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Grothendieck–Serre formula and bigraded Cohen–Macaulay Rees algebras

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Abstract

The Grothendieck–Serre formula for the difference between the Hilbert function and Hilbert polynomial of a graded algebra is generalized for bigraded standard algebras. This is used to get a similar formula for the difference between the Bhattacharya function and Bhattacharya polynomial of two $\mathfrak{m}$-primary ideals I and J in a local ring $(A, \mathfrak{m})$ in terms of local cohomology modules of Rees algebras of I and J. The cohomology of a variation of the Kirby–Mehran complex for bigraded Rees algebras is studied which is used to characterize the Cohen–Macaulay property of bigraded Rees algebra of I and J for two dimensional Cohen–Macaulay local rings.

Keywords: Bhattacharya polynomial; Bigraded Cohen–Macaulay Rees algebras; Bigraded Kirby–Mehran complex; Complete reduction; Grothendieck–Serre formula; Joint reduction; Mixed multiplicities; Ratliff–Rush closure

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1. Introduction

Let $R = \bigoplus_{n \geqslant 0} R_n$ be a finitely generated standard graded algebra over an Artinian local ring R_0. Let λ denote length. The Hilbert function of R, $H(R,n) = \lambda_{R_0}(R_n)$, is given by a polynomial $P(R,n)$ for $n \gg 0$. The Grothendieck–Serre formula expresses the difference $H(R,n) - P(R,n)$ in terms of lengths of graded components of the local cohomology modules of R with support in the irrelevant ideal $R_+ = \bigoplus_{n>0} R_n$ of R. We shall prove a version of this formula in Section 2 for bigraded standard algebras over Artinian local rings. We need this generalization to find necessary and sufficient conditions for the Cohen–Macaulay property of bigraded Rees algebras. These conditions involve the coefficients of the Bhattacharya polynomial of two $\mathfrak{m}$-primary ideals in a local ring $(R, \mathfrak{m})$.

To be more precise, let I and J be $\mathfrak{m}$-primary ideals in a d-dimensional local ring $(R, \mathfrak{m})$. The function $B(r,s) = \lambda(R/I^r J^s)$ is called the Bhattacharya function of I and J [B]. Bhattacharya proved in [B] that this function is given by a polynomial $P(r,s)$ for $r, s \gg 0$. We represent the Bhattacharya polynomial $P(r,s)$ corresponding to $B(r,s)$ by

$$P(r,s) = \sum_{i+j \leqslant d} e_{ij} \binom{r}{i} \binom{s}{j}$$

where $e_{ij} \in \mathbb{Z}$. The integers e_{ij} for which $i + j = d$ were termed as mixed multiplicities of I and J by Teissier and Risler in [T]. We write $e_j(I|J)$ for e_{ij} when $i + j = d$.

The bigraded version of the Grothendieck–Serre formula, proved in Section 2, allows us to express the difference of the Bhattacharya function and Bhattacharya polynomial of two $\mathfrak{m}$-primary ideals I and J in terms of lengths of bigraded components of local cohomology modules of the extended Rees algebra of I and J. This is done in Section 5 of the paper.

In Section 3 we prove some preliminary results about Ratliff–Rush closure of products of ideals. In Section 4 we present a variation on a complex first defined by Kirby and Mehran in [KM]. The cohomology of this complex is related to the local cohomology of Rees algebras of two ideals. An analysis of this relationship yields a formula for the constant term of the Bhattacharya polynomial $P(r,s)$. This formula is used to prove the characterization of Cohen–Macaulay property of bigraded Rees algebras mentioned above.

2. Grothendieck–Serre difference formula for bigraded algebras

We begin by establishing the notation for bigraded algebras. A ring A is called a bigraded algebra if $A = \bigoplus_{r,s \in \mathbb{Z}} A_{(r,s)}$ where each $A_{(r,s)}$ is an additive subgroup of A such that $A_{(r,s)} \cdot A_{(l,m)} \subseteq A_{(r+l,s+m)}$ for all $(r,s), (l,m) \in \mathbb{Z}^2$. We say that A is a standard bigraded algebra if A is finitely generated, as an $A_{(0,0)}$-algebra, by elements of degree $(1,0)$ and $(0,1)$. The elements of $A_{(r,s)}$ are called bihomogeneous of degree (r,s). An ideal I of A is said to be bihomogeneous if I is generated by bihomogeneous elements. The ideal of A generated by elements of degree (r,s), where $r+s \geqslant 1$ is denoted by A_+ and the ideal generated by elements of degree (r,s), where $r,s \geqslant 1$ is denoted by A_{++}. An A-module M is called bigraded if $M = \bigoplus_{r,s\in\mathbb{Z}} M_{(r,s)}$, where $M_{(r,s)}$ are additive subgroups of M satisfying $A_{(r,s)} \cdot M_{(l,m)} \subseteq M_{(r+l,s+m)}$ for all $r,s,l,m \in \mathbb{Z}$. It is known that when $A_{(0,0)}$ is Artinian and M is a finitely generated bigraded A-module, the function $\lambda_{A_{(0,0)}}(M_{(r,s)})$, called Hilbert function of M, is finite for all r,s and coincides with a polynomial for $r,s \gg 0$. In this section we express the difference between the Hilbert function and the Hilbert polynomial in terms of the Euler characteristic of local cohomology modules. For an ideal I in A and an A-module M, let $H_I^i(M)$ denote the ith local cohomology module of M with respect to I. We refer the reader to [BS] for properties of local cohomology modules. Note that when I is a bihomogeneous ideal in a bigraded algebra A and M is a bigraded A-module, the local cohomology modules $H_I^i(M)$ have a natural bigraded structure inherited from A and M.

Throughout this section $(A,\mathfrak{m})$ will denote a d-dimensional Noetherian local ring unless stated otherwise. Let $X = (X_1,\ldots,X_m)$ and $Y = (Y_1,\ldots,Y_n)$ be two sets of indeterminates. Let $R = A[X_1,\ldots,X_m,Y_1,\ldots,Y_n]$. We assign the grading $\deg X_i = (1,0)$ for $i = 1,\ldots,m$ and $\deg Y_i = (0,1)$ for $i = 1,\ldots,n$ so that R is a standard bigraded algebra. We write $R_{(r,s)}$ for the A-module generated by products of monomials of degree r in X and degree s in Y. In the next lemma we establish finite generation over A of the bigraded components of the local cohomology modules of R with respect to X and Y, respectively.

The results in this section are not new. They are folklore in the multigraded case. Lemma 2.1 follows from [CHT, Lemma 2.2 and Corollary 2.3] when A is a field. Theorem 2.3 and Theorem 2.4 follow from [KT, Lemma 4.2 and Lemma 4.3]. We refer the reader to [O2, Lemma 2.1], [Sn, Theorem 9.1] and [K, Section 1].

Although the results in the section are not new, we have provided easy proofs so that these results are accessible to readers not familiar with sheaf cohomology.

Lemma 2.1. *Let $R = A[X_1,\ldots,X_m,Y_1,\ldots,Y_n]$. Then*

(i) $H_X^i(R) = 0$ *for all* $i \neq m$ *and* $H_Y^i(R) = 0$ *for all* $i \neq n$.
(ii) $H_X^m(R)_{(r,s)} = 0$ *for all* $r > -m$ *and,* $H_Y^n(R)_{(r,s)} = 0$ *for all* $s > -n$.
(iii) $H_X^m(R)_{(r,s)}$ *and* $H_Y^n(R)_{(r,s)}$ *are finitely generated* A*-modules for all* $r,s \in \mathbb{Z}$.

Proof. (i) is standard.

(ii) Induct on m. Let $m = 0$. Then $H_{(0)}^0(R) = R = A[Y]$. Therefore, $H_{(0)}^0(R)_{(r,s)} = 0$ for all $r > 0$. Suppose $m > 0$. Let $\overline{R} = R/X_mR$ and $(\overline{X}) =$

$(X_1, \ldots, X_{m-1})$. Consider the short exact sequence

$$0 \longrightarrow R(-1,0) \xrightarrow{.X_m} R \longrightarrow \overline{R} \longrightarrow 0.$$

By the change of ring principle, $H^i_{(X)}(\overline{R}) = H^i_{(\overline{X})}(\overline{R})$. Since $(\overline{X})$ is generated by $m-1$ indeterminates, $H^i_{(\overline{X})}(\overline{R}) = 0$ for all $i \neq m-1$. Therefore we get the following long exact sequence:

$$0 \longrightarrow H^{m-1}_{(\overline{X})}(\overline{R}) \longrightarrow H^m_{(X)}(R)(-1,0) \xrightarrow{.X_m} H^m_{(X)}(R) \longrightarrow 0. \tag{1}$$

By induction hypothesis, for all $r > -m+1$, $H^{m-1}_{(\overline{X})}(\overline{R})_{(r,s)} = 0$. Hence for $r > -m+1$ we get an exact sequence

$$0 \longrightarrow H^m_{(X)}(R)_{(r-1,s)} \xrightarrow{.X_m} H^m_{(X)}(R)_{(r,s)} \longrightarrow 0.$$

Let $z \in H^m_{(X)}(R)_{(r-1,s)}$. Pick the smallest $l \geqslant 1$, such that $X^l_m z = 0$. Then $X_m(zX^{l-1}_m) = 0$. Therefore $z = 0$. Hence $H^m_{(X)}(R)_{(r,s)} = 0$ for all $r > -m$. Similarly one can can show that $H^n_Y(R)_{(r,s)} = 0$ for all $s > -n$.

(iii) We need to show that $H^m_{(X)}(R)_{(r,s)}$ is finitely generated for all $r \leqslant -m$. Apply induction on m. It is clear for $m = 0$. Assume the statement for $m-1$. Now apply decreasing induction on r. When $r = -m+1$, $H^{m-1}_{(\overline{X})}(\overline{R})_{(-m+1,s)} \cong H^m_{(X)}(R)_{(-m,s)}$, by (1) and (ii). By induction hypothesis on m, $H^{m-1}_{(\overline{X})}(\overline{R})_{(-m+1,s)}$ is finitely generated; hence so is $H^m_{(X)}(R)_{(-m,s)}$. Now for $r < -m+1$ we have the short exact sequence

$$0 \longrightarrow H^{m-1}_{(\overline{X})}(\overline{R})_{(r,s)} \longrightarrow H^m_{(X)}(R)_{(r-1,s)} \xrightarrow{.X_m} H^m_{(X)}(R)_{(r,s)} \longrightarrow 0.$$

By induction on r, $H^m_{(X)}(R)_{(r,s)}$ is finitely generated and $H^{m-1}_{(\overline{X})}(\overline{R})_{(r,s)}$ is finitely generated by induction on m. Therefore $H^m_{(X)}(R)_{(r-1,s)}$ is finitely generated. Similarly $H^n_Y(R)_{(r,s)}$ is finitely generated for all $r, s \in \mathbb{Z}$. □

Lemma 2.2.

(i) $H^i_{R_{++}}(R) = 0$ *for all* $i \neq m, n$ *and* $m+n-1$.
(ii) $H^i_{R_{++}}(R)_{(r,s)} = 0$ *for* $r, s \gg 0$ *and* $i \geqslant 0$.
(iii) $H^i_{R_{++}}(R)_{(r,s)}$ *is a finitely generated* A*-module for all* $i \geqslant 0$ *and* $r, s \in \mathbb{Z}$.

Proof. First note that $R_{++} = X \cap Y$ with $X = (X_1, \ldots, X_m)$, $Y = (Y_1, \ldots, Y_n)$. Set $R_+ = X + Y$ and consider the Mayer–Vietoris sequence:

$$\cdots \longrightarrow H^i_{R_+}(R) \longrightarrow H^i_X(R) \oplus H^i_Y(R) \longrightarrow H^i_{R_{++}}(R) \longrightarrow H^{i+1}_{R_+}(R) \longrightarrow \cdots \tag{2}$$

(i) If $i \neq m, n, m+n-1$, $H^i_X(R) = H^i_Y(R) = H^{i+1}_{R_+}(R) = 0$. Hence $H^i_{R_{++}}(R) = 0$ for $i \neq m, n, m+n-1$.

By [Bl, Theorem 2.2.4] and Lemma 2.1, (ii) and (iii) are satisfied by $H^i_X(R)$, $H^i_Y(R)$, and $H^{i+1}_{R_+}(R)$. Hence (ii) and (iii) are satisfied by $H^i_{R_{++}}(R)$. □

Theorem 2.3. *Let $R = \bigoplus_{r,s\geqslant 0} R_{(r,s)}$ be a finitely generated standard bigraded algebra over a Noetherian local ring $R_{00} = (A, \mathfrak{m})$. Let M be a finitely generated bigraded R-module. Then*

(i) $H^i_{R_{++}}(M)_{(r,s)} = 0$ *for all $r, s \gg 0$ and $i \geqslant 0$.*
(ii) $H^i_{R_{++}}(M)_{(r,s)}$ *is a finitely generated A-module for all $r, s \in \mathbb{Z}$ and $i \geqslant 0$.*

Proof. As R is standard bigraded $R \cong A[X_1, \dots, X_m, Y_1, \dots, Y_n]/I$ for a bihomogeneous ideal I. Consider M as a bigraded $S = A[X_1, \dots, X_m, Y_1, \dots, Y_n]$-module. Then by the change of ring principle $H^i_{R_{++}}(M) = H^i_{S_{++}}(M)$ for all $i \geqslant 0$. Therefore, without loss of generality, we may assume that $R = A[X_1, \dots, X_m, Y_1, \dots, Y_n]$. Since M is a finitely generated bigraded R-module, there exists a free R-module $F = \bigoplus_{j=1}^{s} R(m_j)$, $m_j \in \mathbb{Z}^2$, and a short exact sequence of finitely generated bigraded R-modules

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0.$$

Consider the corresponding long exact sequence of local cohomology modules

$$\cdots \longrightarrow H^i_{R_{++}}(K) \longrightarrow H^i_{R_{++}}(F) \longrightarrow H^i_{R_{++}}(M) \longrightarrow H^{i+1}_{R_{++}}(K) \longrightarrow \cdots$$

By Lemma 2.2, (i) and (ii) are true for $H^i_{R_{++}}(F)$. We prove the theorem by decreasing induction on i. Since $H^i_{R_{++}}(M) = 0$ for $i \gg 0$, (i) and (ii) obviously hold for $i \gg 0$. By induction $H^{i+1}_{R_{++}}(K)$ has properties (i) and (ii). Hence $H^i_{R_{++}}(M)$ satisfies (i) and (ii). □

Theorem 2.4. *Let $R = \bigoplus_{r,s\geqslant 0} R_{(r,s)}$ be a finitely generated standard bigraded algebra with $R_{00} = (A, \mathfrak{m})$, an Artinian local ring and let $M = \bigoplus_{r,s\geqslant 0} M_{(r,s)}$ be a bigraded finite R-module. Put $B_M(r,s) = \lambda_A(M_{(r,s)})$. Let $P_M(r,s)$ denote the Hilbert polynomial corresponding to the function $B_M(r,s)$. Then for all $r, s \in \mathbb{Z}$,*

$$B_M(r,s) - P_M(r,s) = \sum_{i\geqslant 0} (-1)^i \lambda_A\big(H^i_{R_{++}}(M)_{(r,s)}\big).$$

Proof. Write $R = A[x_1, \dots, x_m, y_1, \dots, y_n]$ with $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$. We prove the theorem by induction on $m + n$. Suppose $m + n = 0$. Then $M_{(r,s)} = 0$ for $r, s \gg 0$. Hence $P_M(r,s) = 0$. Since $\dim M = 0$, we have $H^i_{R_{++}}(M) = 0$ for all $i > 0$ and $H^0_{R_{++}}(M) = M$. Therefore $B_M(r,s) = \lambda_A(H^0_{R_{++}}(M)_{(r,s)})$.

Now suppose $m + n > 0$. If $m = 0$ or $n = 0$, the result reduces to Theorem 2.2.2 of [Bl]. Let $m > 0$ and $n > 0$. Consider the exact sequence of finitely generated bigraded R-modules

$$0 \longrightarrow K \longrightarrow M(-1,0) \xrightarrow{.x_m} M \longrightarrow C \longrightarrow 0. \tag{3}$$

For any finitely generated bigraded R-module N, define

$$\chi_N(r,s) = \sum_{i \geqslant 0} (-1)^i \lambda_A \big(H^i_{R_{++}}(N)_{(r,s)} \big) \quad \text{and} \quad f_N(r,s) = B_N(r,s) - P_N(r,s).$$

Since $H^i_{R_{++}}(N(-\mu,0))_{(r,s)} = H^i_{R_{++}}(N)_{(r-\mu,s)}$, it follows that $\chi_{N(-\mu,0)}(r,s) = \chi_N(r-\mu,s)$. Thus from (3), we get

$$\chi_M(r-1,s) - \chi_M(r,s) = \chi_K(r,s) - \chi_C(r,s)$$

and

$$f_M(r-1,s) - f_M(r,s) = f_K(r,s) - f_C(r,s)$$

for all $r,s \in \mathbb{Z}$. Let $\overline{R} = R/x_m R \cong A[\bar{x}_1, \ldots, \bar{x}_{m-1}, \bar{y}_1, \ldots, \bar{y}_n]$. Since $x_m K = 0 = x_m C$, we can consider K and C as $\overline{R}$-modules. By the change of ring principle,

$$H^i_{R_{++}}(K) \cong H^i_{\overline{R}_{++}}(K) \quad \text{and} \quad H^i_{R_{++}}(C) \cong H^i_{\overline{R}_{++}}(C)$$

for all $i \geqslant 0$. By induction $f_K(r,s) = \chi_K(r,s)$ and $f_C(r,s) = \chi_C(r,s)$. Therefore we have $\chi_M(r,s) - \chi_M(r-1,s) = f_M(r,s) - f_M(r-1,s)$ for all $(r,s) \in \mathbb{Z}^2$. Consider the exact sequence (3) with the map, multiplication by y_n. Proceeding as in the above case we get that $\chi_M(r,s) - \chi_M(r,s-1) = f_M(r,s) - f_M(r,s-1)$. By Theorem 2.3, $\chi_M(r,s) = 0$ for $r,s \gg 0$ and clearly $f_M(r,s) = 0$ for $r,s \gg 0$. Set $h = \chi_M - f_M$; then $h(r,s) = 0$ for all $r,s \gg 0$ and we have $h(r,s) = h(r-1,s)$, $h(r,s) = h(r,s-1)$ for all r,s. Therefore $h = 0$ and

$$B_M(r,s) - P_M(r,s) = \sum_{i \geqslant 0} (-1)^i \lambda_A \big(H^i_{R_{++}}(M)_{(r,s)} \big). \qquad \square$$

3. Ratliff–Rush closure of products of ideals

Let A be a commutative ring and $K \subset I$ be ideals of A. We say that K is a *reduction* of I if there exists an integer $r \geqslant 1$ such that $I^{r+1} = K I^r$. The smallest integer r satisfying this equation is called the reduction number, $r_K(I)$, of I with respect to K. We say that K is a minimal reduction of I if K is minimal with respect to inclusion among all reductions of I. We refer the reader to [NR] for basic facts about reductions of ideals.

Let $(A, \mathfrak{m})$ be a local ring and I be an ideal of A. The stable value of the sequence $\{I^{n+1} : I^n\}$ is called the Ratliff–Rush closure of I, denoted by $\tilde{I}$. An ideal I is said to be Ratliff–Rush if $\tilde{I} = I$. In this section we discuss the concept of the Ratliff–Rush closure for the product of two ideals.

The following proposition summarizes some basic properties of Ratliff–Rush closure found in [RR].

Proposition 3.1. *Let I be an ideal containing a regular element in a Noetherian ring A. Then*

1. $I \subseteq \tilde{I}$ *and* $\widetilde{(\tilde{I})} = \tilde{I}$.
2. $(\tilde{I})^n = I^n$ *for* $n \gg 0$. *Hence if I is $\mathfrak{m}$-primary, the Hilbert polynomial of I and $\tilde{I}$ are same.*
3. $(\widetilde{I^n}) = I^n$ *for* $n \gg 0$.
4. *If $(x_1, \ldots, x_g)$ is a minimal reduction of I, then* $\tilde{I} = \bigcup_{n \geqslant 0} I^{n+1} : (x_1^n, \ldots, x_g^n)$.

We show that the Ratliff–Rush closure for product of two ideals can be computed from complete reductions, a generalization of reductions of ideals introduced by Rees in [R2].

Let $(A, \mathfrak{m})$ be a d-dimensional local ring. Let $I_1, \ldots, I_r$ be $\mathfrak{m}$-primary ideals of $(A, \mathfrak{m})$. Let (x_{ij}) with $x_{ij} \in I_i$, for all $j = 1, \ldots, d$ and $i = 1, \ldots, r$, be a system of elements in A. Put $y_j = x_{1j} x_{2j} \ldots x_{rj}$, $j = 1, \ldots, d$. Then the system of elements (x_{ij}) is said to be a *complete reduction* of the sequence of ideals $I_1, \ldots, I_r$ if $(y_1, \ldots, y_d)$ is a reduction of $I_1 \ldots I_r$. In [R2] Rees proved the existence of complete reductions when the residue field of A is infinite.

Lemma 3.2. *Let I and J be ideals of A. Then we have*

(i) $\widetilde{IJ} = \bigcup_{r,s \geqslant 0} I^{r+1} J^{s+1} : I^r J^s$.

(ii) $(\widetilde{I^a J^b}) = \bigcup_{k \geqslant 0} I^{a+k} J^{b+k} : I^k J^k$.

(iii) *If I and J are $\mathfrak{m}$-primary ideals with a minimal reduction $(y_1, \ldots, y_d)$ of IJ obtained from a complete reduction of I and J, then*

$$\left(\widetilde{I^a J^b}\right) = \bigcup_{k \geqslant 0} I^{a+k} J^{b+k} : \left(y_1^k, \ldots, y_d^k\right).$$

Proof. (i) Let $x \in \widetilde{IJ}$, then $x I^n J^n \subseteq I^{n+1} J^{n+1}$ for some n. Conversely if $x I^r J^s \subseteq I^{r+1} J^{s+1}$ for some $r, s \geqslant 0$ then for $n = \max\{r, s\}$, $x I^n J^n \subseteq I^{n+1} J^{n+1}$ so that $x \in (\widetilde{IJ})$.

(ii) By (i), $(\widetilde{I^a J^b}) = \bigcup_{r,s \geqslant 0} I^{ar+a} J^{bs+b} : I^{ar} J^{bs}$. Let $z \in (\widetilde{I^a J^b})$ then for some r, s we have $z I^{ar} J^{bs} \subseteq I^{ar+a} J^{bs+b}$. Set $k = \max\{ar, bs\}$. Then $z I^k J^k \subseteq I^{a+k} J^{k+b}$ and hence $z \in I^{a+k} J^{b+k} : I^k J^k$. Let $z I^k J^k \subseteq I^{a+k} J^{b+k}$ for some k. We may assume that $k = nab$ for $n \gg 0$. Therefore $z \in I^{nab+a} J^{nab+b} : I^{nab} J^{nab} \subseteq (\widetilde{I^a J^b})$.

(iii) Suppose $z \in (\widetilde{I^a J^b})$. Then for some k, $z I^k J^k \subseteq I^{a+k} J^{b+k}$, by (ii). Since $(y_1^k, \ldots, y_d^k) \subseteq I^k J^k$, we have $z(y_1^k, \ldots, y_d^k) \subseteq I^{a+k} J^{b+k}$. Let $z y_i^k \in I^{a+k} J^{b+k}$

for $i = 1, \ldots, d$. Let $(\underline{y})$ denote the ideal $(y_1, \ldots, y_d)$. Then $(IJ)^{m+n} = (\underline{y})^m (IJ)^n$ for all $m \geqslant 0$ and $n \geqslant r_0 = r_{(\underline{y})}(IJ)$. Hence $(IJ)^{r+dk} = (\underline{y})^{dk} I^r J^r$ for $r \geqslant r_0$. Therefore,

$$zI^{r+dk} J^{r+dk} = z(\underline{y})^{dk} I^r J^r = \sum_{\sum i_j = dk} z y_1^{i_1} \cdots y_d^{i_d} I^r J^r \subseteq I^{a+dk} J^{b+dk} I^r J^r.$$

Hence $z \in (\widetilde{I^a J^b})$, by (ii). □

Lemma 3.3. *Let I, J be ideals in a Noetherian ring A, M a finite A-module and K an ideal of A generated by M-regular elements. Then there exist $t_1, t_2 > 0$ such that $I^r J^s M :_M K = I^{r-t_1} J^{s-t_2} (I^{t_1} J^{t_2} M :_M K)$ for all $r \geqslant t_1, s \geqslant t_2$.*

Proof. We follow the line of argument in [M, Proposition 11.E]. Let $K = (a_1, a_2, \ldots, a_n)$ where a_i are M-regular. Let S be the multiplicatively closed subset generated by $a_1, \ldots, a_n$. For $j = 1, \ldots, n$ consider the A-submodule $M_j = a_j^{-1} M$ of $S^{-1} M$ and set $L = M_1 \oplus M_2 \oplus \cdots \oplus M_n$. Let Δ_M be the image of the diagonal map $x \mapsto (\frac{x}{1}, \ldots, \frac{x}{1})$ from M to L. Since a_i's are regular $\Delta_M \cong M$. Then,

$$I^r J^s M :_M K = \bigcap_j \bigl(I^r J^s M :_M a_j\bigr) = \bigcap_j \bigl(I^r J^s M_j \cap M\bigr) \cong I^r J^s L \cap \Delta_M.$$

Since L is a finite A-module and Δ_M is a submodule of L, we can apply the generalized Artin–Rees Lemma to get $t_1, t_2 > 0$ such that

$$I^r J^s L \cap \Delta_M = I^{r-t_1} J^{s-t_2} \bigl(I^{t_1} J^{t_2} L \cap \Delta_M\bigr) \quad \text{for all } r \geqslant t_1,\ s \geqslant t_2.$$

Hence

$$I^r J^s M : K = I^{r-t_1} J^{s-t_2} \bigl(I^{t_1} J^{t_2} M : K\bigr) \quad \text{for all } r \geqslant t_1,\ s \geqslant t_2. \qquad \square$$

Lemma 3.4. *Suppose IJ has a reduction generated by regular elements, then for $r, s \gg 0$, $(\widetilde{I^r J^s}) = I^r J^s$.*

Proof. We first show that $I^{r+1} J^{s+1} : IJ = I^r J^s$ for $r, s \gg 0$. Let $(\underline{\boldsymbol{x}}) = (x_1, \ldots, x_g)$ be a reduction of IJ generated by regular elements. Then, $I^n J^n = (\underline{\boldsymbol{x}}) I^{n-1} J^{n-1}$ for $n \gg 0$ and hence $I^{r+1} J^{s+1} = (\underline{\boldsymbol{x}}) I^r J^s$ for $r, s \gg 0$. By setting $M = A$ and $K = (\underline{\boldsymbol{x}})$ in the Lemma 3.3, we get $t_1, t_2 > 0$ such that $I^{r+1} J^{s+1} : (\underline{\boldsymbol{x}}) = I^{r+1-t_1} J^{s+1-t_2} (I^{t_1} J^{t_2} : (\underline{\boldsymbol{x}}))$. Choose r and s large enough so that $r - t_1, s - t_2 \geqslant r_{(\underline{\boldsymbol{x}})}(IJ)$. Then we have

$$\begin{aligned} I^{r+1} J^{s+1} : IJ &\subseteq I^{r+1} J^{s+1} : (\underline{\boldsymbol{x}}) = I^{r+1-t_1} J^{s+1-t_2} \bigl(I^{t_1} J^{t_2} : (\underline{\boldsymbol{x}})\bigr) \\ &= (\underline{\boldsymbol{x}}) I^{r-t_1} J^{s-t_2} \bigl(I^{t_1} J^{t_2} : (\underline{\boldsymbol{x}})\bigr) \subseteq I^r J^s. \end{aligned}$$

Therefore $I^{r+1} J^{s+1} : IJ = I^r J^s$ $\forall r, s \gg 0$. We claim that for all $k \geqslant 1$ and $r, s \gg 0$

$$I^{r+k} J^{s+k} : I^k J^k = I^r J^s.$$

Apply induction on k. The $k=1$ case has just been proved. Let $k>1$. Assume the result for $k-1$. Then

$$I^{r+k}J^{s+k}:I^kJ^k=\left(I^{r+k}J^{s+k}:I^{k-1}J^{k-1}\right):IJ=I^{r+1}J^{s+1}:IJ=I^rJ^s. \qquad \square$$

4. A generalization of the Kirby–Mehran complex

In this section we construct a bigraded analogue of a complex first constructed by Kirby and Mehran in [KM]. We study the cohomology modules of this complex and relate them to those of the bigraded Rees algebras of two ideals. Let $(A,\mathfrak{m})$ be a d-dimensional Noetherian local ring with infinite residue field and I, J be $\mathfrak{m}$-primary ideals of A. Let $\mathcal{R}$ and $\mathcal{R}^*$ be respectively the Rees and the extended Rees algebra of A with respect to I and J. Let $y_1,\ldots,y_n\in IJ$. For $k\geqslant 1$ set $(\underline{y})^{[k]}=(y_1^k,\ldots,y_n^k)$ and $(\underline{yt})^{[k]}=((y_1t_1t_2)^k,\ldots,(y_nt_1t_2)^k)$. Consider the Koszul complex $K^{\cdot}((\underline{yt})^{[k]};\mathcal{R})$:

$$0\to\mathcal{R}\to\mathcal{R}(k,k)^{\binom{n}{1}}\to\cdots\to\mathcal{R}\big((n-1)k,(n-1)k\big)^{\binom{n}{n-1}}\to\mathcal{R}(nk,nk)\to 0.$$

This complex has a natural bigraded structure inherited from $\mathcal{R}$. Write the (r,s)th graded component, $K^{\cdot}_{(r,s)}((\underline{yt})^{[k]};\mathcal{R})$, of this complex:

$$0\to(It_1)^r(Jt_2)^s\to(It_1)^{r+k}(Jt_2)^{s+k\binom{n}{1}}\to\cdots\to(It_1)^{r+nk}(Jt_2)^{s+nk}\to 0.$$

This complex can be considered as a subcomplex of the Koszul complex:

$$K^{\cdot}\big((\underline{y})^{[k]};A\big):\quad 0\longrightarrow A\longrightarrow A^{\binom{n}{1}}\longrightarrow\cdots\longrightarrow A^{\binom{n}{n-1}}\longrightarrow A\longrightarrow 0.$$

Therefore there is map of complexes $0\longrightarrow K^{\cdot}_{(r,s)}((\underline{yt})^{[k]};\mathcal{R})\longrightarrow K^{\cdot}((\underline{y})^{[k]};A)$. Since this inclusion is a chain map, there exists a quotient complex.

Definition 4.1. For $k\geqslant 1$, $r,s\in\mathbb{Z}$, and $n\geqslant 1$ we define the complex $C^{\cdot}(n,k,r,s)$ to be the quotient of the complex $K^{\cdot}((\underline{y})^{[k]};A)$ by the complex $K^{\cdot}_{(r,s)}((\underline{yt})^{[k]};\mathcal{R})$.

We have the short exact sequence

$$0\longrightarrow K^{\cdot}_{(r,s)}\big((\underline{yt})^{[k]};\mathcal{R}\big)\longrightarrow K^{\cdot}\big((\underline{y})^{[k]};A\big)\longrightarrow C^{\cdot}(n,k,r,s)\longrightarrow 0, \qquad (4)$$

One can easily see that $C^{\cdot}(n,k,r,s)$ is the complex

$$0\to A/I^rJ^s\xrightarrow{d_C^0}\big(A/I^{r+k}J^{s+k}\big)^{\binom{n}{1}}\xrightarrow{d_C^1}\cdots\xrightarrow{d_C^{n-1}}\big(A/I^{r+nk}J^{s+nk}\big)\xrightarrow{d_C^n}0.$$

where the differentials are induced by those of Koszul complex $K^{\cdot}(y_1^k,\ldots,y_n^k;A)$. We compute some of the cohomology modules of this complex in the following proposition.

Proposition 4.2. *For all $k \geqslant 1$, $r, s \in \mathbb{Z}$ we have*

(i) $H^0(C^{\cdot}(n,k,r,s)) = I^{r+k}J^{s+k} : (\underline{y}^{[k]})/I^rJ^s$.
(ii) $H^n(C^{\cdot}(n,k,r,s)) = A/(I^{r+k}J^{s+k} + (\underline{y}^{[k]}))$.
(iii) *If $y_1, \ldots, y_n$ is an A-sequence, then*

$$H^{n-1}\big(C^{\cdot}(n,k,r,s)\big) \cong \frac{(\underline{y}^{[k]}) \cap I^{r+nk}J^{s+nk}}{(\underline{y}^{[k]})I^{r+(n-1)k}J^{s+(n-1)k}}.$$

Proof.

$$\text{(i) } H^0(C^{\cdot}(n,k,r,s)) = \ker d_C^0 = \left\{\bar{u} \in A/I^rJ^s \;\middle|\; y_i^k u \in I^{r+k}J^{s+k} \text{ for each } i = 1, \ldots, n\right\} = \frac{I^{r+k}J^{s+k} : (\underline{y})^{[k]}}{I^rJ^s}.$$

$$\text{(ii) } H^n(C^{\cdot}(n,k,r,s)) = \frac{\ker d_C^n}{\operatorname{im} d_C^{n-1}} = \frac{A/I^{r+nk}J^{s+nk}}{(\underline{y})^{[k]} + I^{r+nk}J^{s+nk}/I^{r+nk}J^{s+nk}} \cong \frac{A}{(\underline{y})^{[k]} + I^{r+nk}J^{s+nk}}.$$

(iii) Suppose that $y_1, \ldots, y_n$ is an A-sequence. Consider the Koszul complex

$$K^{\cdot}\big((\underline{y})^{[k]}, A\big): \quad \cdots \longrightarrow A^{\binom{n}{n-2}} \xrightarrow{d_K^{n-2}} A^{\binom{n}{n-1}} \xrightarrow{d_K^{n-1}} \big(y_1^k, \ldots, y_n^k\big) \longrightarrow 0.$$

Since $(y_1^k, \ldots, y_n^k)$ is an A-sequence, this is an exact sequence. Tensoring by $A/I^{r+(n-1)k}J^{s+(n-1)k}$, we get an exact sequence

$$\left(\frac{A}{I^{r+(n-1)k}J^{s+(n-1)k}}\right)^{\binom{n}{n-2}} \xrightarrow{\bar{d}_K^{n-2}} \left(\frac{A}{I^{r+(n-1)k}J^{s+(n-1)k}}\right)^{\binom{n}{n-1}} \xrightarrow{\bar{d}_K^{n-1}} \frac{(\underline{y})^{[k]}}{(\underline{y})^{[k]}I^{r+(n-1)k}J^{s+(n-1)k}} \longrightarrow 0.$$

We have $\operatorname{im} \bar{d}_K^{n-2} = \operatorname{im} d_C^{n-2}$ and a commutative diagram of exact rows

$$\begin{array}{ccccccccc}
0 \longrightarrow & \operatorname{im} \bar{d}_K^{n-2} & \longrightarrow & \left(\frac{A}{I^{r+(n-1)k}J^{s+(n-1)k}}\right)^n & \longrightarrow & \frac{(y_1^k,\ldots,y_n^k)}{(y_1^k,\ldots,y_n^k)I^{r+(n-1)k}J^{s+(n-1)k}} & \longrightarrow 0 \\
& \downarrow \alpha & & \downarrow \mathrm{id} & & \downarrow \gamma & \\
0 \longrightarrow & \ker d_C^{n-1} & \longrightarrow & \left(\frac{A}{I^{r+(n-1)k}J^{s+(n-1)k}}\right)^n & \longrightarrow & \frac{A}{I^{r+nk}J^{s+nk}} &
\end{array}$$

where α is the inclusion map and γ is the natural map. By the Snake lemma, we get

$$\begin{aligned} H^{n-1}\big(C^{\cdot}(n,k,r,s)\big) &\cong \operatorname{coker}\alpha \cong \ker\gamma \\ &\cong \frac{\big(y_1^k,\ldots,y_n^k\big)\cap I^{r+nk}J^{s+nk}}{\big(y_1^k,\ldots,y_n^k\big)I^{r+(n-1)k}J^{s+(n-1)k}}. \qquad \square \end{aligned}$$

For the rest of the section let I and J be $\mathfrak{m}$-primary ideals of A. Let $x_{1j}\in I$ and $x_{2j}\in J$ for $j=1,\ldots,d$ and for $i=1,2,\ldots,d$, set $y_i=x_{1i}x_{2i}$.

Proposition 4.3. *Let $r,s\in\mathbb{Z}$.*

(i) *For all $k\geqslant 1$, there is an exact sequence of A-modules*

$$\begin{aligned} 0 &\to H^0\big((\underline{yt})^{[k]};\mathcal{R}\big)_{(r,s)} \to H^0\big((\underline{y})^{[k]};A\big) \to H^0\big(C^{\cdot}(n,k,r,s)\big) \\ &\to H^1\big((\underline{yt})^{[k]};\mathcal{R}\big)_{(r,s)} \to \cdots. \end{aligned}$$

(ii) *There is an exact sequence of A-modules*

$$\begin{aligned} 0 &\to H^0_{(\underline{yt})}(\mathcal{R})_{(r,s)} \to H^0_{(\underline{y})}(A) \to \varinjlim_k H^0\big(C^{\cdot}(n,k,r,s)\big) \\ &\to H^1_{(\underline{yt})}(\mathcal{R})_{(r,s)} \to \cdots. \end{aligned}$$

Proof. (i) Follows from the long exact sequence of Koszul homology modules corresponding to (4).

(ii) For each i, consider the commutative diagram of complexes

$$\begin{array}{lccccccc} K^{\cdot}\big((y_it_1t_2)^k;\mathcal{R}\big):\ 0 & \longrightarrow & \mathcal{R} & \xrightarrow{(y_it_1t_2)^k} & \mathcal{R} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle y_it_1t_2} & & \\ K^{\cdot}\big((y_it_1t_2)^{k+1};\mathcal{R}\big):\ 0 & \longrightarrow & \mathcal{R} & \xrightarrow{(y_it_1t_2)^{k+1}} & \mathcal{R} & \longrightarrow & 0. \end{array}$$

This gives a map $\bigotimes_{i=1}^n K^{\cdot}((y_it_1t_2)^k;\mathcal{R}) \longrightarrow \bigotimes_{i=1}^n K^{\cdot}((y_it_1t_2)^{k+1};\mathcal{R})$, i.e., we get a map

$$K^{\cdot}\big((\underline{yt})^{[k]};\mathcal{R}\big) \longrightarrow K^{\cdot}\big((\underline{yt})^{[k+1]};\mathcal{R}\big)$$

and its restriction to the (r,s)-th component gives the map

$$K^{\cdot}_{(r,s)}\big((\underline{yt})^{[k]};\mathcal{R}\big) \longrightarrow K^{\cdot}_{(r,s)}\big((\underline{yt})^{[k+1]};\mathcal{R}\big).$$

Thus we obtain a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 \to & K_{(r,s)}\big((\underline{yt})^{[k]};\mathcal{R}\big) & \to & K^{\cdot}\big((\underline{y})^{[k]};A\big) & \to & C^{\cdot}(n,k,r,s) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & K_{(r,s)}\big((\underline{yt})^{[k+1]};\mathcal{R}\big) & \to & K^{\cdot}\big((\underline{y})^{[k+1]};A\big) & \to & C^{\cdot}(n,k+1,r,s) & \to 0. \end{array}$$

Apply $\varinjlim_k$ to the long exact sequence of the cohomology modules to get (ii). □

Corollary 4.4. *Let $(A, \mathfrak{m})$ be Cohen–Macaulay of dimension $d \geqslant 2$ and (x_{ij}), $i = 1, 2$, $1 \leqslant j \leqslant d$, be a complete reduction of (I, J). Let $r, s \in \mathbb{Z}$. Then*

(i) *For all $k \geqslant 0$, we have*

$$H^i\big((\underline{yt})^{[k]}; \mathcal{R}\big)_{(r,s)} \cong H^{i-1}\big(C^{\cdot}(d,k,r,s)\big) \quad \text{for all } 1 \leqslant i \leqslant d-1$$

and an exact sequence of A-modules

$$0 \to H^{d-1}\big(C^{\cdot}(d,k,r,s)\big) \to H^d\big((\underline{yt})^{[k]}; (\mathcal{R})\big)_{(r,s)} \to H^d\big((\underline{y})^{[k]}; A\big) \to H^d\big(C^{\cdot}(d,k,r,s)\big) \to 0.$$

(ii) *There is an isomorphism of A-modules*

$$H^i_{(\underline{yt})}(\mathcal{R})_{(r,s)} \cong \varinjlim_k H^{i-1}\big(C^{\cdot}(d,k,r,s)\big) \quad \text{for all } 1 \leqslant i \leqslant d-1$$

and an exact sequence

$$0 \longrightarrow \varinjlim_k \frac{(\underline{y})^{[k]} \cap I^{r+dk} J^{s+dk}}{(\underline{y})^{[k]} I^{r+(d-1)k} J^{s+(d-1)k}} \longrightarrow H^d_{(\underline{yt})}(\mathcal{R})_{(r,s)} \longrightarrow H^d_m(A) \longrightarrow \varinjlim_k \frac{A}{(\underline{y})^{[k]} + I^{r+dk} J^{s+dk}} \longrightarrow 0.$$

(iii) $H^1_{(\underline{yt})}(\mathcal{R})_{(r,s)} \cong \dfrac{(\widetilde{I^r J^s})}{I^r J^s}$.

Proof. (i) Consider the long exact sequence of cohomology modules corresponding to (4):

$$0 \longrightarrow H^0\big(K^{\cdot}\big((\underline{yt})^{[k]}; \mathcal{R}\big)\big) \longrightarrow H^0\big(K^{\cdot}\big((\underline{y})^{[k]}; A\big)\big) \longrightarrow H^0\big(C^{\cdot}(d,k,r,s)\big) \longrightarrow H^1\big(K^{\cdot}\big((\underline{yt})^{[k]}; \mathcal{R}\big)\big) \longrightarrow \cdots.$$

Since A is Cohen–Macaulay $H^i(K^{\cdot}((\underline{y})^{[k]}; A) = 0$ for all $0 \leqslant i \leqslant d-1$. Hence (i) follows.

(ii) Apply $\varinjlim_k$ to (i).

(iii) By (ii) and Lemma 3.2 we have

$$H^1_{(\underline{yt})}(\mathcal{R})_{(r,s)} \cong \varinjlim_k H^0\big(C^{\cdot}(d,k,r,s)\big) = \varinjlim_k \frac{I^{r+k} J^{s+k} : (\underline{y})^{[k]}}{I^r J^s} = \frac{(\widetilde{I^r J^s})}{I^r J^s}. \quad \square$$

A similar theory can be developed for the extended Rees algebra by setting $I^r = A = J^s$ if $r, s \leqslant 0$ and defining the complex $C^{\cdot}(n,k,r,s)^*$ in a similar way as we defined $C^{\cdot}(n,k,r,s)$. We can prove results similar to Propositions 4.2, 4.3, etc. First we prove a general result relating local cohomology modules of two bigraded algebras which will help us in relating the local cohomology modules of the Rees and the extended Rees algebras.

Proposition 4.5. *Let $R = \bigoplus_{r,s\geqslant 0} R_{(r,s)} \hookrightarrow \bigoplus_{r,s\in\mathbb{Z}} R_{(r,s)} = R^*$ be an inclusion of bigraded algebras over $R_{(0,0)}$, a Noetherian ring. Then*

(i) *For $i>1$, we have $H^i_{R_{++}}(R) \cong H^i_{R_{++}}(R^*)$.*
(ii) *We have an exact sequence*

$$0 \to H^0_{R_{++}}(R) \to H^0_{R_{++}}(R^*) \to R^*/R \to H^1_{R_{++}}(R) \to H^1_{R_{++}}(R^*) \to 0.$$

Proof. Consider the exact sequence of bigraded R-modules:

$$0 \longrightarrow R \longrightarrow R^* \longrightarrow R^*/R \longrightarrow 0. \tag{5}$$

Since R_{++} acts nilpotently on R^*/R,

$$H^0_{R_{++}}(R^*/R) = R^*/R \quad \text{and} \quad H^i_{R_{++}}(R^*/R) = 0 \quad \text{for all } i \neq 0.$$

The proposition follows from the long exact sequence of local cohomology modules derived from (5). □

Corollary 4.6. *Consider the bigraded rings $\mathcal{R} = A[It_1, Jt_2] \hookrightarrow \mathcal{R}^* = A[It_1, Jt_2, t_1^{-1}, t_2^{-1}]$ and $\mathcal{G} = \bigoplus_{r,s\geqslant 0} I^rJ^s/I^{r+1}J^{s+1} \hookrightarrow \mathcal{G}^* = \mathcal{R}^*/t_1^{-1}t_2^{-1}\mathcal{R}^*$. Then*

(i) *For all $i \geqslant 2$ we have the isomorphism $H^i_{\mathcal{R}_{++}}(\mathcal{R}) \cong H^i_{\mathcal{R}_{++}}(\mathcal{R}^*)$ and there is an exact sequence of bigraded $\mathcal{R}$-modules*

$$\begin{aligned} 0 &\longrightarrow H^0_{\mathcal{R}_{++}}(\mathcal{R}) \longrightarrow H^0_{\mathcal{R}_{++}}(\mathcal{R}^*) \longrightarrow \mathcal{R}^*/\mathcal{R} \longrightarrow H^1_{\mathcal{R}_{++}}(\mathcal{R}) \\ &\longrightarrow H^1_{\mathcal{R}_{++}}(\mathcal{R}^*) \longrightarrow 0. \end{aligned}$$

(ii) *For all $i \geqslant 2$ we have $H^i_{\mathcal{G}_{++}}(\mathcal{G}) \cong H^i_{\mathcal{G}_{++}}(\mathcal{G}^*)$ and there is an exact sequence of bigraded $\mathcal{G}$-modules*

$$\begin{aligned} 0 &\longrightarrow H^0_{\mathcal{G}_{++}}(\mathcal{G}) \longrightarrow H^0_{\mathcal{G}_{++}}(\mathcal{G}^*) \longrightarrow \mathcal{G}^*/\mathcal{G} \longrightarrow H^1_{\mathcal{G}_{++}}(\mathcal{G}) \\ &\longrightarrow H^1_{\mathcal{G}_{++}}(\mathcal{G}^*) \longrightarrow 0. \end{aligned}$$

Corollary 4.7. *For all $r,s \geqslant 0$,*

$$H^1_{\mathcal{R}_{++}}(\mathcal{R}^*)_{(r,s)} \cong \frac{(\widetilde{I^rJ^s})}{I^rJ^s}.$$

Proof. Use Corollaries 4.4(iii) and 4.6(i) to get the required result. □

5. The difference formula

In this section we obtain an expression for the difference of Bhattacharya polynomial and Bhattacharya function. The main motivation were results of Johnston–Verma [JV] and C. Blancafort [Bl] which express the difference of Hilbert–Samuel polynomial and Hilbert–Samuel function in terms of the Euler characteristic of the Rees algebra (respectively extended Rees algebra). We have followed Blancafort's elegant line of approach in the proof. However, we prove the theorem only for non-negative integers. The question remains still open for negative integers.

Theorem 5.1. *Let* $\mathcal{R}^* = A[It_1, Jt_2, t_1^{-1}, t_2^{-1}]$. *Then*

(i) $\lambda_A(H^i_{\mathcal{R}_{++}}(\mathcal{R}^*)_{(r,s)}) < \infty$ *for all* $r, s \in \mathbb{Z}$, $i = 0, 1, \ldots, d$.
(ii) $P(r,s) - B(r,s) = \sum_{i=0}^{d}(-1)^i \lambda_A(H^i_{\mathcal{R}_{++}}(\mathcal{R}^*)_{(r,s)})$ *for all* $r, s \geqslant 0$.

Proof. (i) By Theorem 2.3, $H^i_{\mathcal{R}_{++}}(\mathcal{R})_{(r,s)}$ are finitely generated A-modules and they vanish for $r, s \gg 0$. By Lemma 2.2 and Corollary 4.6, $H^i_{\mathcal{R}_{++}}(\mathcal{R}^*)_{(r,s)} = 0$ for all $r, s \gg 0$. We have an exact sequence of bigraded $\mathcal{R}$-modules:

$$0 \longrightarrow \mathcal{R}^*(1,1) \xrightarrow{t_1^{-1}t_2^{-1}} \mathcal{R}^* \longrightarrow \mathcal{G}^* \longrightarrow 0, \tag{6}$$

where $\mathcal{G}^* = \mathcal{R}^*/t_1^{-1}t_2^{-1}\mathcal{R}^*$. By the change of ring principle, $H^i_{\mathcal{R}_{++}}(\mathcal{G}^*) = H^i_{\mathcal{G}_{++}}(\mathcal{G}^*)$ for all $i \geqslant 0$. From the above short exact sequence we obtain the long exact sequence:

$$\begin{aligned} 0 &\longrightarrow H^0_{\mathcal{R}_{++}}(\mathcal{R}^*)_{(r+1,s+1)} \longrightarrow H^0_{\mathcal{R}_{++}}(\mathcal{R}^*)_{(r,s)} \longrightarrow H^0_{\mathcal{G}_{++}}(\mathcal{G}^*)_{(r,s)} \\ &\longrightarrow H^1_{\mathcal{R}_{++}}(\mathcal{R}^*)_{(r+1,s+1)} \longrightarrow \cdots. \end{aligned}$$

We prove (i) by decreasing induction on r and s. Since $H^i_{\mathcal{R}_{++}}(\mathcal{R}^*)_{(r,s)} = 0$ for all $r, s \gg 0$, the result is obviously true for $r, s \gg 0$. Consider the exact sequence

$$\cdots \longrightarrow H^i_{\mathcal{R}_{++}}(\mathcal{R}^*)_{(r+1,s+1)} \longrightarrow H^i_{\mathcal{R}_{++}}(\mathcal{R}^*)_{(r,s)} \longrightarrow H^i_{\mathcal{G}_{++}}(\mathcal{G}^*)_{(r,s)} \longrightarrow \cdots.$$

By induction $H^i_{\mathcal{R}_{++}}(\mathcal{R}^*)_{(r+1,s+1)}$ has finite length. By Theorem 2.3 and Corollary 4.6(ii), $H^i_{\mathcal{G}_{++}}(\mathcal{G}^*)_{(r,s)}$ is a finitely generated $\mathcal{G}_{00}$-module. Since $\mathcal{G}_{00}$ is Artinian, $H^i_{\mathcal{G}_{++}}(\mathcal{G}^*)_{(r,s)}$ has finite length. Therefore $H^i_{\mathcal{R}_{++}}(\mathcal{R}^*)_{(r,s)}$ has finite length.

(ii) For a bigraded module M over the bigraded ring $\mathcal{R}$, set

$$\chi_M(r,s) = \sum_{i \geqslant 0}(-1)^i \lambda_A\bigl(H^i_{\mathcal{R}_{++}}(M)_{(r,s)}\bigr) \quad \text{and} \quad g(r,s) = P(r,s) - B(r,s).$$

Then from the exact sequence (6) we get for all $r, s \geqslant 0$,

$$\begin{aligned}
&\chi_{\mathcal{R}^*(1,1)}(r,s) - \chi_{\mathcal{R}^*}(r,s)\\
&\quad = \chi_{\mathcal{R}^*}(r+1,s+1) - \chi_{\mathcal{R}^*}(r,s)\\
&\quad = -\chi_{\mathcal{G}^*}(r,s) = -\chi_{\mathcal{G}}(r,s) \quad \text{(by Corollary 4.6(ii))}\\
&\quad = P_{\mathcal{G}}(r,s) - H_{\mathcal{G}}(r,s) = P_{\mathcal{G}^*}(r,s) - H_{\mathcal{G}^*}(r,s)\\
&\quad = \big(P(r+1,s+1) - P(r,s)\big) - \big(B(r+1,s+1) - B(r,s)\big)\\
&\quad = g(r+1,s+1) - g(r,s).
\end{aligned}$$

Set $h(r,s) = \chi_{\mathcal{R}^*}(r,s) - g(r,s)$. Then $h(r,s) = h(r-1,s-1)$ for all $r,s \geqslant 0$ and $h(r,s) = 0$ for all $r,s \gg 0$. This clearly implies that $h(r,s) = 0$ for all $r,s \geqslant 0$. □

Corollary 5.2. *Let $(A,\mathfrak{m})$ be a 2-dimensional Cohen–Macaulay local ring and I, J be $\mathfrak{m}$-primary ideals of A. Then for all $r,s \geqslant 0$*

$$P(r,s) - B(r,s) = \lambda\big(H^2_{\mathcal{R}_{++}}(\mathcal{R})_{(r,s)}\big) - \lambda\big(\widetilde{I^rJ^s}/I^rJ^s\big).$$

In particular,

$$e_{00} = \lambda\big(H^2_{\mathcal{R}_{++}}(\mathcal{R})_{(0,0)}\big).$$

Proof. By the previous theorem,

$$P(r,s) - B(r,s) = \lambda\big(H^0_{\mathcal{R}_{++}}(\mathcal{R})_{(r,s)}\big) - \lambda\big(H^1_{\mathcal{R}_{++}}(\mathcal{R})_{(r,s)}\big) + \lambda\big(H^2_{\mathcal{R}_{++}}(\mathcal{R})_{(r,s)}\big).$$

Since I and J are $\mathfrak{m}$-primary, $\mathcal{R}_{++}$ contains a regular element. Therefore $H^0_{\mathcal{R}_{++}}(\mathcal{R}) = 0$. By Proposition 4.6,

$$H^1_{\mathcal{R}_{++}}(\mathcal{R})_{(r,s)} \cong \frac{\widetilde{I^rJ^s}}{I^rJ^s}.$$

Now,

$$e_{00} = P(0,0) - B(0,0) = \lambda\big(H^2_{\mathcal{R}_{++}}(\mathcal{R})_{(0,0)}\big). \qquad \square$$

6. Bigraded Cohen–Macaulay Rees algebras

In the previous section we have established a formula for the difference between the Bhattacharya function and Bhattacharya polynomial. It is interesting to know when is the Bhattacharya function equal to the Bhattacharya polynomial. Here we give a partial answer to this question, in dimension 2. Huneke [H, Theorem 2.1] and Ooishi [O1, Theorem 3.3] gave a characterization for the reduction number of an $\mathfrak{m}$-primary ideal to be at most 1 in terms of $e_0(I)$ and $e_1(I)$. Huckaba and Marley [HM, Corollaries 4.8, 4.10] generalized this result for higher reduction numbers. In particular, they characterized Cohen–Macaulay property of the Rees algebra in terms of the $e_1(I)$. It is natural to ask whether one can characterize the Cohen–Macaulay property of bigraded Rees algebras in terms of coefficients of the Bhattacharya polynomial. The Theorem 6.3 below answers this in dimension 2. A similar characterization for Cohen–Macaulayness of the multi-Rees algebras in higher dimension in terms of Bhattacharya coefficients is not known.

We need another generalization of reductions for two ideals, namely joint reductions. Let A be a commutative ring with identity and let $I_1, I_2, \ldots, I_g$ be ideals of A. A system of elements $(\underline{\boldsymbol{x}}) := (x_1, x_2, \ldots, x_g)$, where $x_i \in I_i$, is said to be a *joint reduction* of the sequence of ideals $(I_1, I_2, \ldots, I_g)$ if there exist positive integers $d_1, d_2, \ldots, d_g$ such that

$$x_1 I_1^{d_1-1} I_2^{d_2} \cdots I_g^{d_g} + \cdots + x_g I_1^{d_1} \cdots I_{g-1}^{d_{g-1}} I_g^{d_g-1} = I_1^{d_1} \cdots I_g^{d_g}.$$

We say that the sequence of ideals $(I_1, \ldots, I_g)$ has *joint reduction number zero* if

$$x_1 I_2 \cdots I_g + \cdots + x_g I_1 \cdots I_{g-1} = I_1 I_2 \cdots I_g.$$

We first prove a general property of the Bhattacharya coefficients.

Lemma 6.1. *Let $(A, \mathfrak{m})$ be a 1-dimensional Cohen–Macaulay local ring with infinite residue field. Let I and J be $\mathfrak{m}$-primary ideals of A. Then*

(i) $P(r+1, s) - H(r+1, s) \geqslant P(r, s) - H(r, s)$ *and* $P(r, s+1) - H(r, s+1) \geqslant P(r, s) - H(r, s)$.
(ii) $\lambda(A/I) \geqslant e_{10} + e_{00}$ *and* $\lambda(A/J) \geqslant e_{01} + e_{00}$.

Proof. Let $(x) \subseteq I$ be a reduction of I. Then

$$\begin{aligned} P(r+1, s) - H(r+1, s) &= e_{10}(r+1) + e_{01}s + e_{00} - \lambda\big(A/I^{r+1}J^s\big) \\ &= P(r, s) + e_{10} - \lambda\big(A/I^{r+1}J^s\big) \\ &\geqslant P(r, s) + \lambda\big(A/(x)\big) - \lambda\big(A/xI^rJ^s\big) \\ &= P(r, s) - \lambda\big((x)/xI^rJ^s\big) \\ &= P(r, s) - H(r, s). \end{aligned}$$

Similarly one can prove that $P(r, s+1) - H(r, s+1) \geqslant P(r, s) - H(r, s)$. From (i) it is clear that $P(r, s) - H(r, s) \leqslant 0$ for all r, s. Putting $(r, s) = (1, 0)$ and $(r, s) = (0, 1)$, we get (ii). □

Lemma 6.2. *Let $(A, \mathfrak{m})$ be a 2-dimensional Cohen–Macaulay local ring and I, J be $\mathfrak{m}$-primary ideals of A. Then $\lambda(A/I) \geqslant e_{10}$ and $\lambda(A/J) \geqslant e_{01}$.*

Proof. Let (x, y), where $x \in I$ and $y \in J$, be a joint reduction of (I, J). Choose the joint reduction such that x is superficial for I and J. Let $\overline{}$ denote "modulo x". Let $\overline{H}(r, s)$ and $\overline{P}(r, s)$ denote the Bhattacharya function and Bhattacharya polynomial of the $\overline{\mathfrak{m}}$-primary ideals $\bar{I}$ and $\bar{J}$ of $\overline{A} = A/(x)$.

Claim. $\overline{P}(r,s) = P(r,s) - P(r-1,s)$.

From the following exact sequence

$$0 \longrightarrow I^rJ^s : x/I^rJ^s \longrightarrow A/I^rJ^s \xrightarrow{x} A/I^rJ^s \longrightarrow A/\big(I^rJ^s, x\big) \longrightarrow 0,$$

$\lambda(I^rJ^s : x/I^rJ^s) = \lambda(A/(I^rJ^s, x))$. Then for all $r, s \gg 0$,

$$\begin{aligned}
\overline{P}(r,s) &= \lambda\big(A/\bar{I}^r\bar{J}^s\big) = \lambda\big(A/(I^rJ^s, x)\big) = \lambda\big(I^rJ^s : x/I^rJ^s\big) \\
&= \lambda\big(I^{r-1}J^s/I^rJ^s\big) \quad \text{(since } x \text{ is superficial for } I \text{ and } J\text{)} \\
&= P(r,s) - P(r-1,s).
\end{aligned}$$

Therefore

$$\begin{aligned}
\overline{P}(r,s) &= e_{20}\left[\binom{r}{2} - \binom{r-1}{2}\right] + e_{11}\big(r-(r-1)\big)s + e_{10}\big(r-(r-1)\big) \\
&= e_{20}(r-1) + e_{11}s + e_{10} = e_{20}r + e_{11}s + e_{10} - e_{20}.
\end{aligned}$$

Since $\dim \overline{A} = 1$, by Lemma 6.1, $\lambda(\overline{A}/\bar{I}) \geqslant e_{20} + (e_{10} - e_{20})$. Hence $\lambda(A/I) \geqslant e_{10}$. Similarly one can prove that $\lambda(A/J) \geqslant e_{01}$. □

Theorem 6.3. *Let $(A, \mathfrak{m})$ be a 2-dimensional Cohen–Macaulay local ring and I, J be $\mathfrak{m}$-primary ideals of A. Let $P(r,s) = \sum_{i+j\leqslant 2} e_{ij}\binom{r}{i}\binom{s}{j}$ be the Bhattacharya polynomial of I and J corresponding to the function $B(r,s) = \lambda(A/I^rJ^s)$. Then the following conditions are equivalent*:

(1) *$e_{10} = \lambda(A/I)$ and $e_{01} = \lambda(A/J)$.*
(1′) *$e_{10} \geqslant \lambda(A/I)$ and $e_{01} \geqslant \lambda(A/J)$.*
(2) *$P(r,s) = B(r,s)$ for all $r, s \geqslant 0$.*
(3) *The joint reduction number of (I, J) is zero, $r(I) \leqslant 1$ and $r(J) \leqslant 1$.*
(4) *The Rees ring $A[It_1, Jt_2]$ is Cohen–Macaulay.*

Proof. The equivalence of (1) and (1′) is clear from Lemma 6.2. First we show that hypotheses in (1) imply that the joint reduction number of (I, J) is zero. By [V, Theorem 3.2], it is enough to show that $e_1(I|J) = \lambda(A/IJ) - \lambda(A/I) - \lambda(A/J)$. By Corollary 5.2

$$e_{00} = \lambda\big(H^2_{\mathcal{R}_{++}}(\mathcal{R})_{(0,0)}\big),$$
$$e_1(I|J) + e_{10} + e_{01} + e_{00} - \lambda(A/IJ) = \lambda\big(H^2_{\mathcal{R}_{++}}(\mathcal{R})_{(1,1)}\big) - \lambda\big(\widetilde{IJ}/IJ\big).$$

Let (y_1, y_2) be a reduction of IJ coming from a complete reduction of (I, J). It follows from the long exact sequence of local cohomology modules corresponding to the short exact sequence

$$0 \longrightarrow \mathcal{R}^*(-1,-1) \xrightarrow{.y_1t_1t_2} \mathcal{R}^* \longrightarrow \mathcal{R}^*/y_1t_1t_2\mathcal{R}^* \longrightarrow 0$$

and Corollary 4.6, that for all $r, s \in \mathbb{Z}$

$$\lambda\big(H^2_{\mathcal{R}_{++}}(\mathcal{R})_{(r+1,s+1)}\big) \leqslant \lambda\big(H^2_{\mathcal{R}_{++}}(\mathcal{R})_{(r,s)}\big).$$

Therefore

$$e_1(I|J) + e_{10} + e_{01} + e_{00} - \lambda\big(A/\widetilde{IJ}\big) \leqslant e_{00}.$$

Hence

$$\begin{aligned} e_1(I|J) &\leqslant \lambda\big(A/\widetilde{IJ}\big) - \lambda(A/I) - \lambda(A/J) \\ &\leqslant \lambda(A/IJ) - \lambda(A/I) - \lambda(A/J). \end{aligned}$$

By the isomorphism $A/I \oplus A/J \cong (a,b)/aJ + bI$ for any regular sequence (a,b) where $a \in I$, and $b \in J$, it follows that

$$e_1(I|J) \geqslant \lambda(A/IJ) - \lambda(A/I) - \lambda(A/J).$$

Therefore

$$e_1(I|J) = \lambda(A/IJ) - \lambda(A/I) - \lambda(A/J).$$

Since the joint reduction number of (I, J) is zero, by [V, Theorem 3.2], for all $r, s \geqslant 1$

$$\lambda\big(A/I^r J^s\big) = \lambda\big(A/I^r\big) + e_1(I|J)rs + \lambda\big(A/J^s\big).$$

Write

$$\begin{aligned} \lambda\big(A/I^r\big) &= e_0(I)\binom{r}{2} + e_1(I)r + e_2(I) \quad \text{and} \\ \lambda\big(A/J^s\big) &= e_0(J)\binom{s}{2} + e_1(J)s + e_2(J). \end{aligned}$$

The reader may note that this way of writing the Hilbert polynomials of I and J is different from the way in which the Hilbert polynomial is usually written. Therefore the first Hilbert coefficient $e_1(I)$ appearing in the formulas above is different from the $e_1(I)$ appearing in papers of, for example, Huneke and Ooishi. Therefore, for $r, s \gg 0,$ we have,

$$\begin{aligned} P(r,s) &= e_0(I)\binom{r}{2} + e_1(I|J)rs + e_0(J)\binom{s}{2} + e_1(I)r + e_1(J)s \\ &\quad + e_2(I) + e_2(J). \end{aligned}$$

By assumption $e_1(I) = \lambda(A/I)$ and $e_1(J) = \lambda(A/J)$. By the Huneke–Ooishi theorem [H], for $d = 2$ we have $r(I) \leqslant 1$, $e_2(I) = 0$ and $r(J) \leqslant 1$, $e_2(J) = 0$. This proves (3) as well as (2). The statement (2) $\Rightarrow$ (1) is obvious. The equivalence of (2) and (3) follows from [V, Theorem 3.2] and [H, Theorem 2.1]. The equivalence of (3) and (4) follows from [Hy, Corollary 3.5] and Goto–Shimoda Theorem [GS]. □

The following example shows that a naive generalization of Theorem 6.3 does not work for $d > 2$.

Example 6.4. Let $A = k[\![x, y, z]\!]$, $I = (x^2, xy, y^2, z)$, and $J = (x, y^3, z)$. Then (x^2, y^2, z) is a reduction of I with reduction number 1. One can also check that $IJ = (x, z)I + y^2 J = xI + (y^2, z)J$. Therefore $r(I) = 1$, $r(J) = 0$ and joint reduction number of (I, J) is zero. One can see from computations on Macaulay 2 [GrS] that depth $\mathcal{R} = 4$. But dim $\mathcal{R} = 5$. Therefore $\mathcal{R}$ is not Cohen–Macaulay.

Example 6.5. Consider the plane curve $f = y^2 - x^n = 0$. Put $A = \mathbb{C}[\![x, y]\!]$ and $\mathfrak{m} = (x, y)A$. Let J denote the Jacobian ideal (f_x, f_y) of $f = 0$. Then $r(J) = r(\mathfrak{m}) = 0$. Moreover, $y\mathfrak{m} + xJ = \mathfrak{m}J$. Therefore by the previous theorem, the Bhattacharya polynomial of $\mathfrak{m}$ and J is given by the formula

$$\lambda\big(A/\mathfrak{m}^r J^s\big) = \binom{r}{2} + rs + (n-1)\binom{s}{2} + r + (n-1)s \quad \text{for all } r, s \geqslant 0.$$

Example 6.6. We give an example to show that neither of the conditions in (1) of Theorem 6.3 can be dropped to get the conclusions (2) and (3). Let $(A, \mathfrak{m})$ denote a 2-dimensional regular local ring. Let $\mathfrak{m} = (x, y)$ and $I = (x^3, x^2y^4, xy^5, y^7)$. Then $I\mathfrak{m} = x^3\mathfrak{m} + yI$. By [V, Theorem 3.2], we get

$$\begin{aligned}\lambda\big(A/\mathfrak{m}^r I^s\big) &= \lambda\big(A/\mathfrak{m}^r\big) + e_1(\mathfrak{m}|I)rs + \lambda\big(A/I^s\big)\\ &= \binom{r+1}{2} + o(I)rs + \lambda\big(A/I^s\big).\end{aligned}$$

In the above equation $o(I)$ denotes the $\mathfrak{m}$-adic order of I which is 3. The fact that $e_1(\mathfrak{m}|I) = o(I)$ is proved in [V]. We now calculate the Hilbert polynomial of I.

The ideal $J = (x^3, y^7)$ is a minimal reduction of I and $JI^2 = I^3$ and $\lambda(I^2/JI) = 1$. By a result of Sally [S], $\lambda(R/I^n) = P_I(n)$ for all $n > 1$. Here $P_I(n)$ denotes the Hilbert polynomial of I corresponding to the Hilbert function $\lambda(A/I^n)$. By using Macaulay 2 [GrS], we find that $\lambda(A/I) = 16$, $\lambda(A/I^2) = 52$, $\lambda(A/I^3) = 109$. Therefore the Hilbert polynomial

$$P_I(n) = 21\binom{n+1}{2} - 6\binom{n}{1} + 1.$$

Hence the Bhattacharya polynomial is

$$\begin{aligned}P(r, s) &= \binom{r+1}{2} + 3rs + 21\binom{s+1}{2} - 6\binom{s}{1} + 1\\ &= \binom{r}{2} + 3rs + 21\binom{s}{2} + \binom{r}{1} + 15\binom{s}{1} + 1.\end{aligned}$$

Therefore $e_{01} = 15 < \lambda(R/I)$. Notice that the constant term of the Bhattacharya polynomial is non-zero.

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