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K-TH SHORTEST COLLISION-FREE PATH PLANNING*

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ABSTRACT-A hierarchical approach is formulated for the shortest collision-free path construction problem. The new concepts of k-th neighborhood, k-th visibility graph and k-th shortest path are introduced. The proposed approach generalizes some earlier algorithms and allows for incremental improvement of the planned path.

1. INTRODUCTION

The class of problems of finding the collision-free paths for a moving object in the Euclidean plane cluttered with obstacles is known as collision-free path planning. We assume that the obstacles are represented by f disjoint convex polygons with n vertices in total. Furthermore, we assume that the two query points s and t describing the optimal collision free path do not lie in the interior of any polygon and that no three vertices are co-linear. We now try to solve the Shortest Collision-Free (SCF) path problem which is formulated as follows: given two points s and t, and f obstacles all belonging to some connected region in $IR^{2,}$ construct the SCF path connecting the points s and t in the shortest possible time.

The most popular method for collision-free planning is probably the one using the visibility space concept [Lozano-Perez, 1981]. It is based on the idea to shrink the object into a single point while at the same time expanding the obstacles according to the shape of the object. The visibility graph indicates all collision-free straight line paths among the vertices of the expanded polygonal obstacles. The search for the minimum path is subsequently performed on the visibility graph.

The main disadvantage of this method is the high demands for the preprocessing time. There exists an inverse relation between the amount of time invested in the initial computations and the resulting path optimality. Knowledge about the nature of the above tradeoff would allow the planner to choose the proper path planning algorithm according to the planning and the path traversal time requirements. Such a knowledge is also important in order to perform sensitivity analysis on the computed path [Bellman and Kalaba, 1960].

First, we note that the SCF path must consist only of the edges of the polygons and of the supporting lines between them [Rohnert, 1986]. Next, we introduce a hierarchy of visibility graphs using a concept of neighborhood. A corresponding hierarchy of the shortest collision-free paths is obtained subsequently.

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2. HIERARCHICAL ALGORITHM

Definition 2.1: Let P_1 and P_2 denote two obstacles (convex polygons). Define the distance between two polygons P_1 and P_2 via the relation

$$dist\,(P_1,P_2) = \min_{\substack{x\in\delta P_1 \\ y\in\delta P_2}} dist(x,y)$$

where $dist(x,y)$ is the Euclidean distance between two points x, $y\in IR^2$ and δP stands for the boundary of polygon P. Note, that $dist\,(P_1,P_2)$ can be computed in $O(log\ n_1 + log\ n_2)$ where n_1, and n_2 are the number of vertices of P_1 and P_2 respectively [Edelsbrunner, 1985].

Definition 2.2: Let Θ denote the set of the given polygonal obstacles. Let $P\in\Theta$. The set $N_1(P)$ of polygons $P_1 \ldots P_m\in\Theta$, that are closer to P than to any other polygon in Θ, constitutes the 1-neighborhood of the polygon P. Define $N_k(P)$, the k-th neighborhood of P, by the union of all 1-neighborhoods of all $P_j \in N_{k-1}(P)$, excluding P_j, i.e.,

$$N_k(P) \overset{\Delta}{=} \bigcup_{i=1}^{m} N_1\left(P_i^{(k-1)}\right) - N_{k-1}(P)$$

where $P_i^{(k-1)}$ belongs to the (k-1)-th neighborhood of P for every $i = 1,\ldots,m$, and

$$\bigcup_{i=1}^{m} P_i^{(k-1)} = N_{k-1}(P)\ .$$

Definition 2.3: Define V_0, the 0-visibility graph by $V_0 \overset{\Delta}{=} \Theta \cup \{s,t\}$. Define V_k, the k-th visibility graph, to be the union of the (k-1)-th visibility graph and of the set of all useful supporting segments between every polygon $P\in\Theta$ and every polygon $P^{(k)}$ in its k-th neighborhood. (A supporting segment s is useful if $s \cap int\,P = \emptyset, \forall P\in\Theta$.) More formally, let $S_k(j)$ denote the set of all useful supporting segments between polygon P_j and all the polygons belonging to its k-th neighborhood $N_k(P_j)$, and let $S_k \overset{\Delta}{=} \bigcup_{j=1}^{f'} S_k(j)$, where $f' \overset{\Delta}{=} |\ V_0\ | = f{+}2$. Then $V_k = V_{k-1} \cup S_k$.

Algorithm (Hierarchical SCF Path Planning):

Start with the 1-visibility graph.
repeat

Construct all the useful supporting segments within the current hierarchy level;
Solve the SCF Path Problem in the resulting visibility graph;
Increment the hierarchy level

until last hierarchy level is reached.

Definition 2.4: The SCF path resulting from the Algorithm at the k-th hierarchy level (or, in other words, from the k-th visibility graph), is called the k-th SCF path. (Figures 1, 2).

3. COMPUTATIONAL COMPLEXITY

The most natural way to view the k-th visibility graph is through the available proximity computation mechanisms since the k-th visibility graph is defined via the neighborhood concept. Let us associate an internal point T_i with every polygon $P_i \in \Theta$. By constructing the Voronoi diagram [Preparata and Shamos, 1985] on the set $\{T_i\}$ and using the edges of the diagram as pointers defining the neighbors, one can prove the following lemmas:

Lemma 3.1: $| N_k(i) | = 0(f), \quad k=0,...,f, \; i=1,...,f.$

Lemma 3.2: $| E(V_k) | = 0(kf+n)$.

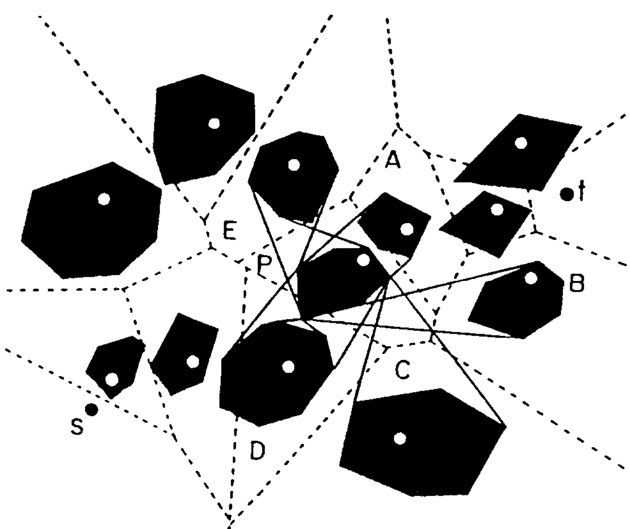

Figure 1. 1-visibility graph of P.

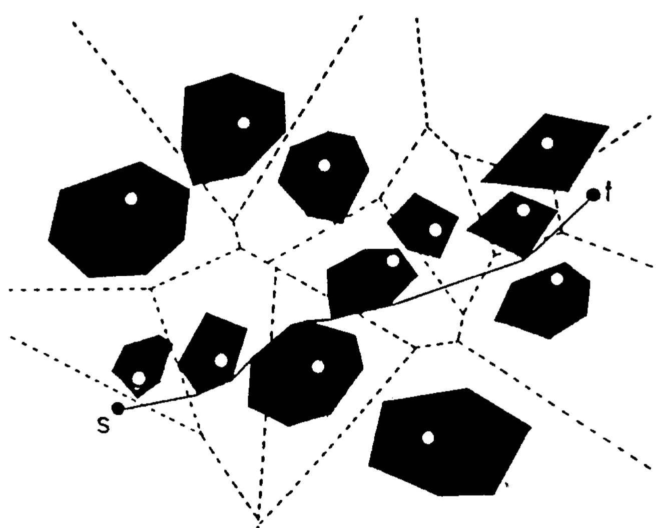

Figure 2. 1-optimal path from s to t.

Corollary 3.1: $|E(V_f)| = 0(f^2 + n)$.

Now, using the above two lemmas, the technique developed by Rohnert [Rohnert, 1986, proofs of lemmas 3 and 5], and mathematical induction, one can obtain

Theorem 3.1: The k-th visibility graph V_k can be constructed in $0(f \cdot k \log n + n)$ time and $0(f \cdot k + n)$ space.

Corollary 3.2: The $4f(f-1)/2$ supporting segments of convex polygons can be constructed in $0(n+f^2)$ space and $0\left(f^2 \log \frac{n}{f}\right)$ time [Lemma 4, Rohnert, 1986].

Corollary 3.3: The non-useful supporting segments in V_1 can be eliminated in $0(n)$ time and $0(f)$ space [Lemma 7, Lirov, 1987].

Corollary 3.4: The supporting lines between the neighboring polygonal obstacles can be computed in $0\left(f \log \frac{n}{f}\right)$ time [Theorem 2, Lirov, 1987].

And finally, by applying Dijkstra's algorithm [Fredman and Tarjan, 1984], we obtain our main result:

Theorem 3.2: The k-th SCF path can be computed in $0(k \cdot f + n \log n)$ time after $0(k \cdot f \log n + n)$ preprocessing.

Corollary 3.5: The SCF path can be computed in $0(n+f^2)$ space and $0(f^2 + n \log n)$ time after $0(n+f^2 \log n)$ preprocessing [Theorem, Rohnert, 1986].

Corollary 3.6: 1-optimal path can be computed in $0(n+f)$ space and $0(f+n \log n)$ time after $0(n+f \log n)$ preprocessing [Theorem 3, Lirov, 1987].

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