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ABELIAN GROUPS $\aleph_0$-CATEGORICAL OVER A SUBGROUP*

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An abelian group A is said to be $\aleph_0$-categorical over its subgroup B when there is a unique countable model of the theory of A with distinguished subgroup B for any possible choice of countable dinstinguish subgroup. We give necessary and sufficient conditions for an abelian group to be $\aleph_0$-categorical over one of its subgroups. Furthermore we give an axiomatization of such theories in terms of some first-order invariants and show that these invariants can have any value as long as they satisfy some minor conditions. With these results we obtain a new proof of Hodges' decomposition theorem (Corollary 2.6). Finally, in the case of torsion-free abelian groups we conclude that A is $\aleph_0$-categorical over its subgroup B iff $B=mA$ for some integer m.

1. Introduction

Let L be a first-order language and $L(P)$ be the language obtained from L by adding a unary predicate P. Pairs of the form (A,B) will represent the $L(P)$-structure formed by the L-structure A with its substructure B as realization of the predicate P. Following [2] $(A,B)\times(C,D)$ will be the structure $(A\times C, B\times D)$. It is important to remember that a direct decomposition of a structure does not always induce a direct decomposition of one of its substructures. Hence a direct decomposition of (A,B) means, in general, much more than just a decomposition of A.

Definition. An L-structure A is said to be $\aleph_0$*-categorical over one of its substructures* B, if A is countable and furthermore if for all countable $L(P)$-structures (C,D) and (C',D') elementarily equivalent to (A,B) such that $D=D'$ there exists an isomorphism of C to C' extending $D=D'$ (i.e. its restriction to D is the identity).

Remark. In the above definition it is possible that the L-structures involved are finite. This case is not excluded because a finite L-structure does not have to be

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$\aleph_0$-categorical over all of its substructures. Nevertheless it is possible to give a simple description of $\aleph_0$-categoricity over a substructure, when the substructure is finite.

Proposition 1.1. *Let A and B be L-structures and suppose B is finite. The following two conditions are equivalent*:

(i) *A is $\aleph_0$-categorical over B*,

(ii) (a) *(A,B) is $\aleph_0$-categorical as an $L(P)$-structure and*

(b) *every automorphism of B can be extended to an automorphism of A.*

Proof. Suppose (i) is satisfied. Take (C,D) countable and elementarily equivalent to (A,B). Since B is finite, it is clear that B and D are isomorphic. By Proposition 3.4 of [3], there exists an isomorphism between A and C extending the one between B and D. It follows that (A,B) and (C,D) are isomorphic as $L(P)$-structures and (ii)(a) is proved.

(ii)(b) follows directly from Proposition 3.4 of [3].

Suppose now that (ii) is satisfied. Let (C,D) and (C',D') be countable and elementarily equivalent to (A,B) with $D=D'$. By (ii)(a) one finds an $L(P)$-isomorphism $\alpha:(C,D)\to(C',D')$. Let β be the restriction of α to $D=D'$. By (ii)(b) $\beta^{-1}:D'\to D'$ can be extended to an isomorphism $\gamma:C'\to C'$ since by (ii)(a) (A,B) and (C,D) are isomorphic. Hence $\gamma\circ\alpha:C\to C'$ is an isomorphism and its restriction to D is the identity. This completes the proof. □

It follows from the previous result that if A is finite, one has just to check if (ii)(b) is satisfied in order to have $\aleph_0$-categoricity over B.

In this paper we will restrict ourselves to abelian groups. Hence L will be the language of the theory of abelian groups and $L(P)$ will be the language obtained from L by adding a unary predicate P. Abelian groups will be symbolized by capital letters and pairs of the form (A,B) will represent the $L(P)$-structure formed by the abelian group A with its subgroup B as realization of the predicate P. As is usual in algebra we will speak of finite direct sums instead of products. Hence $(A,B)\oplus(C,D)$ will be the structure $(A\oplus C, B\oplus D)$. The term group will mean here abelian group.

Hodges [2, Corollary 2.2] proved that if A is an abelian group $\aleph_0$-categorical over B then A/B is a bounded abelian group. Furthermore Pillay [5, Corollary 11] proved that in the general case if A is a structure $\aleph_0$-categorical over B then the intersection of any $L(P)$-definable subset (without parameters) of (A,B) with B is L-definable in B (again without parameters). In this paper we show (Theorem 2.3) that in fact an abelian group A is $\aleph_0$-categorical over its subgroup B if and only if A/B is bounded and $B\cap p^nA$ is definable without parameters in B (see the following for the notation). Furthermore we give (Theorem 2.5) for such A and B a simple axiomatization of the theory of (A,B) in terms of the L-theory of B and a finite number of $L(P)$-sentences. From this we deduce (Corollary 2.6) a new proof of the tight-decomposition theorem of Hodges [2, Theorem 4.6 and 5.2]. In the case where

A is torsion-free, we prove that A is $\aleph_0$-categorical over B if and only if $B=mA$ for some natural number m.

2. Extending isomorphisms

The greatest problem in the study of $\aleph_0$-categoricity over a subgroup is to find a way to extend the isomorphism between the subgroups. It is clear that such a thing is impossible in complete generality, but using a result of [2], one can see that in the case of $\aleph_0$-categoricity over a subgroup it is possible to use the procedure algebraists use to show Ulm's theorem.

We now need some notation and terminology.

Definition. Let A be a group and p be a prime number. We define by induction on the ordinals the following subgroups of A.

$$p^0A=A, \qquad p^{\alpha+1}A=\{px : x\in p^\alpha A\},$$
$$p^\alpha A=\bigcap_{\beta\in\alpha} p^\beta A \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Definition. The *p-height* $h_p(a)$ of an element a of A is the ordinal α such that $a\in p^\alpha A\setminus p^{\alpha+1}A$ if it exists and infinity otherwise. If there is a danger of confusion we will write $h_p^A(a)$ to make clear that the height is computed in A.

Definition. Let A be a group, p be a prime number and α be an ordinal. $p^\alpha A[p]$ will be the subgroup of $p^\alpha A$ formed by its elements of order p.

Definition. Let A and C be groups and B and D be subgroups of A and C respectively. An isomorphism $\alpha : B\to D$ is said to *preserve height* if $h_p(b)=h_p(\alpha(b))$ for every b in B and every prime number p. It is important to understand that the heights are computed in A and C respectively.

The following lemma is just a reformulation of a result of [4] (see also [1, Lemma 77.1]). I state it here in a form which will be useful for our purpose.

Lemma 2.1. *Let A and C be groups and B and D be subgroups of A and C respectively. Let $\alpha : B\to D$ be a height preserving isomorphism. Suppose also that*

$$\begin{aligned}&\dim_p(p^\sigma A[p]/(B+p^{\sigma+1}A)\cap p^\sigma A[p])\\ &\quad =\dim_p(p^\sigma C[p]/(D+p^{\sigma+1}C)\cap p^\sigma C[p])\end{aligned} \tag{1}$$

for every prime number p and every ordinal σ. ($\dim_p$ *is the dimension of the vector space over the p element field.*)

Let now p be a prime number, a be an element of A, ϱ be an ordinal, b and b' be elements of B such that

$$pa = b', \qquad \varrho = h_p(a/B) = h_p(a+b),$$

where $h_p(a/B)$ denotes the height of the coset of a in A/B. Suppose also that b is chosen so that, if there is an element y of B such that $\varrho = h_p(a+y)$ and $h_p(pa+py) > \varrho+1$, then $h_p(pa+pb) > \varrho+1$.

Then there exists an element c in C such that

$$pc = \alpha(b'), \qquad \varrho = h_p(c/D) = h_p(c+\alpha(b)) \tag{2}$$

and for every c in C satisfying (2) there is a height preserving isomorphism $\alpha^*: \langle B, a\rangle \to \langle D, c\rangle$ extending α such that $\alpha(a) = c$ and

$$\dim_p(p^\sigma A[p]/(\langle B,a\rangle + p^{\sigma+1}A) \cap p^\sigma A[p])$$
$$= \dim_p(p^\sigma C[p]/(\langle D,c\rangle + p^{\sigma+1}C) \cap p^\sigma C[p])$$

for every prime number p and every ordinal σ.

Proof. See the proof of Lemma 77.1 of [1]. □

Definition. A group G is said to be *bounded* if there is a natural number n such that $nG = 0$. The *bound* of G is then the smallest such natural number.

Corollary 2.2. *Suppose A and C are countable and satisfy the hypothesis of Lemma* 2.1 *and furthermore that A/B and C/D are bounded. Then there exists an isomorphism $\beta: A \to C$ extending α such that $\beta(a) = c$.*

Proof. The isomorphism between A and C is constructed by a back and forth argument.

Enumerate $A = \{a_i : i \in \omega\}$, $C = \{c_i : i \in \omega\}$. Take the first i such that a_i is not in $\langle B, a\rangle$. Since A/B is bounded, it follows that a_i is the finite sum of elements of A which are p-elements modulo $\langle B, a\rangle$ for various prime numbers p. Since adding successively these elements to $\langle B, a\rangle$ will generate a subgroup of A containing a_i, it is possible to suppose without loss of generality that a_i is a p-element modulo $\langle B, a\rangle$ for some prime number p. Let a_i be an element of order p^n modulo B for some natural number n. Since A/B is bounded the p-height of $p^{n-1}a_i$ modulo $\langle B, a\rangle$ is smaller than ω and there exists an element b in $\langle B, a\rangle$ such that $h_p(p^{n-1}a_i/\langle B, a\rangle) = h_p(p^{n-1}a_i + b)$ and furthermore such that if there is a y in $\langle B, a\rangle$ such that $h_p(p^{n-1}a_i/\langle B, a\rangle) = h_p(p^{n-1}a_i + y)$ and $h_p(p^n a_i + py) > h_p(p^{n-1}a_i/\langle B, a\rangle) + 1$, then b already satisfies this condition.

So $p^{n-1}a_i$ satisfies the same conditions as the a in Lemma 2.1 where $b' = p^n a_i$, α^* is in place of α and $\langle B, a\rangle, \langle D, c\rangle$ are in place of B, D. Therefore there exists an element c_j in C and a height preserving isomorphism $\alpha^{**}: \langle B, a, p^{n-1}a_i\rangle \to \langle D, c, c_j\rangle$ extending α^* and also

$$\dim_p(p^\sigma A[p]/(\langle B, a, p^{n-1}a_i\rangle + p^{\sigma+1}A) \cap p^\sigma A[p])$$
$$= \dim_p(p^\sigma C[p]/(\langle D, c, c_j\rangle + p^{\sigma+1}C) \cap p^\sigma C[p])$$

for every prime number p and every ordinal σ.

Now a_i is of order p^{n-1} modulo $\langle B, a, p^{n-1}a_i\rangle$ and hence iterating the last steps one can show that there is some height preserving isomorphism between a subgroup B' of A containing $\langle B, a\rangle$ and a_i, and a subgroup D' of C containing $\langle D, c\rangle$ extending α^* and such that

$$\dim_p(p^\sigma A[p]/(B' + p^{\sigma+1}A) \cap p^\sigma A[p])$$
$$= \dim_p(p^\sigma C[p]/(D' + p^{\sigma+1}C) \cap p^\sigma C[p])$$

for every prime number p and every ordinal σ.

Proceeding in the same way with the first element of C which is not in D' one gets a back and forth procedure which gives an isomorphism between A and C extending α^* and the proof is completed. □

It is now possible to give necessary and sufficient conditions for $\aleph_0$-categoricity over a subgroup.

Theorem 2.3. *A is $\aleph_0$-categorical over B if and only if*
(i) *A/B is a bounded group,*
(ii) *$B \cap p^n A$ is L-definable (without parameters) in B for every prime number p and natural number n such that p^n divides the bound of A/B.*

Proof. Suppose A is $\aleph_0$-categorical over B. (i) follows from Corollary 2.2 of [2] (it is not necessary that B is finite).

Since p^nA is definable in A, (ii) follows from Corollary 11 of [5].

Suppose now that (i) and (ii) hold. Let (C, D) and (C, D') be countable and elementarily equivalent to (A, B) and let $D = D'$. We now have to check a few things in order to apply the above lemma.

Firstly we must show that $D = D'$ is height preserving. We first show that $D \cap p^\omega C = p^\omega D$, where p is a prime.

The inclusion $p^\omega D \subseteq D \cap p^\omega C$ is obvious. Suppose now that d is an element of $D \cap p^\omega C$. Let n be any natural number. Take m to be the largest natural number such that p^m divides the bound of C/D, which exists by (i). Since d is in $p^\omega C$, there exists an element c of C such that $d = p^{n+m}c$. Hence c is a p-element modulo D and by definition of p^m, $p^m c$ is in D. It follows that d is in p^nD. The result holding for any natural number n, it is clear that $D \cap p^\omega C = p^\omega D$.

This proves that for any d in D, if $h_p^C(d) \geq \omega$ then $h_p^D(d) = h_p^C(d)$. Since $D = D'$

and since C'/D' is also bounded (by elementary equivalence), it follows that $D=D'$ preserves height equal or greater than ω.

Now it suffices to show that $D=D'$ preserves height smaller than ω.

By (ii) one knows that $D\cap p^nC$ is definable in D if p^n divides the bound of C/D. If p^n does not divide this bound, let m be as before. We will check that $D\cap p^nC = p^{n-m}(D\cap p^mC)$. The right-hand side is obviously included in the left-hand one. Take now an element d in D such that $d=p^nc$ for some c in C. Since c is a p-element modulo D, by definition of m, p^mc is in D. Hence from $d=p^{n-m}(p^mc)$ it follows that d is in $p^{n-m}(D\cap p^mC)$ and the above equality is proved.

So for every prime number p and every natural number n we have that $D\cap p^nC$ is definable in D (without parameters). Since by elementary equivalence $D'\cap p^nC'$ is definable by the same formula in $D=D'$, it follows that $D=D'$ preserves height.

We now have to check that the condition (1) of Lemma 2.1 holds. Let p be a prime number and let m be as before. Let c be an element of $p^\sigma C[p]$ for σ an ordinal greater than m. Since there is no p-element of order greater than p^m in C/D, it follows that c is in D. Since by elementary equivalence the bound of C'/D' is equal to the bound of C/D it is now sufficient to check (1) for the p^n such that n is smaller or equal to m. But elementary equivalence takes care of this, since C and C' are countable.

The result now follows from Lemma 2.1 and Corollary 2.2. □

Using Corollary 2.2 it is also possible to prove the following result which is a little generalization of Theorem 4.2 of [2].

Corollary 2.4. *Let A be $\aleph_0$-categorical over B. For any integer n there exists a finite set of $L(P)$-formulas $\{\varphi_i(\bar{x},\bar{y})\colon i\in I\}$ such that for any n-type (in the language $L(P)$) over B which is realized in A, there is an $i\in I$ and a tuple $\bar{b}$ in B such that $\varphi_i(\bar{x},\bar{b})$ isolates the type in question.*

Proof. Let B' be any subgroup of A containing B such that B'/B is finite. We will consider the following structure $(A,B,b)_{b\in B'}$. Also every type considered will be in the language $L(P)$.

Let $\bar{a}=(a_1,\dots,a_n)$ be an n-tuple of A. The type of $\bar{a}$ over B' is isolated if and only if for every $i=1,\dots,n$ the type of a_i over $\langle B',a_1,\dots,a_{i-1}\rangle$ is isolated. Note also that since A/B is bounded $\langle B',a_1,\dots,a_{i-1}\rangle/B$ is finite for $i=1,\dots,n$. Furthermore if for every $i=1,\dots,n$ the type of a_i over $\langle B',a_1,\dots,a_{i-1}\rangle$ is isolated by the formula $\psi_i(x,a_1,\dots,a_{i-1},\bar{b}_i)$ where $\bar{b}_i$ is in B', then $\bigwedge_{i=1}^n \psi_i(x_i,x_1,\dots,x_{i-1},\bar{b}_i)$ isolates the type of $\bar{a}$ over B'.

In the same way for any element a of A which is a p-element modulo B' the type of a over B' is isolated if and only if the type of p^ia over $\langle B',p^{i-1}a\rangle$ is isolated for any $i=1,\dots,s$ where p^s is the order of a. Furthermore $\langle B',p^{i-1}a\rangle/B$ is finite, since A/B is bounded. Here also if for every $i=1,\dots,s$ the type of p^ia over $\langle B',p^{i-1}a\rangle$ is isolated by the formula $\eta_i(x, p^{i-1}a_1, \bar{b}_i')$ where $\bar{b}_i'$ is in B', then $\bigwedge_{i=1}^{s} \eta_i(p^i x, p^{i-1}x, \bar{b}_i')$ isolates the type of a over B'.

Hence the result will follow if we can prove that for any subgroup B' of A containing B and such that B'/B is finite there exists a finite set of L-formulas $\{\varphi_i(x, \bar{y})\}$ such that for any a in A for which pa is in B', then the type of a over B' is isolated by a formula of the form $\varphi_i(x, \bar{b})$ for some $\bar{b}$ in B'.

Take such an a. If a is in B' the result is obvious. Suppose now that a is not in B'. Take m to be equal to the p-height of a modulo B'. Let b be an element of B' such that m equals the p-height of $a+b$ and suppose also that if there is an element b' in B' such that $h_p(a+b')=m$ and if the p-height of $pa+pb'$ is strictly greater than $m+1$ then the p-height of $pa+pb$ is already of this kind.

Then the following set of sentences isolates the type of a over B'.

$$px = pa, \qquad m = h_p(x/B') = h_p(x+b).$$

To see that this is true let $p(x)$ be a type over B' containing these formulas. Take $(C, D, b)_{b \in B'}$ to be a countable elementary extension of $(A, B, b)_{b \in B'}$ such that for some c in C, c realizes $p(x)$. Applying Corollary 2.2 with $\langle B', D\rangle$ in place of B, one finds an automorphism of C leaving $\langle B', D\rangle$ fixed pointwise which send a to c. Therefore $p(x)$ is the type of a over B'.

Hence the type of a over B' is isolated. Furthermore since p and m in the above formula are smaller than the bound of A/B there are only finitely many such formulas and the proof is completed. □

One natural question to ask at this point is whether any definable subgroups can occur as the subgroups $B \cap p^n A$ for some A $\aleph_0$-categorical over B. The following result shows that under a few obvious conditions every choice is possible.

Definition. Let A be $\aleph_0$-categorical over B and let b be the bound of A/B.

The set $\{\varphi_{p,n}(x): \varphi_{p,n}(x)$ defines $A \cap p^n B$ in $B\}$ where the p^n are the powers of primes dividing b is called the *valuation* of (A, B).

Furthermore, the set $\{k_{p,n}: p^n$ is a power of a prime dividing $b\}$ where

$$k_{p,n} = \dim_p(p^n A[p]/(B + p^{n+1}A) \cap p^n A[p])$$

is called the *set of Ulm–Kaplansky dimensions of* (A, B).

Theorem 2.5. *Let A be $\aleph_0$-categorical over B, $\{\varphi_{p,n}\}$ be its valuation, $\{k_{p,n}\}$ be the set of its Ulm–Kaplansky dimensions and b be the bound of A/B. Then the following $L(P)$-sentences axiomatize* Th(A, B).

$$B \cap p^n A = \varphi_{p,n}(B) \quad \textit{for } \varphi_{p,n} \textit{ in the valuation.} \tag{3}$$

$$\bigcap_{p,n} \varphi_{p,n}(B) = bA, \tag{4}$$

where the intersection is taken over the valuation.

$$\dim_p p^n A[p]/(B+p^{n+1}A) \cap p^n A[p] = k_{p,n}, \tag{5}$$

where $k_{p,n}$ are the Ulm–Kaplansky dimensions.

$$\{\psi^P\colon \psi \text{ is an L-sentence true in } B\}, \tag{6}$$

where ψ^P is the relativization of the formula ψ to the predicate P.

Moreover, given any countable group B, any natural number b, any set of formulas $\{\varphi_{p,n}\colon p^n$ is a power of a prime dividing b$\}$ and any set of countable cardinals $\{k_{p,n}\colon p^n$ is a power of a prime dividing b$\}$ such that

$$\varphi_{p,n}(B) \text{ is a subgroup of } B, \tag{3'}$$

$$p^n B \subseteq \varphi_{p,n}(B), \tag{4'}$$

$$p\varphi_{p,n}(B) \subseteq \varphi_{p,n+1}(B), \tag{5'}$$

for any $\varphi_{p,n}$ and $\varphi_{p,n+1}$ in the above set, there exists a countable group A containing B such that A is $\aleph_0$-categorical over B and (3), (4) *and* (5) *are verified.* (*It is easy to check that* (3′), (4′) *and* (5′) *are consequences of* (3). *Hence they are necessary conditions for such a result to hold.*)

Proof. Let A be $\aleph_0$-categorical over B. By Theorem 2.3 A/B is bounded; let b be its bound. By Theorem 2.3 it is possible to find formulas such that (3) is verified. Since the bound of A/B is b (4) follows by (3).

Define now the $k_{p,n}$ simply as they are stated in (5). It is clear that (A,B) satisfy (3), (4), (5) and (6).

To show that (3), (4), (5) and (6) axiomatize $\mathrm{Th}(A,B)$, take a countable model (C,D) of (3), (4), (5) and (6). Since D is elementarily equivalent to B, if B is finite one can assume that $D=B$, otherwise one can easily find elementary chains of pairs of countable groups (A_i,B_i), (C_i,D_i) for $i\in\omega$ such that

$$(A_0,B_0)=(A,B), \qquad (C_0,D_0)\equiv(C,D),$$

$$B_i\subseteq D_i \quad \text{for } i\in\omega, \qquad D_i\subseteq B_{i+1} \quad \text{for } i\in\omega.$$

Hence the union of these chains are countable and have the same subgroup. So without loss of generality we can assume that $D=B$.

As in the proof of Theorem 2.3 one can show that $B=D$ is height preserving and the analogue of condition (1) of Lemma 2.1 is satisfied. Hence by Corollary 2.2 (remember that by Lemma 2.1 there is an element c in C satisfying the conditions of the corollary) there is an isomorphism between A and C extending $B=D$, hence (A,B) and (C,D) are isomorphic and the proof of the first part is completed.

To prove the second part, we will proceed in a way similar to the proof of the Theorem 1 of [7]. It is not possible to proceed exactly in the same way since we want (5) to hold.

Consider the set of all couples of the form (p^n,y) where p^n is a power of a prime dividing b and y is an element of $\varphi_{p,n}(B)$. Let F be the free group on those pairs.

Take K to be a maximal subgroup of $F\oplus B$ under the following conditions.

(a) $K \supseteq \langle p^n(p^n, y)-y; y\in \varphi_{p,n}(B)$, p^n is a power of a prime dividing $b\rangle$,

(b) $K\cap B=0$,

(c) $B\cap (K+p^n(F\oplus B))=\varphi_{p,n}(B)$ where p^n is a power of a prime dividing b.

To show that there is (via Zorn's lemma) such a maximal K, it suffices, since the above properties are preserved under union of chains, to check that they are satisfied by

$$K_1 = \langle p^n(p^n, y)-y; y\in \varphi_{p,n}(B),\ p^n \text{ is a power of a prime dividing } b\rangle.$$

K_1 obviously satisfies properties (a) and (b). To check (c) suppose y is an element of $\varphi_{p,n}(B)$. Then $p^n(p^n, y)-y$ is in K_1 (by (a)) and since $y=p^n(p^n, y)-[p^n(p^n, y)-y]$, y is in $K_1+p^n(F\oplus B)$. If now y is in $B\cap (K_1+p^n(F\oplus B))$, write y as $k+p^nu$ where k is in K_1 and u is in $F\oplus B$. Remembering that the canonical projection of y on F must be 0, that F is free and that (3′), (4′) and (5′) hold one gets that y is the sum of an element of p^nB with one of $\varphi_{p,n}(B)$. Hence by (3′) and (4′) one gets that y is in $\varphi_{p,n}(B)$.

Therefore there is a maximal K with properties (a), (b) and (c) holding. We will now show that for such a K the group $A_1=(F\oplus B)/K$ has property (3), (4) and also that

$$\dim_p p^nA_1[p]/(B+p^{n+1}A_1)\cap p^nA_1[p] = 0$$

for p^n a power of a a prime dividing b, where the embedding of B in A_1 is canonical (it is an embedding by (b)). Firstly (3) is an obvious consequence of (c) while (4) will be satisfied as soon as $bA_1\subseteq B$ will be checked. Since this last inclusion is a consequence of (a), (4) holds.

The only remaining property is the last one. To show this take a_1 in $p^nA_1[p]$ for some p and n satisfying the usual conditions. Take now K' to be $\langle K, \bar a_1\rangle$ where $\bar a_1$ is a representative of a_1 in $F\oplus B$. K' obviously satisfies (a). If $\langle K, \bar a_1\rangle\cap B$ is non-trivial, then $\bar a_1$ is in $K+B$, since $K\cap B=0$. Hence a_1 is in B, therefore a_1 is in $B+p^{n+1}A_1$.

Suppose now that $K'\cap B=0$. Condition (c) is then equivalent to the left to right inclusion only, since the other one follows as before from (a). Let y be in $B\cap(K'+p^n(F\oplus B))$ without being in $K+p^n(F\oplus B)$. It follows that $\bar a_1$ is in $B+p^nF+K$, therefore a_1 is in $B+p^nA_1$ and (c) holds.

To have the result define A_2 to be the direct sum of $k_{p,n}$ many copies of the cyclic group of order p^{n+1}. Define A as $A_1\oplus A_2$, i.e. $(A,B)=(A_1,B)\oplus(A_2,0)$. It is now obvious that (A,B) satisfy (3), (4) and (5). By Theorem 2.3 A is $\aleph_0$-categorical over B and the proof is completed. □

We actually get in a different way the following result of [2]. (See [2, Theorem 4.6].)

Corollary 2.6. *A is $\aleph_0$-categorical over B if and only if*

$$(A,B) = (A_1,B)\oplus(A_2,0),$$

where A_1 is $\aleph_0$-categorical over B,

$$\dim_p p^n A_1[p]/(B+p^{n+1}A_1)\cap p^n A_1[p] = 0$$

for every prime number p and every natural number n such that p^n divides the bound of A_1/B and A_2 is a bounded group. This decomposition is unique up to isomorphism.

Proof. Let A be $\aleph_0$-categorical over B. By Theorem 2.5 there exists a group A_1 $\aleph_0$-categorical over B such that $B\cap p^n A = B\cap p^n A_1$ also such that

$$\dim_p p^n A_1[p]/(B+p^{n+1}A_1)\cap p^n A_1[p] = 0$$

for every prime number p and every natural number n such that p^n divides the bound of A/B. Define A_2 to be the direct sum of

$$\dim_p p^n A[p]/(B+p^{n+1}A)\cap p^n A[p]$$

many copies of the cyclic group of order p^{n+1} for every prime numbers p and every natural number n, such that p^n divides the bound of A/B.

Now by Theorem 2.5 (A,B) and $(A_1,B)\oplus(A_2,0)$ are elementarily equivalent, hence isomorphic and the first part of the proof is completed.

The other direction follows from Theorem 2.3. Furthermore the decomposition is unique since (A_1,B) is the only pair having the same valuation as (A,B) and zero Ulm–Kaplansky invariants and A_2 is characterized up to isomorphism by

$$\dim_p p^n A_2[p]/p^{n+1}A_1[p] = \dim_p p^n A[p]/(B+p^{n+1}A)\cap p^n A[p].$$

This completes the proof. □

Now that the problem of describing $\aleph_0$-categorical groups over a subgroup has been reduced to existence of some definable subgroups, it will be shown in the next section that in torsion-free case this leads to a nice characterization of $\aleph_0$-categoricity over a subgroup.

3. The torsion-free case

Proposition 3.1. *Let G be a torsion-free group. A subgroup H of G is L-definable (without parameters) if and only if $H=mG$ for some natural number m.*

Proof. Let $\varphi(x)$ be an L-formula defining H in G. It is well known (see [6, Theorem 2.$\mathbb{Z}$ 1(b)]) that $\varphi(x)$ is equivalent to a boolean combination of formulas of the form $p^i \mid p^j x$ and $nx=0$ where i, j and n are natural numbers and p is a prime number ($p^i \mid p^j x$ meaning p^i divides $p^j x$). Since G is torsion-free this reduces to $\varphi(x)$ being a boolean combination of formulas of the form $p^i \mid x$ for natural numbers i and prime numbers p.

Therefore, if for some elements g of G and h of H we have that $p^i \mid h$ if and only if $p^i \mid g$ for any natural number i and prime number p, then g is also in H.

We will now use the following general idea. Suppose a_1, a_2 are elements of a group and p is a prime number. If $h_p(a_1)=m_1$ and $h_p(a_2)>m_1$, then $h_p(a_1+a_2)=m_1$, since it is greater or equal to m_1 and if it was strictly greater, then p^{m_1+1} would divide a_1 (because it already divides a_2).

Let now $p_1,\dots,p_n$ be the prime numbers occurring in $\varphi(x)$ and let $p_1^{k_1},\dots,p_n^{k_n}$ be their maximal powers occurring in $\varphi(x)$. Therefore for g in G and h in H if $p_i^{s_i} \mid g$ if and only if $p_i^{s_i} \mid h$ for $i=1,\dots,n$ and $s_i=1,\dots,k_i$ then g is also in H. Let m_i be the minimum of $\{h_{p_i}(h)\colon h\in H\}$; it is an ordinal number.

Define $m=\prod_{i=1}^n p_i^{t_i}$, where t_i is the minimum of $\{k_i,m_i\}$. It is obvious that $H\subseteq mG$. We will now prove that the reverse inclusion holds.

Let h_i be an element of H of p_i-height equal to m_i, for $i=1,\dots,n$. For Δ a subset of $\{1,\dots,n\}$ define

$$h_\Delta=\sum_{j\in\Delta}^{n}\left(\prod_{i\neq j}p_i^{t_i+1}\right)h_j.$$

Where the product over the empty set is equal to 1 and the sum over the empty set is equal to 0. We can now compute the p_i-height of h_Δ.

$$h_{p_i}(h_\Delta)=\begin{cases}t_i & \text{if } t_i=m_i \text{ and } i\in\Delta,\\ \text{greater or equal to } t_i & \text{if } t_i=k_i \text{ and } i\in\Delta,\\ \text{strictly greater than } t_i & \text{if } i\notin\Delta.\end{cases}$$

In particular if $I=\{1,\dots,n\}$ we have the following.

$$h_{p_i}(h_I)=\begin{cases}t_i & \text{if } t_i=m_i,\\ \text{greater or equal to } t_i & \text{if } t_i=k_i.\end{cases}$$

Take now g in mG and let

$$\Gamma=\{i\colon 1\le i\le n, h_{p_i}(g)>t_i\}.$$

Notice that if i is not in Γ then $h_{p_i}(g)=t_i$, since g is in mG. We therefore have the following.

$$h_{p_i}(h_\Gamma+g)=\begin{cases}t_i & \text{if } t_i=m_i \text{ and } i\in\Gamma,\\ \text{greater or equal to } t_i & \text{if } t_i=k_i \text{ and } i\in\Gamma,\\ t_i & \text{if } i\notin\Gamma.\end{cases}$$

The last line come from the fact that g is in mG.

Hence it now follows that $p_i^{s_i}\mid(h_\Gamma+g)$ if and only if $p_i^{s_i}\mid h_I$ for $i=1,\dots,n$ and $s_i=1,\dots,k_i$.

Hence $h_\Gamma+g$ is in H. It follows that $mG=H$ and the proof is completed. □

Theorem 3.2. *Let A be torsion-free. A is $\aleph_0$-categorical over B if and only if $B = mA$ for some natural number m.*

Proof. Let A be as stated. By Theorem 2.3 A/B is bounded, hence $nA \subseteq B$ for some natural number n. Furthermore again by Theorem 2.3 nA is L-definable in B, therefore by Proposition 3.1 $nA = mB$ for some natural number m. Since A is torsion-free, division is unique and we can assume that $(n, m) = 1$. Hence $B = nB + mB = nB + nA = nA$.

The converse follows trivially from Theorem 2.3 and the proof is completed. □

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