, p)}(X) := \sum_{\substack{I = \{1, 2, \ldots, n-p\} \atop \# I = \# \Delta = d}} \{ \det Y_{(I, \Delta)}^{(d)} \}^2,$$

for $X = \begin{pmatrix} 0 & -Y \\ Y & 0 \end{pmatrix} \in \mathfrak{M}_p$,  

(7.6)

where $Y_{(I, \Delta)}^{(d)}$ is a $d \times d$ submatrix of $Y$ defined by $Y_{(I, \Delta)}^{(d)} := ( y_{\tilde{I}, \tilde{\Delta}} )_{\tilde{I} \in I, \tilde{\Delta} \in \Delta}$. We see easily that the above polynomial $F_{(n, p)}(X)$ on $\mathfrak{M}_p$ is Ad-$K_p$-invariant. Indeed we have the expansion

$$\det(I + \lambda X) = 1 + P_1^{(n, p)}(X) \lambda^2 + P_2^{(n, p)}(X) \lambda^4 + \cdots + P_r^{(n, p)}(X) \lambda^{2r},$$

for $X \in \mathfrak{M}_p$,  

(7.7)

where $r = \text{rank } G/K_p = \min\{ p, n-p \}$. 

30 TOMOYUKI KAKEHI
Remark 7.2. (1) For the same reason as in Remark 2.2, the polynomial $F^{(n,p)}(\lambda)$ on $\mathfrak{H}_p$ can be regarded as the principal symbol of $\Phi^{(n,p)}_k$. (2) The algebra $\mathcal{D}(G/K_p)$ of all invariant differential operators on $G/K_p$ is generated by $\{\Phi^{(n,p)}_1, \Phi^{(n,p)}_2, \ldots, \Phi^{(n,p)}_r\}$. (3) In particular, the second order operator $\Phi^{(n,p)}_1$ coincides with the Laplacian on $G/K_p$ with respect to the standard $G$-invariant metric on $G/K_p$. (4) The construction of the above generators is similar to the one given by Gonzalez and Helgason [GH] in the case of real affine Grassmann manifolds.

Similarly as Proposition 2.3, we have

Proposition 7.3. We assume that $s := \text{rank } G/K_p < r := \text{rank } G/K_p$. Then the image $\text{Im } R^*_{p}$ of the Radon transform $R^*_{p} : C^\omega(G/K_p) \rightarrow C^\omega(G/K_p)$ is included in the kernel $\text{Ker } \Phi^{(n,p)}_{r+1}$ of the invariant differential operator $\Phi^{(n,p)}_{r+1}$ on $G/K_p$. Namely, $\text{Im } R^*_{p} \subseteq \text{Ker } \Phi^{(n,p)}_{r+1}$.

8. REPRESENTATION OF REAL GRASSMANN MANIFOLDS

In this section, we discuss the representation of the real Grassmann manifold $G/K_p = SO(n)/SO(p) \times O(n - p)$, similarly as in Section 3. The situation is quite similar as in the complex case except for the case $n = 2p$.

Let $r := \text{rank } G/K_p = \min\{p, n - p\}$ and $v := \text{rank } G = [n/2]$. Then, any highest weight of the real Grassmann manifold $G/K_p = SO(n)/SO(p) \times O(n - p)$ is written as follows.

Case I. $2r < n$. $$(2l_1, 2l_2, \ldots, 2l_s, 0, \ldots, 0) \in \mathbb{R}^r,$$ where $l_1, l_2, \ldots, l_s \geq 0$.

Case II. $2r = n$. $$(2l_1, 2l_2, \ldots, 2l_s) \in \mathbb{R}^r,$$ where $l_1, l_2, \ldots, l_s, l_{s+1} \geq 0$.

We denote by $A(l_1, \ldots, l_s)$ the above highest weight, and by $V^{(n,p)}(l_1, \ldots, l_s)$ the irreducible eigenspace of the Laplacian $A_{\mathcal{G}_p}$ on $G/K_p$ corresponding to the highest weight $A(l_1, \ldots, l_s)$.

Similarly as in the complex case, the following theorem holds.

Theorem 8.1 (Grinberg [Gri2], Gonzalez [Gon3]). (1) Let $F(t_1, \ldots, t_s)$ be a symmetric polynomial. Then, there exists an invariant differential operator $D$ on $G/K_p = SO(n)/SO(p) \times O(n - p)$ such that the eigenvalue of $D$ on the irreducible eigenspace $V^{(n,p)}(l_1, \ldots, l_s)$ is given by $F(\chi_1 + a_1, \chi_2 + a_2, \ldots, \chi_s + a_s)$, where $\chi_j$ and $a_j$ are given respectively by

$$\chi_j := l_j \left( l_j + \frac{n - 2j}{2} \right),$$

$$a_j := \frac{1}{4} \left( \frac{1}{4} \frac{j^2}{4} - \frac{1}{4} nj + \frac{1}{16} n^2 \right) = \left( \frac{2j - n}{4} \right)^2. \quad (8.1)$$
In particular, for the kth elementary symmetric polynomial \( S_k(t_1, ..., t_r) \) defined by (3.3), there exists an invariant differential operator \( \Box_k^{(n, p)} \) on \( G/K_p \) such that

\[
\Box_k^{(n, p)} |_{\text{rad} (t_1, ..., t_r)} = S_k(\chi_1 + a_1, \chi_2 + a_2, ..., \chi_r + a_r). \tag{8.2}
\]

(II) If \( n = 2r \), there exists an invariant differential operator \( \Box_r^{(2r, p)} \) on \( G/K_p \) such that

\[
\Box_r^{(2r, p)} |_{\text{rad} (t_1, ..., t_r)} = \left( t_1 + \frac{r-1}{2} \right) \left( t_2 + \frac{r-2}{2} \right) \cdots \left( t_{r-1} + \frac{1}{2} \right) t_r. \tag{8.3}
\]

Next, we consider the radial part of the above differential operator \( \Box_k^{(n, p)} \) and of the differential operator \( \Phi_k^{(n, p)} \) defined in Section 7.

Let \( a_p \) be the maximal abelian subalgebra of \( \mathfrak{m}_p \) defined by

\[
a_p := \{ H(t) = H(t_1, ..., t_r) = t_1 H_1 + \cdots + t_r H_r \in \mathfrak{m}_p : \quad t = (t_1, ..., t_r) \in \mathbb{R}^r \}, \tag{8.4}
\]

where \( H_r := E_{r+1, r} - E_{r, r+1} \). We introduce the lexicographical order \( \prec \) on \( a_p \) such that \( H_1 > H_2 > \cdots > H_r > 0 \). We identify \( a_p \) with \( \mathbb{R}^r \) by the mapping \( a_p \ni H(t) \mapsto t \in \mathbb{R}^r \). Moreover, we regard the highest weight \( \lambda(t_1, ..., t_r) \) as an element of \( a_p \) by the mapping \( \lambda(t_1, ..., t_r) \mapsto t_1 H_1 + \cdots + t_r H_r \).

Let \( W(G, K_p) \) be the Weyl group of the symmetric space \( G/K_p = SO(n)/SO(p) \times SO(n-p) \). Then, if \( 2r < n \), \( W(G, K_p) \) is the set of all the linear transformations \( w : \mathbb{R}^r \rightarrow \mathbb{R}^r \) of the form

\[
w : (t_1, ..., t_r) \mapsto (e_1 t_{\sigma(1)}, ..., e_r t_{\sigma(r)}), \quad e_j = \pm 1, \quad \sigma \in \mathfrak{S}_r. \tag{8.5}
\]

If \( 2r = n \), \( W(G, K_p) \) is the set of all the linear transformations \( w : \mathbb{R}^r \rightarrow \mathbb{R}^r \) of the form (8.5) with the condition \( e_1 \cdot e_2 \cdots e_r = 1 \). Let us choose a Weyl chamber \( \mathcal{A}^+ \subset a_p \) as

\[
\mathcal{A}^+ := \left\{ (t_1, ..., t_r) \in \mathbb{R}^r ; 0 < t_j < \frac{\pi}{2}, t_1 > t_2 > \cdots > t_r \right\}.
\]

Then, similarly as Proposition 3.5 and Proposition 3.6, we have the following two propositions.

**Proposition 8.2.** The invariant differential operator \( \Box_k^{(n, p)} \) on \( G/K_p \) defined by (8.2) is of order \( 2k \). The radial parts of \( \Box_k^{(n, p)} \) and \( \Box_r^{(2r, p)} \) are written respectively in the form
rad(□_{k}^{(n,p)}) = (-1)^{k} 2^{-2k} S_{k} \left( \frac{\partial^2}{\partial t_{1}^{2}}, \ldots, \frac{\partial^2}{\partial t_{r}^{2}} \right) + \text{lower order terms},

rad(□_{r}^{(2r,r)}) = (-\sqrt{-1})^{r} 2^{-r} \frac{\partial^{r}}{\partial t_{1} \partial t_{2} \cdots \partial t_{r}} + \text{lower order terms}.

**Proposition 8.3.** The radial parts of the generalized John type invariant differential operators \( \Phi_{k}^{(n,p)} \) and \( \Phi_{r}^{(2r,r)} \) are written respectively in the form

rad(\( \Phi_{k}^{(n,p)} \)) = (-1)^{k} 2^{-2k} S_{k} \left( \frac{\partial^2}{\partial t_{1}^{2}}, \ldots, \frac{\partial^2}{\partial t_{r}^{2}} \right) + \text{lower order terms},

rad(\( \Phi_{r}^{(2r,r)} \)) = (-\sqrt{-1})^{r} 2^{-r} \frac{\partial^{r}}{\partial t_{1} \partial t_{2} \cdots \partial t_{r}} + \text{lower order terms}.

9. RANGE THEOREM—REAL CASE

In this section, we show the range theorem for the Radon transform \( R_{q}^{p} \) on the compact real Grassmann manifold \( Gr(q,n,\mathbb{R}) \). As we will see later, in the case \( 2p = n \), the situation is quite different. In fact, the uniqueness of the range-characterizing operator holds in the case \( 2p < n \), whereas in the case \( 2p = n \) such uniqueness no longer holds. (See Theorem 9.1(E) and Proposition 9.7.)

By the same argument as in the complex case, we obtain the following theorem.

**Theorem 9.1.** We assume that \( s := \text{rank } Gr(q,n,\mathbb{R}) < r := \text{rank } Gr(p,n,\mathbb{R}) \) and \( 2r < n \). For the generalized John type invariant differential operator \( \Phi_{r+1}^{(n,p)} \) on the real Grassmann manifold \( Gr(p,n,\mathbb{R}) \), the following (A), (B), (C), (D), and (E) hold.

(A) The eigenvalue of \( \Phi_{r+1}^{(n,p)} \) on the irreducible eigenspace \( Y^{(n,p)}(l_{1}, \ldots, l_{r}) \) is given by the formula

\[
\Phi_{r+1}^{(n,p)}|_{Y^{(n,p)}(l_{1}, \ldots, l_{r})} = \sum_{k=0}^{r+1} (-1)^{r+1-k} T_{r+1-k}(a_{s+1}, \ldots, a_{r}) S_{k}(z_{1} + \ldots + z_{s} + a_{r}),
\]

where \( T_{j} \) and \( S_{j} \) are the symmetric polynomials given respectively by (3.3) and (4.2), and where \( j_{j} = l_{1}(l_{1} + (n-2j)/2) \) and \( a_{j} = (1/4) j^{2} - (1/4) nj + (1/16) n^{2} = (1/16)(2j - n)^{2} \). (See (8.1).)
(B) $\Phi_{s+1}^{(n, p)}$ is expressed as
\[
\Phi_{s+1}^{(n, p)} = \sum_{k=0}^{s+1} (-1)^{s+1-k} T_{s+1-k}(a_s, \ldots, a_r) \square_k^{(n, p)},
\]
where $\square_k^{(n, p)}$ is the 2k-th order invariant differential operator on $Gr(p, n, \mathbb{R})$ defined by (8.2).

(C) $\Phi_{s+1}^{(n, p)}|_{\psi_n^{(l_1, \ldots, l_r)}} = 0$ if and only if $l_{s+1} = \cdots = l_r = 0$.

(D) The range $\text{Im } R_p^s$ of the Radon transform $R_p^s$ is identical with the kernel of $\Phi_{s+1}^{(n, p)}$, that is, $\text{Im } R_p^s = \text{Ker } \Phi_{s+1}^{(n, p)}$.

(E) Let $P$ be an invariant differential operator on $Gr(p, n, \mathbb{R})$ satisfying the following two conditions.

(El) $\text{Im } R_p^s \subset \text{Ker } P$.

(EII) The radial part of $P$, $\text{rad}(P)$, is of the form
\[
\text{rad}(P) = (-1)^{s+1} S_{s+1} \left( \frac{1}{4} \frac{\partial^2}{\partial l_1^2} \cdots \frac{1}{4} \frac{\partial^2}{\partial l_r^2} \right) + \text{lower order terms}.
\]

Then $P$ coincides with $\Phi_{s+1}^{(n, p)}$.

Moreover, as in (5.1), we define an invariant differential operator $\Psi_d^{(n, p)}$ on $Gr(p, n, \mathbb{R})$ by
\[
\Psi_d^{(n, p)} := \sum_{k=0}^{d} (-1)^{d-k} T_{s+1-k}(a_d, \ldots, a_r) \square_k^{(n, p)}. \tag{9.1}
\]

Then, similarly as in the complex case, the above theorem follows from the next proposition.

**Proposition 9.2.** We assume that $s := \text{rank } Gr(q, n, \mathbb{R}) < r := \text{rank } Gr(p, n, \mathbb{R})$, then the above invariant differential operator $\Psi_d^{(n, p)}$ (1 $\leq d \leq r$) satisfies the following (A), (B), and (C).

(A) $\Psi_{s+1}^{(n, p)} R_p^s = R_p^s \Psi_{s+1}^{(n, p)}$, \quad (B) $\text{Im } R_p^s = \text{Ker } \Psi_{s+1}^{(n, p)}$.

(C) If $2r < n$, $\Psi_d^{(n, p)} = \Phi_d^{(n, p)}$.

On the other hand, in the case $2r = n$, our method in the complex case does not work. This is due to the fact that the algebra of invariant differential operators $D(GK_r)$ is generated by $\Phi_j^{(n, r)}$ (1 $\leq j \leq r-1$) and $\Phi_r^{(n, r)}$. Here we note that the order of $\Phi_j^{(n, r)}$ is $r$. (See (7.6).) Let us recall that, in the proof of Theorem 5.1, it is essential to prove $\Psi_d^{(n, p)} = \Phi_d^{(n, p)}$. However, in the case $2r = n$, this part does not go well for the following reason. We put $Q := P - \Psi_{s+1}^{(n, p)}$. Then, by the argument of radial parts, we see that the
order of $Q$ is less than $2r + 1$. However, if $n = 2r$ and $r < 2(s + 1) < 2r$, the operator $Q$ may be expressed as a polynomial of $\Psi_1^{(2r)}$, ..., $\Psi_r^{(2r)}$ and $\Phi^{(2r)}$. Because of the existence of $\Phi^{(2r)}$, we can no longer conclude that $Q = 0$. Therefore, we have to change the method.

In the case $2r = n$, the key is to show that the kernel of $\Phi^{(2r)}$ includes the kernel of $\Phi^{(2r)}$. Then we can reduce the argument to the case $2r < n$. We are grateful to Professor F. Gonzalez for suggesting this idea.

Let us admit the following three propositions.

**Proposition 9.3.** (1) $\Phi_r^{(2r)} = R_r^{(2r)}$. (2) $\text{Im } R_r^{(2r)} = \text{Ker } \Phi_r^{(2r)}$. Here $\bigtriangleup_r^{(2r)}$ is the invariant differential operator on $\text{Gr}(r, 2r; \mathbb{R})$ defined by (8.3).

**Proposition 9.4.** If $s < r$ and $\varphi \in C^\infty(\text{Gr}(r, 2r; \mathbb{R}))$ satisfies the differential equation $\Phi_s^{(2r)} \varphi = 0$, then $\varphi$ satisfies $\Phi_s^{(2r)} \varphi = 0$.

**Proposition 9.5.** If $s + 1 \leq r - 1$, $\Phi_s^{(2r)} R_{s+1}^{(2r)} = R_{s+1}^{(2r)} \Phi_s^{(2r)} = 0$.

We will prove these propositions later.

From now on, we assume that $2 \leq s + 1 \leq r - 1$.

Let us take a function $\varphi \in C^\infty(\text{Gr}(r, 2r; \mathbb{R}))$ satisfying the differential equation $\Phi_s^{(2r)} \varphi = 0$. Then, by Proposition 9.4, we have $\Phi_s^{(2r)} \varphi = 0$. Thus, by Proposition 9.3(2), there exists a function $\varphi_1 \in C^\infty(\text{Gr}(r - 1, 2r; \mathbb{R}))$ such that $\varphi = R_r^{(2r)} \varphi_1$. Therefore, by Proposition 9.5, we have $0 = \Phi_s^{(2r)} \varphi = R_s^{(2r)} \Phi_s^{(2r)} \varphi = R_s^{(2r)} \Phi_s^{(2r-1)} \varphi_1$. Since $R_s^{(2r)}$ is injective, we have $\Phi_s^{(2r-1)} \varphi_1 = 0$. Here we note that $2(r - 1) < 2r = n$. We can apply Theorem 9.1(D) to $\Phi_s^{(2r-1)}$. Thus, there exists a function $f \in C^\infty(\text{Gr}(r, 2r; \mathbb{R}))$ such that $\varphi_1 = R_{s+1}^{(2r)} f$. Therefore, we have $R_s^{(2r)} f = R_s^{(2r-1)} R_{s+1}^{(2r)} f = R_s^{(2r-1)} \varphi_1 = \varphi$, which means that $\text{Im } R_s^{(2r)} = \text{Ker } \Phi_s^{(2r)}$. Combining the above argument with Proposition 7.3 and Proposition 9.3, we obtain the following.

**Theorem 9.6.** We assume that $s := \text{rank } \text{Gr}(q, n; \mathbb{R}) < r := \text{rank } \text{Gr}(p, n; \mathbb{R})$. Even if $2r = n$ ($\iff 2p = n$), the range $\text{Im } R_p^{(r)}$ of the Radon transform $R_p^{(r)}$ is identical with the kernel of the generalized John type invariant differential operator $\Phi_s^{(r)}$, that is, $\text{Im } R_p^{(r)} = \text{Ker } \Phi_s^{(r)}$.

However, in the case $2r = n$, the uniqueness of the range characterizing operator does not hold in the sense of Theorem 5.1(E). In fact, we have the following.

**Proposition 9.7.** Let $P_s^{(2r)} := \Psi_{s+1}^{(2r)} + \sqrt{-1} \tau \Phi_s^{(2r)}$ for $\tau \in \mathbb{R}$. Assume that $r < 2(s + 1) < 2r$. Then the invariant differential operator $P_s^{(2r)}$
characterizes the range of $R_{q, q}$, that is, $\text{Im } R_{q} = \text{Ker } P_{(1)}^{(2r, r)}$ for any real number $r$. In addition, the radial part of $P_{(1)}^{(2r, r)}$, $\text{rad}(P_{(1)}^{(2r, r)})$ is of the form

$$\text{rad}(P_{(1)}^{(2r, r)}) = (-1)^{s+1} 2^{-2(s+1)} S_{s+1} \left( \frac{\partial^2}{\partial t_1^2}, \ldots, \frac{\partial^2}{\partial t_f^2} \right) + \text{lower order terms.}$$

Proof. It suffices to show that $P_{(1)}^{(2r, r)} |_{y_1, y_2, \ldots, l_f} = 0 \iff l_s + 1 = \cdots = l_f = 0$. Thus, the assertion follows from Proposition 9.2 and Proposition 9.3.

Now we go into the proofs of Proposition 9.3-5.

Proof of Proposition 9.3. It follows immediately from Theorem 8.2(II) that $\Box (2r, r) |_{y_1, y_2, \ldots, l_f} = 0$ if and only if $l_s = 0$. Thus, by the density argument combined with the Grinberg's inversion formula, we have $\text{Im } R_{s-1} = \text{Ker } \Box (2r, r)$. Therefore, it suffice to prove (1). Due to Proposition 8.2(2) and Proposition 8.3(2), the principal part of rad($\Phi_{(2r, r)}$) coincides with that of rad($\Psi_{m, p}^{(2r, r)}$), which means that the differential operator $Q := \phi_{(2r, r)}^{(2r, r)} - \Box (2r, r)$ is of order at most $r - 1$. Thus, $Q$ can be written as a polynomial of $\Psi_{j}^{(2r, r)}$, ..., $\Psi_{m, p}^{(2r, r)}$. (Here $m = [r/2]$ the maximum integer less than $r/2$.) In addition, by Proposition 7.3 and Proposition 9.2(B), $\text{Im } R_{s-1} = \text{Ker } (\phi_{(2r, r)}^{(2r, r)} - \Box (2r, r)) = \text{Ker } Q$. Therefore, by the same procedure as in the proof of Theorem 5.1, we can conclude that $Q = 0$.

Proof of Proposition 9.4. By the definition of $\Phi_{s+1}^{(2r, r)}$ (see (7.5) and (7.6)), it suffices to show that

$$M^{(d)}_{(J, B)} \varphi = 0, \quad \text{for } I, A, \# I = \# A = d$$

$$\implies M^{(d+1)}_{(J, B)} \varphi = 0, \quad \text{for } J, B, \# J = \# B = d + 1. \quad (a)$$

Without loss of generality, we may assume that $J = B = \{1, 2, \ldots, d + 1\}$. Then, (a) is easily proved by the equality $M^{(d)}_{(J, B)} = \sum_{k=1}^{d+1} (-1)^{k-1} Y_{ik} M^{(d)}_{(J, B\setminus \{k\})}$, where $Y_{ik}$ is defined by $Y_{ik} f(g) := \left( \frac{\partial}{\partial y_{ik}} \right) f(g \exp X) |_{X=0}$. (Here $X$ is a matrix of the form (7.1)).

Proof of Proposition 9.5. By the argument of radial parts, the order of $(\Phi_{s+1}^{(2r, r)} - \Psi_{s+1}^{(2r, r)}) \leq 2s + 1$. Thus, we can put

$$\Phi_{s+1}^{(2r, r)} - \Psi_{s+1}^{(2r, r)} = F_1 (\Psi_{1}^{(2r, r)}, \ldots, \Psi_{s}^{(2r, r)}),$$

$$F_2 (\Psi_{s+1}^{(2r, r)}),$$

for some polynomials $F_j$ $(j = 1, 2)$. Then, by Proposition 9.2(B) and Proposition 7.3, we have

$$0 = (\Phi_{s+1}^{(2r, r)} - \Psi_{s+1}^{(2r, r)}) R^q_r$$

$$= F_1 (\Psi_{1}^{(2r, r)}, \ldots, \Psi_{s}^{(2r, r)}) \Phi_{s+1}^{(2r, r)} R^q_r + F_2 (\Psi_{s+1}^{(2r, r)}) R^q_r.$$
Moreover, by Proposition 9.3, we see easily that $\Phi^{(2r,r)}_s R^*_s = 0$. Thus we have $F_s (\psi^{(2r,r)}, ... , \psi^{(2r,r)}) R^*_s = 0$. Then, by Proposition 9.2(A), $R^*_s F_s (\psi^{(2r,r)}, ... , \psi^{(2r,r)}) = 0$. Thus, by the injectivity of $R^*_s$, we have $F_s (\psi^{(2r,r)}, ... , \psi^{(2r,r)}) = 0$. Since $\psi^{(2r,r)}, ... , \psi^{(2r,r)}$ are algebraically independent, this equality shows that $F_s = 0$. Therefore, by Proposition 9.2 and (9.2), we have

$$\Phi^{(2r,r)}_{s+1} R^{-1}_{s+1} = \psi^{(2r,r)}_{s+1} R^{-1}_s + F_s (\psi^{(2r,r)}_{s+1}) \Phi^{(2r,r)}_{s+1} R^{-1}_s$$

$$= \psi^{(2r,r)}_{s+1} R^{-1}_s = R^{-1}_s \psi^{(2r,r)}_{s+1} = R^{-1}_{s+1} \Phi^{(2r,r)}_{s+1}.$$

**10. INVERSION FORMULA—REAL CASE**

In this section, we give the inversion formula for the Radon transform on the real Grassmann manifold. The method is almost the same as in the complex case. However, we need the parity condition to state the inversion formula explicitly. In fact, if $p – q$ is even, the explicit form of the inversion formula for the Radon transform: $R^*_s : C^\infty(G/K_q) \to C^\infty(G/K_p)$ can be described in terms of generalized John type operators on $G/K_q$. (Here $G = SO(n)$ and $K_q = S(O(d) \times O(n-d)).$)

Similarly as in Section 6, the starting point is the following result given by Grinberg.

**Theorem 10.1 (Grinberg [Gr3]).** We assume that $2d + 1 \leq n$. Then we have

$$R^*_d \prod_{j=1}^{d} f = \frac{\Gamma((d+1)/2) \Gamma((n-d)/2)}{\Gamma(1/2) \Gamma((n-2d)/2)} \prod_{j=1}^{d} \frac{\Gamma(l_j + (d+1)/2) \Gamma(l_j + (n-d)/2)}{\Gamma(l_j + (d+2)/2) \Gamma(l_j + (n-f-d+1)/2)} f$$

for $f \in V^{(n,d)}(l_1, ..., l_d)$.

Since $R^*_d \prod_{j=1}^{d} R^*_d = R^*_d \prod_{j=1}^{d} R^*_d R^*_d$, we have the following

**Proposition 10.2.** We assume that $2d + 3 \leq n$. Then, we have

$$R^*_d \prod_{j=1}^{d} f = \frac{d! 2^{-2d} \prod_{j=1}^{d} (n-2d-2+j)}{\prod_{j=1}^{d} (l_j + (d+1)/2)(l_j + (n-d)/2)} f$$

for $f \in V^{(n,d)}(l_1, ..., l_d)$. 
Similarly as in Section 6, using Theorem 10.1 and Proposition 10.2, we can calculate the Schur constant of the $G$-equivariant mapping $R^p_p R^q_p$ on the irreducible eigenspace $V^{v, q}(l_1, ..., l_s)$, that is, we obtain the following.

**Theorem 10.3 (Inversion Formula—Diagonalized Version).** We assume that $s := \text{rank } G/K_q \leq r := \text{rank } G/K_p$ and that $|p - q|$ is even. Then the following inversion formula holds:

$$\left\{ \prod_{x \leq n - s + |p - q|} C_{[x, s]} \prod_{k = 1}^s (Z_k + a_k + v_s) \right\} R^p_p R^q_p|_{V^{v, q}(l_1, ..., l_s)} = 1,$$

where $\chi_j$ and $a_j$ are given by (8.1) and where $C_{[x, s]}$ and $v_s$ are given respectively by

$$C_{[x, s]} = \frac{2^{2x}(x - 2 - s)! (n - x - s)!}{(x - 2)! (n - x)!}, \quad v_s = -\frac{1}{4} \left( \frac{x - 1}{2} \right)^2.$$

Moreover, by the same argument as in the proof of Theorem 6.6, we have

**Theorem 10.4 (Inversion Formula—Explicit Version).** We assume that $s := \text{rank } G/K_q \leq r := \text{rank } G/K_p$ and that $|p - q|$ is even. Then the following inversion formula holds for the Radon transform $R^p_p$ on the compact real Grassmann manifold $G/K_q = \text{Gr}(q, n, \mathbb{R})$,

$$\left\{ \prod_{x \leq n - s + |p - q|} \sum_{k = 0}^s \frac{2^{2x}(x - 2 - k)! (n - x - k)!}{(x - 2)! (n - x)!} \Phi_k^{[v, q]} \right\} R^p_p R^q_p = I,$$

on $C^\infty(G/K_q), \Phi_k^{[v, q]}$ (1 $\leq k \leq s$) is the generalized John type operator on $\text{Gr}(q, n, \mathbb{R})$ given by (7.4). (Replace $p$ by $q$ and $d$ by $k$ in the expression (7.4).)

We remark that the above inversion formula is the generalization of the Helgason's inversion formula for the Radon transform on the real projective space $\mathbb{P}^n\mathbb{R}$. (See Helgason [H4].) We also remark that in the case $p - q$ is odd it is difficult to obtain such an explicit inversion formula. Nevertheless, we can prove the existence of the inversion formula. (See Gonzalez [Gon3].)
11. RADIAL PART FORMULA

In this section, as an application of Theorem 5.1, we give the explicit expression of the radial part of a generalized John type invariant differential operator on the complex Grassmann manifold $Gr(p, n, \mathbb{C})$. Therefore, throughout this section, $G$ and $K_p$ denote $SU(n)$ and $SU(p) \times SU(n-p)$, respectively.

The following theorem is essentially due to Hoogenboom [Ho].

**Theorem 11.1 (Hoogenboom [Ho]).** The algebra

$$\{ \text{rad}(P) \in \mathcal{D}(\mathfrak{a}^+)_{W(G, K_p)}; P \in \mathcal{D}(G/K_p) \}$$

is generated by the following operators $U_j (1 \leq j \leq r := \text{rank } G/K_p)$.

$$U_j := 2^{-2i} \frac{1}{\omega} S_j \left( \frac{\partial^2}{\partial t_1^2} + \frac{\sigma_j}{\sigma} \frac{\partial}{\partial t_1}, ..., \frac{\partial^2}{\partial t_r^2} + \frac{\sigma_j}{\sigma} \frac{\partial}{\partial t_r} \right) \omega,$$

where $\omega := \prod_{j<k} (\cos 2t_j - \cos 2t_k)$, $\sigma := \prod_{j=1}^r \sin 2t_j \sin^{3(n-r)} t_j$, and where $S_j$ denotes the $j$th elementary symmetric polynomial.

We note that in (11.1) the function $\sigma_j/\sigma = 2 \cot 2t_j + 2(n-r) \cot t_j$ is a function of one variable $t_j$. We also remark that Hoogenboom [Ho] deals with the case of non-compact complex Grassmann manifolds. However, we see that with a slight modification his method works in the compact case as well.

Let us take an invariant differential operator $Y_j \in \mathcal{D}(G/K_p)$ such that $\text{rad}(Y_j) = U_j$. Then, the first step is to calculate the eigenvalue of $Y_j$ on each eigenspace $V(n, p)(l_1, ..., l_r)$.

We introduce the following equivalence relation $\sim$.

Let $f_1$ and $f_2$ be two Fourier series on the Weyl chamber $\mathfrak{a}^+$ of the form

$$f_1(t) = \sum_{\lambda \in Z(G, K_p), \lambda \in A} C_{\lambda} \exp \sqrt{-1}(\lambda, H(t)),$$

$$f_2(t) = \sum_{\lambda \in Z(G, K_p), \lambda \in A} \tilde{C}_{\lambda} \exp \sqrt{-1}(\lambda, H(t)),$$

where $Z(G, K_p)$ is the weight lattice given by (3.6). We write $f_1 \sim f_2$ when $A_1 = A_2$ and $C_{A_1} = \tilde{C}_{A_2}$ ($\neq 0$).

Then the following lemma is easily seen.
Lemma 11.2. The following relations hold.

\[ \sigma_j \sim 2 \sqrt{-1} (n + 1 - 2r) \sigma, \quad \omega_j \sim 2 \sqrt{-1} (r - f) \omega, \]  
\[ \frac{\partial}{\partial t_j} \theta(t; l_1, \ldots, l_r) \sim 2 \sqrt{-1} l_j \theta(t; l_1, \ldots, l_r), \]  

(11.2) where \( \theta(t; l_1, \ldots, l_r) \) is the restriction of the zonal spherical function \( \Theta_{\mathcal{A}(l_1, \ldots, l_r)} \) to the Weyl chamber \( \mathfrak{a}^+ \). (See Section 3)

After a straightforward computation using the above lemma, we have

\[ U_d \theta(t; l_1, \ldots, l_r) \sim S_d(\chi_1 + \sigma_1, \ldots, \chi_r + \sigma_r) \theta(t; l_1, \ldots, l_r), \]  

(11.4) where \( \chi_j = (r - f)(n + 1 - r - f) \) and \( \chi_j \) is given by (3.2).

We note that \( \chi_j + r(n + 1 - r) = a_j \). On the other hand, due to Theorem 3.3(3), we have

\[ U_d \theta(t; l_1, \ldots, l_r) = \left[ D_{d; \chi_j}(l_1, \ldots, l_r) \right]_{| \mathfrak{a}^+} (t) \]  

(11.5)

Comparing the leading coefficient of the right hand side of (11.4) with that of (11.5), we see that the eigenvalue \( \Phi_{d; \sigma}^{11.3} \) of \( Y_d \) on the irreducible eigenspace \( V^{(n, r)}(l_1, \ldots, l_r) \) is \( S_d(\chi_1 + \sigma_1, \ldots, \chi_r + \sigma_r) \).

The second step is to express the operator \( \Phi_{d; \sigma}^{11.3} \) in terms of \( \Phi_{d; \sigma}^{11.3} \). By (4.1), Proposition 4.7, and Theorem 5.1(A), we have

\[ \left\{ \sum_{k=1}^d (-1)^{d-k} T_{d-k}(\sigma_1, \ldots, \sigma_r) Y_d \right\} \bigg|_{V^{(n, r)}(l_1, \ldots, l_r)} \]  

\[ = \sum_{k=0}^d (-1)^{d-k} T_{d-k}(\sigma_1, \ldots, \sigma_r) S_k(\chi_1 + \sigma_1, \ldots, \chi_r + \sigma_r) \]  

\[ = \Phi_{d; \sigma}^{11.3} \bigg|_{V^{(n, r)}(l_1, \ldots, l_r)}. \]  

(11.6)

For any eigenspace \( V^{(n, r)}(l_1, \ldots, l_r) \), (11.6) holds. Thus we have

\[ \Phi_{d; \sigma}^{11.3} = \sum_{k=0}^d (-1)^{d-k} T_{d-k}(\sigma_1, \ldots, \sigma_r) Y_k. \]  

(11.7)

Taking the radial parts of the both sides of (11.7), we obtain the main theorem in this section.
Theorem 11.3 (Radial Part Formula). The radial part of the generalized John type operator $\Phi^{(n, p)}_d$ $(1 \leq d \leq r)$ is given by the formula,

$$\text{rad} (\Phi^{(n, p)}_d) = \sum_{k=0}^d (-1)^{d-k} T_{d-k}(\sigma_d, ..., \sigma_r) U_k,$$

where $\sigma_j = (r-j)(n+1-r-j)$ and the differential operator $U_k$ on the Weyl chamber $\mathcal{W}^+$ is given by (11.1). (For the definition of $T_j$, see (4.2).)

As we mentioned in Remark 2.2, the algebra $\mathcal{D}(G/K_p)$ of invariant differential operators on $G/K_p$ is generated by the set $\{\Phi^{(n, p)}_1, \Phi^{(n, p)}_2, ..., \Phi^{(n, p)}_r\}$. Therefore, the above theorem determines the mapping $\text{rad} : \mathcal{D}(G/K_p) \to \mathcal{D}(\mathcal{W}^{+})_{W(G, K_p)}$. Moreover, we can say that the above theorem determines indirectly the Harish-Chandra isomorphism $\gamma : \mathcal{D}(G/K_p) \to \mathcal{D}(\mathcal{W}^{+})_{W(G, K_p)}$. Here $\mathcal{D}(\mathcal{W}^{+})_{W(G, K_p)}$ denotes the algebra of all $W(G, K_p)$-invariant polynomials on $\mathcal{W}^{+}$.

12. FINAL REMARKS

(1) Another Expression of the Eigenvalue Formula. By making use of Proposition 4.5, we can rewrite the eigenvalue formulas in Theorem 5.1(A) and Theorem 9.1(A). Namely we have the following.

Theorem 12.1. Let $r := \text{rank} \text{Gr}(p, n, \mathbb{F}) = \min \{p, n-p\}$. (i) In the case $\mathbb{F} = \mathbb{C}$, we take $\chi_j = l_j(l_j + n + 1 - 2j)$ and $a_j = j^2 - (n + 1 - j + n$. (ii) In the case $\mathbb{F} = \mathbb{R}$ we take $\chi_j = l_j(l_j + n - 2j)/2$ and $a_j = (1/4) j^2 - (1/4)n j + (1/16) n^2$. Moreover, in this case, we assume that $2r < n$.

Then the eigenvalue $\Phi^{(n, p)}_{s+1} |_{\chi^{(l_1, ..., l_t)}}$ of the generalized John type invariant differential operator $\Phi^{(n, p)}_{s+1}$ $(1 \leq s \leq r-1)$ on $\text{Gr}(p, n, \mathbb{F})$ is given by the formula

$$\Phi^{(n, p)}_{s+1} |_{\chi^{(l_1, ..., l_t)}} = \sum_{J \subset \{1, 2, ..., r\}, \# J = s+1} \chi^J + \sum_{J \subset \{1, 2, ..., r\}, 1 \leq \# J \leq s} A[s, J] \chi^J, \quad (12.1)$$

$$A[s, J] = \sum_{t \in \mathcal{J}} \prod_{s=1}^t (a_{J(s, 1)} - a_{J(s, 2)}). \quad (12.2)$$

Here, in (12.1), $\chi^J$ denotes the product $\chi_{J(1)} \chi_{J(2)} \cdots \chi_{J(m)}$ for $J = \{J(1), J(2), ..., J(m)\} (m = \# J, 1 \leq J(1) < J(2) < \cdots < J(m) \leq r)$. In (12.2), $J^1 := \{1, 2, ..., s\} \setminus J, J^2 := J \setminus \{1, 2, ..., s\}, \mu := \# J^1 v := \# J^2, \text{ and } 1 := s + 1 - m = \mu + 1 - r.$
For the definition of the set $P$ and the mapping $I \mapsto I^*$, see Notation B in Section 4.

(II) Range Characterization in the Quaternionic Case. Let $\text{Gr}(p, n; \mathbb{H})$ be the quaternionic Grassmann manifold of all $p$-dimensional subspaces in $\mathbb{H}^n$. As in the real or complex case, we can define a Radon transform $R_p : C^\infty(\text{Gr}(q, n; \mathbb{H})) \to C^\infty(\text{Gr}(p, n; \mathbb{H}))$. (Take $\mathbb{R} = \mathbb{H}$ in (1.1) and in (1.2).) Then, by the same argument as in the complex case, we see that the analogous eigenvalue formula and the analogous range theorem as Theorem 5.1(A) and (D) hold in the quaternionic case. More precisely, the results are stated as follows.

**Theorem 12.2.** Let $s := \text{rank } \text{Gr}(q, n; \mathbb{H})$ and $r := \text{rank } \text{Gr}(p, n; \mathbb{H})$ respectively. A highest weight occurring in $C^\infty(\text{Gr}(p, n; \mathbb{H}))$ is of the form $(l_1, l_2, \ldots, l_r, 0, \ldots, 0) \in \mathbb{R}^n$, where $l_j \in \mathbb{Z}$, $(1 \leq j \leq r)$ and $l_1 \geq l_2 \geq \cdots \geq l_r \geq 0$. Let $V^{(n, p)}(l_1, \ldots, l_r)$ be the corresponding irreducible eigenspace of the standard Laplacian on $\text{Gr}(p, n; \mathbb{H})$. Moreover, let $\chi_j := l_j(2l_j + 2n - 4j + 3)$ and $\alpha_j := 4l_j^2 - 2(2n + 3)j$ $(1 \leq j \leq r)$. We assume that $s < r$. Then there exists a $(s + 1)$st order invariant differential operator $\Psi^{(n, p)}_{s + 1}$ on $\text{Gr}(p, n; \mathbb{H})$ such that $\text{Im } R_p = \text{Ker } \Psi^{(n, p)}_{s + 1}$. The eigenvalue $\Psi^{(n, p)}_{s + 1} | v^{(n, p)}(l_1, \ldots, l_r)$ of the operator $\Psi^{(n, p)}_{s + 1}$ on the irreducible eigenspace $V^{(n, p)}(l_1, \ldots, l_r)$ is given by the same formula as in Theorem 5.1(A) for the above $\chi_j$ and $\alpha_j$.

Furthermore, the uniqueness result such as Theorem 5.1(E) holds. This is essentially due to the fact that both $\text{Gr}(p, n; \mathbb{H})$ and $\text{Gr}(p, n; \mathbb{C})$ have the same Weyl group structure.

(III) Reproducing Operators in Inversion Formulas. There are various kinds of Radon transforms. Thus, there are various kinds of inversion formulas such as Helgason's inversion formulas. However, all of them are of the form

\[ \{ \text{elliptic (pseudo-) differential operator} \} R^* = I, \]

where $R^*$ denotes the dual transform of the Radon transform $R$. More precisely, the above reproducing operator is a polynomial of the Laplacian or a fractional power of the Laplacian. (See, for example, Helgason [H4, Chap. (II)].) Therefore, one may expect that reproducing operators appearing in inversion formulas are always elliptic. However, our inversion formulas (Theorem 6.6 and Theorem 10.4) assert that in general cases the highest order part of a reproducing operator is not necessarily elliptic. In fact, the corresponding part of the inversion formula in Theorem 6.6 or in Theorem 10.4 is some power of the 2th order John type operator $\Phi^{(n, p)}_{s + 1}$, which is obviously non-elliptic if $s \geq 2$. 

(IV) **Other Results.** As we remarked in the introduction, recently similar range characterizations were obtained by Oshima [O] and by Tanisaki [Tan] from a quite different point of view. In [O], Oshima studies boundary value problems for $G(n, \mathbb{R})$ by making use of the Poisson transform, which is closely related to the proof of the so-called Helgason conjecture. As an application of the results of the above problems, he shows that the range of the Radon transform on a real Grassmann manifold is characterized by generalized Capelli operators. In addition, Tanisaki [Tan] deals with Radon transforms on compact Hermitian symmetric spaces, using the theory of $D$-modules. According to [Tan], the ranges of the above Radon transforms are characterized by a certain third order system of differential equations. Their range-characterizing systems are expressed in terms of determinantal type of differential equations. In this sense, their characterizations are similar to ours. However, what we would like to stress here is that our approach is based on the elementary but detailed calculus of invariant differential operators which results not only in our range theorems but also in the explicit inversion formulas and other important formulas in the harmonic analysis on Grassmann manifolds.

(V) **Conjecture.** There are many results in range characterization problems. Therefore, one may think of the following question: To what extent can one generalize range theorems for Radon transforms?

Let us state our conjectures.

**Conjecture 1.** Let $G$ be a compact Lie group and $G/L$ and $G/K$ be homogeneous spaces. We can define a $G$-equivariant Radon transform $R: C^\infty(G/L) \to C^\infty(G/K)$ for the canonical double fibration $G/L \leftarrow G \rightarrow G/K$. Assume the following conditions. (a) $\dim G/L < \dim G/K$. (b) There exists a continuous linear mapping $S: C^\infty(G/K) \to C^\infty(G/L)$ such that $SR = I$ on $C^\infty(G/L)$. Then the range of $R$ is characterized by some $G$-invariant differential operator on $G/K$.

**Conjecture 2.** Let $X$ and $Y$ be compact analytic manifolds. Let $R$ be a continuous linear mapping from $C^\infty(X)$ to $C^\infty(Y)$. Assume the following three conditions. (a) $\dim X < \dim Y$. (b) There exists a continuous linear mapping $S$ from $C^\infty(Y)$ to $C^\infty(X)$ such that $SR = I$ on $C^\infty(X)$. (c) $R$ maps a constant function on $X$ to a constant function on $Y$. Then the range of $R$ is characterized by some (pseudo-) differential operator on $Y$.

The second conjecture suggests that if a closed subspace of $C^\infty(Y)$ is isomorphic to $C^\infty(X)$ for another lower-dimensional manifold $X$ then such a closed subspace is separated as the kernel of some (pseudo-) differential operator on $Y$. 

43 INTEGRAL GEOMETRY
We remark that Guillemin and Sternberg [GuS] partly answers the above question under some strong conditions.

ACKNOWLEDGMENTS

I would like to thank Professor Fulton Gonzalez and Professor Sigurdur Helgason for their valuable suggestions. I am also grateful to my teacher Professor Chiaki Tsukamoto for his encouragement.

REFERENCES


